Compact Topological Inverse Semigroups

Oleg Gutik

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Abstract

A locally compact commutative band with open principal ideals is a zero-dimensional scattered space. Cardinal invariants of locally compact and compact commutative bands with open principal ideals are investigated. A base of the topology of a compact commutative band with open principal ideals is determined and the structure of such compact semilattices in which every principal lower set is a chain is described. Every first countable compact inverse semigroup with open right (left, two-sided) principal ideals is metrizable.

Keywords: Topological inverse semigroup, commutative band, \(\alpha\)-semilattice, glt-semilattice, bopi-semigroup

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1. Introduction

A topological inverse semigroup is a Hausdorff topological space together with a continuous multiplication and an inversion.

We follow the terminology of [1], [3], [5], and [6].

Let \( S \) be a topological inverse semigroup and \( E \) the band of \( S \). We define the maps \( \phi : S \to E \) and \( \psi : S \to E \) by the formulae \( \phi(x) = xx^{-1} \) and \( \psi(x) = x^{-1}x \).

By \( \Omega \) denote the class of all ordinal numbers. Put \( \Omega(\alpha) = \{\beta \in \Omega \mid \beta \leq \alpha\} \) for all \( \alpha \in \Omega \). The set \( \Omega(\alpha) \) is well-ordered by the natural order \( \leq \): \( \gamma \leq \beta \) if \( \Omega(\gamma) \subseteq \Omega(\beta) \) for each \( \gamma, \beta \in \Omega(\alpha) \). By \( \omega \) we denote the first infinite ordinal. Further we identify all cardinals with their corresponding initial ordinals. The successor cardinal of \( \lambda \) is denoted by \( \lambda^+ \).

By \( |X|, w(X), d(X), \chi(X), c(X) \) we denote cardinality, weight, density, character, and cellularity of a topological space \( X \), respectively.

Let \( X \) and \( Y \) be topological spaces. A continuous map \( f : X \to Y \) is called pseudo-open if for each point \( y \in Y \) and for each open neighbourhood \( U \) of the set \( f^{-1}(y) \) the interior of the set \( f(U) \) contains \( y \) [2]. Recall from [12] that a continuous map \( f : X \to Y \) is a bi-quotient map if, whenever \( y \in Y \) and \( U \) is a cover of \( f^{-1}(y) \) by open subsets of \( X \), then finitely many \( f(U) \), with \( U \in \mathcal{U} \), cover a neighbourhood of \( y \) in \( Y \).

Definition 1.1. A topological inverse semigroup \( S \) is called a semigroup with open right (left, two-sided) principal ideals if the set \( aS \) (resp., \( Sa, SaS \)) is open in \( S \) for each \( a \in S \).

We define a semigroup \( S \) to be a semigroup with right (left) pseudo-open (resp., bi-quotient) translations iff for every \( a \in S \) the map \( r_a : S \to S \) (\( l_a : S \to S \)) determined by \( r_a(x) = ax \) (\( l_a(x) = xa \)) is pseudo-open (resp., a bi-quotient) map. A semigroup together with pseudo-open (resp., bi-quotient) right and left translations we call a semigroup with pseudo-open (resp., bi-quotient) translations.
Further a band is an idempotent semigroup.

Let $E$ be a commutative band (a semilattice). For $e, f \in E$ we write $e \leq f$ if and only if $ef = fe = e$; this defines a partial order on $E$. An idempotent $e \in E$ is called maximal if $ef \neq e$ for each $f \in E \setminus \{e\}$. Further, by $\text{Max } E$ we denote the subset of all maximal idempotents of $E$. Obviously, all ideals are open in $E$ if all principal ideals are open in $E$.

Denote

$$L(e) = \{ f \in E \mid ef = fe = f \}$$

and

$$M(e) = \{ f \in E \mid ef = fe = e \}$$

for every $e \in E$.

Obviously, there exists a compact commutative band with open principal ideals which is not a semigroup with open translations: it is sufficient to consider $J = \{1, 2\} \cup \{1 - \frac{1}{n} \mid n \in \mathbb{N}\}$ with the natural topology and the multiplication: $e * f = \min\{e, f\}$. Evidently, $J$ is the countable compact linearly ordered commutative band with open principal ideals; but translations are not open in $J$.

