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An *n*-dimensional Ambarzumian type theorem for Dirac operators

Márton Kiss

Department of Mathematical Analysis, Institute of Mathematics, Budapest University of Technology and Economics, H 1111 Budapest, Műegyetem rkp. 3-9, Hungary

E-mail: mkiss@math.bme.hu

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Abstract
We consider the one-dimensional Dirac operator with continuous self-adjoint $n \times n$ matrix potential. In some special cases the potential can be reconstructed from one spectrum. The proof is based on the properties of a special variable (see lemma 2.3).

1. Introduction

Inverse spectral theory, which aims to reconstruct the potential from spectral data, began with a work of Ambarzumian [1] in 1929. He investigated the Schrödinger operator with Neumann boundary conditions, and found that if its spectrum consists of 0 and infinitely many other square numbers, then the potential is zero. In contrast to this, in general two spectra are needed (see Borg [2]), that is why special cases when one spectrum is sufficient are named after Ambarzumian. Detailed historical notes can be found in Horváth [4].

For operators with matrix potentials we refer to the works [6–9].

Consider on $[0, \pi]$ the eigenvalue problem

$$\psi_2' + Q(x)\psi_1 + m\psi_1 = \lambda\psi_1, \tag{1.1}$$

$$-\psi_1' + Q(x)\psi_2 - m\psi_2 = \lambda\psi_2 \tag{1.2}$$

with the boundary conditions

$$\psi_2(0) = \psi_2(\pi) = 0, \tag{1.3}$$

where the values of ψ_1 and ψ_2 are vectors of length n, Q is a self-adjoint (not necessarily real) $n \times n$ matrix potential and $m > 0$.

In the case $n = 1$ equations (1.1), (1.2) describe a relativistic electron with electrostatic potential $Q(x)$ (see [10]). Otherwise the spin of the described particle is greater. A more general problem is now under investigation, see the remarks at the end.

Let $\Psi(x)$ denote the n linearly independent solutions of (1.1) and (1.2) with $\Psi(0) = \begin{bmatrix} I \\ 0 \end{bmatrix}$. Let us denote by Ψ_1 and Ψ_2 the first and the last n rows of Ψ, respectively. Then λ is an eigenvalue iff $\Psi_2(\pi)$ is singular, and its multiplicity equals n-rank$(\Psi_2(\pi))$ (see also [3]). In the case when $Q = 0$, we can easily calculate the eigenvalues and eigenfunctions of (1.1)–(1.3). All the eigenvalues are of multiplicity n and

$$\begin{aligned} &\Psi(x,\lambda_k) = \begin{bmatrix} \cos kxI \\ \frac{\lambda_k - m}{k}\sin kxI \end{bmatrix}, \qquad \lambda_k = \pm\sqrt{m^2+k^2}, \quad k = \pm 1, \pm 2, \ldots, \\ &\Psi(x,m) = \begin{bmatrix} I \\ 0 \end{bmatrix}, \qquad \lambda_0 = m. \end{aligned} \tag{1.4}$$

Note that for a large $|\nu|$, $\lambda_\nu = \nu + o(1)$.

The main theorem of this paper reads as follows:

Theorem 1.1. *Suppose that the elements of Q are in $C[0,\pi]$ and the spectrum of the eigenvalue problem (1.1)–(1.3) is the same as in (1.4). Then $Q = 0$ on $[0,\pi]$.*

An analogue of this statement is already known for Schrödinger operators [3]. For Dirac operators with scalar potentials, Horváth in [4] proved a similar theorem with the assumption $m < 1/2$. In this paper a different method is used: we investigate the eigenvalues of a special variable defined in lemma 2.3.