We shall prove that any locally compact commutative band with open principal ideals is a zero-dimensional scattered space. A base of a topology of a compact commutative band with open principal ideals is determined and some cardinal invariants of such bands are investigated. The structure of a compact semilattice with open principal ideals in which every principal lower set is a chain is described. It is proved that: every first countable compact inverse semigroup whose band is a semigroup with open principal ideals, is metrizable.

This paper is a continuation of previous author’s studies [9].

2. Compact commutative bands with open principal ideals

Lemma 2.1. Let $E$ be a locally compact commutative band with open principal ideals. Then $E$ is a zero-dimensional topological space.

Proof. Let $e$ be an idempotent of $E$ and $U(e)$ an open neighbourhood of $e$. Then there exists a neighbourhood $V(e)$ such that $\overline{V(e)}$ is a compact subset of $E$ and $\overline{V(e)} \subseteq U(e)$. The set $L(e) \cap \overline{V(e)}$ is compact and $L(e) \cap V(e)$ is open in $E$.

Consider a compact subspace $N(e) = (L(e) \cap \overline{V(e)}) \setminus (L(e) \cap V(e))$ of $E$. The family $\gamma = \{ L(f) \mid f \in N(e) \}$ is an open cover of $N(e)$. Then there exists a finite subcover $\gamma' \subseteq \gamma$ of $N(e)$. Thus, $O(e) = L(e) \setminus (\cup \gamma')$ is an open compact neighbourhood of the element $e$ and $O(e) \subseteq U(e)$. Hence, $E$ is a zero-dimensional topological space.

Definition 2.2. [11] A semilattice which has a basis of subsemilattices is called Lawson semilattice.

Corollary 2.3. Every compact commutative band with open principal ideals is a Lawson semilattice.

This follows from Lemma 1 and Theorem II.1.5 of [6].
Proposition 2.4. If $E$ is a compact commutative band with open principal ideals, then $\text{Max } E$ is a finite subset of $E$.

A topological space is scattered if it does not contain a nonempty dense-in-itself subspace [5].

Lemma 2.5. If $E$ is a locally compact commutative band with open principal ideals, then $E$ is a scattered topological space.

Proof. We shall prove that every non-empty closed subset of $E$ has an isolated point.

Let $A$ be a closed subset of $E$ and $e \in A$. By Lemma 2.1, there exists an open compact neighbourhood $U(e)$ of $e$ such that $U(e) \subseteq L(e)$. Denote $e_1 = e$. The subset $A_1 = A \cap U(e)$ is both compact and open in $A$. We shall prove that $A_1$ has an isolated point. If $A_1$ is a singleton, then $\{x\} = A_1$ and $x$ is an isolated point of $A_1$. In the other case, put $A_2 = A_1$ and $e_2$ is any point of $A_2 \setminus \{e_1\}$. Let $\lambda$ be the cardinality of $A \setminus \{e_1\}$. By transfinite induction we construct compact subsets $A_\alpha$ and choose points $e_\alpha$ for $2 < \alpha < \lambda^+$ as follows. If $\alpha$ is not a limit ordinal, then set $A_\alpha = L(e_{\alpha-1}) \cap A_1$ and let $e_\alpha$ be any element of $A_\alpha \setminus \{e_{\alpha-1}\}$. If $\alpha$ is a limit ordinal, the set $A_\alpha = \bigcap \{L(e_\beta) : \beta < \alpha\} \cap A_1$ and $e_\alpha$ is any point of $A_\alpha$. If for some $2 < \alpha < \lambda^+$ the set $A_\alpha$ is singleton, then $\{x\} = A_\alpha$. Hence, $\{x\} = L(x) \cap A_1$ and $x$ is an isolated point of $A_1$. In the other case the set $A^* = \bigcap_{\alpha < \lambda^+} A_\alpha$ is nonempty and any two disjoint idempotents of $A^*$ are not comparable. If $A^*$ is a singleton, then $\{x\} = A^*$. In the other case choose any point $x \in A^*$. Then $\{x\} = L(x) \cap A_1$ and $x$ is an isolated point of the set $A_1$.

Definition 2.6. [6] An element $e$ of $E$ is called a local minimum if there exists an open neighbourhood $U$ of $e$ such that $L(e) \cap U \subseteq M(e)$.

The set of all local minima of $E$ will be denoted by $K(E)$. If $E$ is a commutative band with open principal ideals, then $K(E)$ is a set of isolated points of $E$.

Theorem 2.7. Let $E$ be a locally compact commutative band with open principal ideals. Then

(i) $K(E)$ is dense in $E$;

(ii) $c(E) = d(E) = |K(E)| = w(E) = |E|$.

Proof. By Lemmas 2.1 and 2.5, the band $E$ has a base $B$ whose elements are compact and such that each element of $B$ contains an isolated point in $E$. Thus $K(E)$ is dense in $E$.

By Proposition 1.21 [4] and Lemma 2.5 we get $w(E) = |E|$. The equality $w(E) = |K(E)|$ follows from [7] (see: p.168A). A set of all local minima of $E$ is dense in $E$, thus $d(E) \leq |K(E)|$. Since every point of $K(E)$ is isolated in $E$, we have $c(E) \geq |K(E)|$. Thus, $c(E) = d(E) = |K(E)|$.  

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Theorem 2.8. Let $E$ be a compact commutative band with open principal ideals. Then the following conclusions hold:

(i) any subset of pairwise non-comparable idempotents in $K(E)$ is finite;
(ii) the family $B(x) = \{L(x) \cap M(e) \mid e \in K(E) \cap L(x)\}$ is a base of the topology of $E$ in a point $x \in E$;
(iii) $\chi(E) = c(E) = |K(E)| = |E|$.

Proof. Let $Y$ be any subset of pairwise non-comparable elements of $K(E)$. Consider the family of open sets $\gamma = \{M(x) \subseteq E \mid x \in Y\}$.

Put $G = \bigcup_{y \in Y} M(x)$. Obviously, for any $y \in E \setminus G$ the inclusion $L(y) \subseteq E \setminus G$ holds and thus $E \setminus G$ is a open set in $E$. Evidently, that the family $\gamma^* = \gamma \cup \{E \setminus G\}$ is a open cover of the space $E$. Note that any point $x \in Y$ lies only in $M(x) \in \gamma^*$. Since the band $E$ is compact, we have the cover $\gamma^*$ has a finite subcover. Thus, $Y$ is finite.

To prove statement (ii), show that for every idempotent $x \in E$ and any its neighbourhood $V(x)$ there exists $e \in K(E) \cap L(x)$ such that $U_e(x) = L(x) \cap M(e) \subseteq V(x)$.

If $x \in K(E)$, then $e = x$.

Suppose $x \in E \setminus K(E)$. Since $K(E)$ is dense in $E$, there exists an idempotent $e_0 \in K(E) \cap L(x) \cap V(x)$. The set $B = U_{e_0}(x) \setminus V(x)$ is compact. Then the open cover $\gamma_B = \{L(x) \mid x \in B\}$ of $B$ has a finite subcover $\gamma_B^* = \{L(x_i) \mid x_i \in B, i = 1, \ldots, n\}$. The set $U_{e_0}(x) \setminus (L(x_1) \cup \ldots \cup L(x_n))$ is open in $E$ and $K(E)$ is dense in $E$, hence there exists $e \in K(E) \cap L(x)$ such that $e \in U_{e_0}(x) \setminus (L(x_1) \cup \ldots \cup L(x_n))$. Thus

$$U_e(x) \subseteq U_{e_0}(x) \setminus (L(x_1) \cup \ldots \cup L(x_n)) \subseteq V(x) \setminus B \subseteq V(x).$$

Statement (iii) follows from Lemma 2.5, Theorem 1.25 [4] and statement (ii) of Theorem 2.7.

Proposition 2.9. There exists no structure of a topological commutative band with open principal ideals on the one-point Alexandroff compactification of an uncountable discrete space.

Proof. Let $A(X)$ be a one-point Alexandroff compactification of an uncountable discrete space $X$ and $\{a\} = A(X) \setminus X$.

Suppose the contrary.

Let $a$ be the identity of $A(X)$. We denote $NO(e) = A(X) \setminus M(e)$ for all $e \in A(X)$. If $e \in A(X) \setminus \{a\}$, then $M(e)$ is clopen, and hence $NO(e)$ is finite.