2. The proof

Lemma 2.1.

$$\Psi_1^*\Psi_2 = \Psi_2^*\Psi_1. \tag{2.1}$$

Proof. $\Psi_1^*(0)\Psi_2(0) = \Psi_2^*(0)\Psi_1(0)$ and $(\Psi_1^*\Psi_2)' = -\Psi_2^*(\lambda I - Q + mI)\Psi_2 + \Psi_1^*(\lambda I - Q - mI)\Psi_1 = (\Psi_2^*\Psi_1)'$. □

Corollary 2.2.

$$(\Psi_1 + \mathrm{i}\Psi_2)^*(\Psi_1 + \mathrm{i}\Psi_2) = \Psi^*\Psi = (\Psi_1 - \mathrm{i}\Psi_2)^*(\Psi_1 - \mathrm{i}\Psi_2), \tag{2.2}$$

and these matrices are invertible.

Proof. Equation (2.2) follows immediately from the preceding lemma. Through the unicity of the solutions Ψ has a rank of n at every point, thus $\Psi^*\Psi$ is invertible. Consequently neither of $(\Psi_1 \pm \mathrm{i}\Psi_2)$ can be singular. □

Lemma 2.3. *Let $E = (\Psi_1 + \mathrm{i}\Psi_2)(\Psi_1 - \mathrm{i}\Psi_2)^{-1}$. Then for a real λ E is unitary.*

Proof. From (2.2) we can easily check that $E^* = E^{-1}$. □

Lemma 2.4.

$$E' = \mathrm{i}E(\lambda I - Q) + \mathrm{i}(\lambda I - Q)E - m(I + E^2). \tag{2.3}$$

Proof. Using the definition of E and $(B^{-1})' = -B^{-1}B'B^{-1}$, the statement follows. □

Lemma 2.5. *λ is an eigenvalue of the problem (1.1)–(1.3) iff* $+1$ *is an eigenvalue of* $E(\pi, \lambda)$*, and they have the same multiplicity.*

Proof. Let λ be an eigenvalue of the problem (1.1)–(1.3). Then its multiplicity equals n-rank$(\Psi_2) = n$-rank$((\Psi_1 + i\Psi_2) - (\Psi_1 - i\Psi_2)) = n$-rank$(E - I)$, which is exactly the multiplicity of the eigenvalue 1 of E. □

Lemma 2.6. *For a fixed real λ there exist n functions, differentiable in x, representing the repeated eigenvalues of E.*

Proof. Fix λ. For every x the eigenvalues of E are semisimple, thus by theorem 5.4 of chapter II of Kato [5] the unordered n-tuple of the repeated eigenvalues is differentiable. Then by theorem 5.6 of the same chapter there exist n complex-valued functions representing the eigenvalues of E in $[0, \pi]$. □

Lemma 2.7. *For a fixed real λ there exist n real-valued functions $\varphi_1, \varphi_2, \dots, \varphi_n$, differentiable in x, such that* $e^{i\varphi_k}$ *are the repeated eigenvalues of E, and* $\varphi_k(0) = 0$.

Proof. The statement follows from the fact that E is unitary and $E(0) = I$, using the preceding lemma. □

Lemma 2.8. *Let* $e^{i\varphi_{k_1}(x)} = e^{i\varphi_{k_2}(x)} = \cdots = e^{i\varphi_{k_j}(x)}$ *and let $P(x)$ be the eigenprojection of $E(x)$ corresponding to these eigenvalues. Then* $\varphi'_{k_1}(x), \varphi'_{k_2}(x), \dots, \varphi'_{k_j}(x)$ *are the repeated eigenvalues of* $P[2(\lambda I - Q) - m(E + E^{-1})]P$ *in* $Ran(P)$.

Proof. By theorem 5.4 of chapter II of Kato [5], $(e^{i\varphi_{k_l}(x)})'$ $(l = 1, 2, \dots, j)$ are the repeated eigenvalues of $P[iE(\lambda I - Q) + (\lambda I - Q)iE - mi(I + E^2)]P$, thus by $EP = PE = e^{i\varphi_{k_l}(x)}P$ the statement of the lemma follows. □

Corollary 2.9. *Let* $\Phi = \sum_{j=1}^{n} \varphi_j$. *Then*

$$\Phi'(x) = \mathrm{Tr}[2(\lambda I - Q) - m(E + E^{-1})]. \tag{2.4}$$

Corollary 2.10. *Let $f(x)$ be continuous in* $[0, \pi]$ *(and λ be real). Then*

$$\int_0^{\pi} f(x)\, e^{\pm i\varphi_k(x,\lambda)} \to 0 \tag{2.5}$$

if $|\lambda| \to \infty$.