Let $e_1$ be an idempotent of $A(X) \setminus \{a\}$. Since $M(e_1)$ is infinite, there exists $e_2 \in M(e_1) \setminus \{\{e_1\} \cup \{a\}\}$ such that $e_1 < e_2$. Further, by induction for every integer $n \geq 3$ choose an idempotent $e_n \in M(e_{n-1}) \setminus \{\{e_{n-1}\} \cup \{a\}\}$. We obtain

$$e_1 < e_2 < \ldots < e_n < \ldots, \quad NO(e_1) \subset NO(e_2) \subset \ldots \subset NO(e_n) \subset \ldots$$

If $\bigcup_{n \in \mathbb{N}} NO(e_n) = A(X) \setminus \{a\}$, then $X$ is countable. In the other case, for any choice of points $\{e_n\}_{n \in \mathbb{N}}$ there exists an idempotent $e \in A(X) \setminus (\bigcup_{n \in \mathbb{N}} NO(e_n)) = \ldots$.
\[ \cap_{n \in \mathbb{N}} M(e_n). \] By the construction, \( \{e_1, \ldots, e_n \ldots \} \subseteq NO(e) \), but \( NO(e) \) is finite; a contradiction. Thus, if \( a \) is the identity of \( A(X) \), then \( A(X) \) and \( X \) are countable.

Suppose \( a \) is a non-zero idempotent and \( a \) is not the identity. Then \( a \) is the identity of \( L(a) \). Since \( L(a) \) is clopen, we have \( |L(a)| > \omega \), a contradiction with our preceding proof.

Suppose \( a \) is a zero of \( A(X) \). Then \( a \) is an isolated point of \( A(X) \). Contradiction.

The following Example shows that for every cardinal \( \lambda \) there exists a compact commutative band with open principal ideals \( E \) such that \( |E| = \lambda \).

**Example 2.10.** Let \( \alpha \) be an ordinal. Put

\[ \mathcal{B} = \{(x, y) = \{z \in \Omega(\alpha) \mid y < z \leq x\} \mid x, y \in \Omega(\alpha) \& y < x\} \cup \{0\}, \]

where 0 is the order type of the empty set. Let \( \tau_\Omega \) be the determined by the base \( B \) topology on \( \Omega(\alpha) \). By Theorem 6.9 [1], \((\Omega(\alpha), \tau_\Omega)\) is a Hausdorff compact topological space. We define a semigroup operation on \( \Omega(\alpha) \) by: \( \beta \ast \gamma = \min\{\beta, \gamma\} \) for all \( \beta, \gamma \in \Omega(\alpha) \).

Obviously, \((\Omega(\alpha), \ast, \tau_\Omega)\) is a topological commutative band with open principal ideals.

**Definition 2.11.** A topological semilattice \( E \) is called an \( \alpha \)-semilattice if \( E \) is topologically isomorphic to \((\Omega(\beta), \ast, \tau_\Omega)\) for some ordinal \( \beta \).

3. Compact glt-semilattices with open principal ideals

A commutative band \( E \) is called linearly ordered (well-ordered) if the multiplication induces on \( E \) a linear order (a well-order).

**Definition 3.1.** A commutative band \( E \) is called a glt-semilattice iff for each \( x \in E \) the subband \( L(x) \) is linearly ordered.

Since the band \( E \) is compact and for all \( x \in E \) the set \( L(x) \) is clopen we obtain

**Lemma 3.2.** Let \( E \) be a linearly ordered compact commutative band with open principal ideals. Then \( E \) is well-ordered.

**Proposition 3.3.** Every well-ordered semilattice \( E \) is algebraically isomorphic to a subsemilattice of \((\Omega(\alpha), \min)\) for some \( \alpha \in \Omega \).

**Proof.** Since the cardinality of \( E \) is bounded, by Theorem 3.11' [1] the well-ordered set \( E \) is similar to some interval of \( \Omega(\alpha) \) (where \( \alpha \geq |E|^+ \)). This similar map we denote by \( f \). Obviously, \( f \) is an algebraic isomorphism of \( E \) into \((\Omega(\alpha), \min)\).

**Theorem 3.4.** Every linearly ordered compact commutative band \( E \) with open principal ideals is an \( \alpha \)-semilattice.

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Proof. By Lemma 3.2, the band \( E \) is well-ordered set and by Proposition 3.3, there exists an algebraic isomorphism \( f : E \rightarrow \Omega(\delta) \) (where \( \delta \geq |E|^+ \)). Obviously, \( E \) has an identity \( 1 \) and we put \( f(1) = \beta \in \Omega(\delta) \). It is easy to see that \((f(E), \min)\) is \( \alpha \)-semilattice and the isomorphism \( f : E \rightarrow \Omega(\delta) \) is continuous.

Corollary 3.5. Every compact glt-semilattice with open principal ideals is a tree. \( \blacksquare \)

Theorem 3.6. Every compact glt-semilattice \( E \) with open principal ideals is a finite direct sum of \( \alpha \)-semilattices.

Proof. The subset of maximal idempotents of \( E \) is finite and we put \( \text{Max} E = \{e_1, \ldots, e_n\} \). Put \( \tilde{L}(e_1) = L(e_1) \) and for all \( i = 2, \ldots, n \) define \( \tilde{L}(e_i) = L(e_i) \setminus (\bigcup_{k=1}^{n-1} \tilde{L}(e_k)) \). Obviously, \( E = \bigoplus_{i=1}^{n} \tilde{L}(e_i) \) and \( \tilde{L}(e_i) \) for each \( i = 1, \ldots, n \) are clopen \( \alpha \)-semilattices as subsemilattices. \( \blacksquare \)

4. Topological inverse semigroup with restrictions on translations

Theorem 4.1. Let \( S \) be a topological inverse semigroup and \( E \) be the band of \( E \). Then the following conditions are equivalent:

(i) \( S \) is a semigroup with open right principal ideals;
(ii) \( S \) is a semigroup with open left principal ideals;
(iii) \( E \) is a semigroup with open principal ideals;
(iv) \( S \) is a semigroup with pseudo-open translations;
(v) \( E \) is a semigroup with pseudo-open translations;
(vi) \( S \) is a semigroup with bi-quotient translations;
(vii) \( E \) is a semigroup with bi-quotient translations.

Proof. Implications (i) \( \Rightarrow \) (iii), (ii) \( \Rightarrow \) (iii), (iv) \( \Rightarrow \) (v), (vi) \( \Rightarrow \) (vii) and (vii) \( \Rightarrow \) (v) are trivial.

(iii) \( \Rightarrow \) (i). We shall prove that for any \( a \in S \) the equality \( aS = \varphi^{-1}(aa^{-1}E) \) holds. Let \( x \) be an element of the set \( \varphi^{-1}(aa^{-1}E) \), then \( xx^{-1} \in aa^{-1}E \) and there exists an idempotent \( e \in E \) such that \( xx^{-1} = aa^{-1}e \). Hence, \( x = aa^{-1}ex \) and \( x \in aS \). Thus, \( \varphi^{-1}(aa^{-1}E) \subseteq aS \).

Now we shall prove the inclusion \( aS \subseteq \varphi^{-1}(aa^{-1}E) \). Let \( x \) be an element of the set \( aS = aa^{-1}S \), then there exists an element \( y \in S \) such that \( x = aa^{-1}y \). Thus,

\[
xx^{-1} = aa^{-1}y(aa^{-1}y)^{-1} = aa^{-1}y^{-1}aa^{-1} = aa^{-1}aa^{-1}y^{-1} = \]
\[
aa^{-1}y^{-1} \in aa^{-1}E
\]

and we obtain \( aS = \varphi^{-1}(aa^{-1}E) \). Since for every \( a \in S \) the set \( aa^{-1}E \) is open in \( E \), therefore \( aS \) is open in \( S \). Thus \( S \) is a semigroup with open right principal ideals.

The implication (iii) \( \Rightarrow \) (ii) follows from the equality \( Sa = \psi^{-1}(a^{-1}aE) \) for each \( a \in S \).

(iii) \( \Rightarrow \) (v). Suppose \( E \) is a semigroup with open principal ideals. Let \( e \) be an element of \( E \) and \( U \) an open neighbourhood of \( M(e) \). Then \( e \in L(e) \cap U \subseteq eU \) and, hence, \( E \) is a semigroup with pseudo-open translations.

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Suppose \( E \) is a semigroup with pseudo-open translations. Let \( e \) be an element of \( E \). Since \( f \in \text{Int}(fE) \) for every \( f \in E \), we see that \( L(e) = \bigcup_{f \leq e} \text{Int}(fE) \) is open in \( E \).