Proof.

$$2(\lambda - \|Q\| - m) \leqslant \frac{d}{dx}\varphi_k(x, \lambda) \leqslant 2(\lambda + \|Q\| + m), \tag{2.6}$$

by $\|E + E^{-1}\| \leqslant 2$ and the preceding lemma. Thus Riemann's lemma yields the statement. □

Corollary 2.11.

$$\Phi(\pi, \lambda) = 2n\lambda\pi - 2\,\mathrm{Tr} \int_0^{\pi} Q + o(1) \tag{2.7}$$

if λ is real and $|\lambda| \to \infty$.

Proof. Integrate (2.4) from 0 to π. □

We defined the functions φ_k for a fixed λ, thus we do not know whether they are continuous in λ or not. But through (2.4) their sum must be continuous (moreover, holomorphic) in λ.

The proof of theorem 1.1. All eigenvalues have a multiplicity of n, thus λ is an eigenvalue of (1.1)–(1.3) iff for $1 \leqslant k \leqslant n$ $\varphi_k(\pi,\lambda)$ is a multiple of 2π. By the continuity of $\Phi(\pi,\lambda)$, if λ moves along the real line, $\Phi(\pi,\lambda)$ cannot change more than $2n\pi$ between two neighbouring eigenvalues. Using (2.7) and the fact that for a large ν, $\lambda_\nu = \nu + o(1)$,

$$\Phi(\pi,\lambda_\nu) - \Phi(\pi,\lambda_{-\nu}) = 4n\nu\pi + o(1). \tag{2.8}$$

For an eigenvalue λ, $\Phi(\pi,\lambda)$ must be a multiple of 2π, thus for a large ν

$$\Phi(\pi,\lambda_\nu) - \Phi(\pi,\lambda_{-\nu}) = 4n\nu\pi, \tag{2.9}$$

and by that, $\Phi(\pi,\lambda)$ changes exactly 2π between two neighbouring eigenvalues. Then through (2.7) the following must hold for all eigenvalues:

$$\Phi(\pi,\lambda_\nu) = 2n\nu\pi - 2\,\mathrm{Tr}\int_0^\pi Q \tag{2.10}$$

(and $\mathrm{Tr}\int_0^\pi Q$ must be an integer times π, too). Especially, for $\nu = 0$ we get

$$\Phi(\pi,m) = -2\,\mathrm{Tr}\int_0^\pi Q. \tag{2.11}$$

On the other hand, from (2.4),

$$\Phi(\pi,m) = 2nm\pi - 2\,\mathrm{Tr}\left[\int_0^\pi Q + \int_0^\pi m\frac{E(\cdot,m)+E^{-1}(\cdot,m)}{2}\right]. \tag{2.12}$$

Comparing these equations,

$$\mathrm{Tr}\int_0^\pi \frac{E(\cdot,m)+E^{-1}(\cdot,m)}{2} = n\pi. \tag{2.13}$$

This is impossible, except the case when $E(x,m) = I$. But then $\Psi_2(x,m) = 0$, thus $\Psi_1(x,m) = I$. Writing this to (1.1), $Q = 0$ arises, as we asserted. □

3. Remarks

Remark 1. If $m = 0$, theorem 1.1 does not remain true. Counterexamples for the one-dimensional case can be found in Horváth [4].

Remark 2. The potential cI with a constant c can also be reconstructed from its spectrum. We need only apply theorem 1.1 to the Dirac operator with potential $Q - cI$.

Remark 3. In the case of Schrödinger operators, it is enough to know a part of the spectrum, i.e., the ground state and infinitely many other eigenvalues. In the Dirac case the situation is a bit different: we need to know the eigenvalue m and (as seen from (2.9)) infinitely many positive and infinitely many negative eigenvalues.

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