(v) \( \Rightarrow \) (iv). Suppose \( E \) is a semigroup with pseudo-open translations. We shall prove that \( S \) is a semigroup with right pseudo-open translations (the proof that \( S \) is a semigroup with left pseudo-open translations is analogous). Since (i)\( \Leftrightarrow \) (v), \( S \) is a semigroup with open right principal ideals. Define a map \( h_a : S \to S \) by the formula \( h_a(x) = ax \). Let \( b \in aS \) and \( U \) be any neighbourhood of the set \( A = h_a^{-1}(b) \). Obviously, \( A \cap a^{-1}S \neq \emptyset \). Since the map \( h_a|_{a^{-1}S} : a^{-1}S \to aS \) is a topological isomorphism, the set \( A \cap a^{-1}S \) is a singleton. Put \( \{t\} = A \cap a^{-1}S \). The set \( U \cap a^{-1}S \) is open such that \( S \) is a semigroup with right open principal ideals. Since the map \( h_a|_{a^{-1}S} : a^{-1}S \to aS \) is a topological isomorphism, the set \( h_a(U \cap a^{-1}S) = h_a|_{a^{-1}S}(U \cap a^{-1}S) \) is open in \( S \). Thus, \( \text{Int}(h_a(U)) \neq \emptyset \) and \( b \in \text{Int}(h_a(U)) \).

The proof of implication (vii) \( \Rightarrow \) (vi) is analogous to the proof of (v) \( \Rightarrow \) (iv).

(iii) \( \Rightarrow \) (vii). Suppose \( E \) be a semigroup with open principal ideals. Let \( e \) be an element of \( E \) and \( U \) an open neighbourhood of \( e \). Then \( e \in U \cap L(e) \subseteq eU \). Thus, any translation in \( E \) is a bi-quotient map.

The following Example shows there exists a compact inverse semigroup \( S \) with a finite band \( E(S) \) of idempotents (hence, \( E(S) \) is a semigroup with open translations), but \( S \) is not a semigroup with open translations.

**Example 4.2.** Let \( G \) be a compact nondiscrete topological group. Put \( S = G^1 \) and suppose the identity of \( S \) is isolated in \( S \).

**Definition 4.3.** A topological inverse semigroup \( S \) is called a *bopi-semigroup* iff the band \( E(S) \) of \( S \) is a semigroup with open principal ideals.

**Theorem 4.4.** Let \( S \) be a Clifford topological inverse semigroup. Then \( S \) is a bopi-semigroup if and only if \( S \) is a semigroup with open two-sided principal ideals.

**Proof.** Obviously, every Clifford topological inverse semigroup with open two-sided principal ideals is a bopi-semigroup.

Suppose \( S \) is a bopi-semigroup and \( E \) the band of \( S \). Since \( S \) is a Clifford inverse semigroup, the maps \( \varphi : S \to E \) and \( \psi : S \to E \) coincide. We shall prove that \( SaS = \varphi^{-1}(aa^{-1}E) \) for every \( a \in S \). If \( x \in SaS \), then there exist \( t, s \in S \) such that \( x = sat \). By Theorem II.2.6 [13], we have

\[
x x^{-1} = sat(s a t)^{-1} = s a t^{-1} a^{-1} s^{-1} = s a a^{-1} t t^{-1} s s^{-1} = a a^{-1} t t^{-1} s s^{-1} = a a^{-1} S.
\]

Thus, \( SaS \subseteq \varphi^{-1}(aa^{-1}E) \).

Let \( x \in \varphi^{-1}(aa^{-1}E) \). Then \( xx^{-1} = x^{-1} x \in aa^{-1}E \). There exists an idempotent \( e \in E \) such that \( xx^{-1} = aa^{-1} e \); hence, \( x = xx^{-1} x = xx^{-1} aa^{-1} e x \) and \( x \in SaS \). Thus, \( \varphi^{-1}(aa^{-1}E) \subseteq SaS \).

\[
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\]
Example 4.5 shows that there exists a topological inverse semigroup $S$ with two-sided open principal ideals such that $S$ is not a bopi-semigroup.

**Example 4.5.** Let $S$ be an inverse semigroup and $a, b \notin S$. A semigroup $C(S)$ is generated by the set $S \cup \{a, b\}$ and is defined by the following equalities: $ab = 1, as = a, sb = b$ and by equalities in $S$. If $S$ has the identity then the identity of $C(S)$ is the identity of $S$. In the other case the identity of $C(S)$ is an accessory identity of $S$ (see: [3], § 1.1). Any element of $C(S)$ is uniquely represented by $b^i a^j, t \in S \cup \{1\}, i, j \in \mathbb{N} \cup \{0\}$.

Let $S$ be a topological inverse semigroup. If $S$ has no identity let $S^1 = S \cup \{1\}$ be a semigroup with an isolated accessory identity. Let $B$ be a base of the topology on $S^1$. A topology $\tau$ on $C(S)$ is determined by the base

$$B = \{b^i U a^j | U \in B, i, j \in \mathbb{N} \cup \{0\}\}.$$  

By Corollary 1 [10] $C(S)$, $tau$) is a simple topological inverse semigroup and $S$ is topologically isomorphically imbedded into $(C(S), \tau)$. The semigroup $(C(S), \tau)$ is called the **Bruck semigroup over $S$**.

Let $S$ be a topological inverse semigroup and the band of $S$ is not a bopi-semigroup. Put $C(S)$ be the Bruck semigroup over $S$. Then $C(S)SC(S) = C(S)$ for all $s \in C(S)$ and hence all two-sided principal ideals are open in $C(S)$. However, the principal ideals of the band of $C(S)$ are not open in the band.

Further, $S$ is a compact inverse semigroup and $E(S)$ is the band of $S$. For all $e, f \in E(S)$ define $H(e, f) = \{x \in S | xx^{-1} = e, x^{-1}x = f\}$ and $H(e) = H(e, e)$.

The following theorem affirmatively answers a conjecture of B. Bokalo (see: [8]) for the class of compact inverse bopi-semigroups.

**Theorem 4.6.** Every first countable compact inverse bopi-semigroup is metrizable.

**Proof.** Let $S$ be as in the statement. The band $E(S)$ is a first countable space. Let $e, f \in E(S)$. If $H(e, f) \neq \emptyset$, then $H(e, f)$ is homeomorphic to the metrizable subgroup $H(e)$ and, hence, $H(e, f)$ is a metrizable compactum. Theorem 2.8 implies $|E(S)| = \chi(E(S)) \leq \omega$, hence, $S$ is a countable union of metrizable compacta and by the Arhangelskii Theorem (see Theorem 3.2.20 [5]) $S$ is metrizable.

**Definition 4.7.** A topological inverse semigroup $S$ is called an $\alpha$-semigroup (a glt-semigroup) iff the band of $S$ is an $\alpha$-semilattice (a glt-semilattice).

Theorem 3.4 implies

**Corollary 4.8.** Every compact inverse bopi-semigroup with linearly ordered band is an $\alpha$-semigroup.

Theorem 3.6 implies

**Corollary 4.9.** Every compact Clifford inverse glt-bopi-semigroup is a finite direct sum of $\alpha$-semigroups.
Proposition 4.10. There exists no structure of topological inverse bopi-semigroup on the one-point Alexandroff compactification of an uncountable discrete space.

Proof. Let $\mathcal{A}(X)$ be the one-point Alexandroff compactification of an uncountable discrete space $X$ and $\{a\} = \mathcal{A}(X) \setminus X$. Suppose there exists a structure of a topological inverse bopi-semigroup on $\mathcal{A}(X)$. The band of the semigroup $\mathcal{A}(X)$ is denoted by $E$. Since the inversion is a homeomorphism, $a$ is an idempotent of $\mathcal{A}(X)$. The maximal subgroups $H(e)$ and sets $H(e, f) \ (e, f \in E)$ are finite. Hence $|E| = |\mathcal{A}(X)| > \omega$.

Every point of $E \setminus \{a\}$ is isolated in $\mathcal{A}(X)$ and $E$ is closed in $\mathcal{A}(X)$. Therefore, the topological space $E$ is homeomorphic to the one-point Alexandroff compactification of an uncountable discrete space. A contradiction with Proposition 2.9.

Problem (I.V. Protasov). Describe compact topological spaces which admit a structure of commutative band with open principal ideals.

Remark 4.11. The finite Cartesian product of semigroups with open right (resp. left, two-sided) principal ideals is a semigroup with open right (resp. left, two-sided) principal ideals.

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References


Department of Algebra
Institute of Applied Problems of Mechanics
and Mathematics of National Academy of
Science of Ukraine
3b, Naukova Str., Lviv, 79601, UKRAINE
e-mail: ogutik@iapmm.lviv.ua

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