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Invited Lectures
and
Abstracts of Research Reports

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Preface

The main aim of summer schools is to form an atmosphere of creative communication between leading experts in various fields of mathematics and students and young researchers.

The first Summer school in Topological Algebra and Functional Analysis was held on July 22–31, 2003 in the village of Kozyova, Lviv region, in one of picturesque places of Carpathian mountains. The School was organized by the Lviv National University, mainly by topologists from the Faculty of Mechanics and Mathematics. Among the lectors there were T. Banakh, M. Gorbachuk, I. Guran, O. Lopushans’kyi, V. Maslyuchenko, V. Kadets, T. Radul, V. Sushchans’kyi, A. Zahorodnyuk, M. Zarichnyi. About 70 post-graduate students and young researchers participated in the work of the School.

In 2004, the framework of summer school as well as its place was changed: the second Summer school in Algebra and Topology was held on August 2–14, 2004 in Dolyna, Ivano-Frankivs’k region. The following lectures were delivered:

- Ya. Prytula “History of Lviv mathematics”;
- V. Sharko “$L–2$ cohomology of Smooth Manifolds and applications”;
- A. Plichko “Automatic continuity of group homomorphisms”;
- S. Maksimenko “Morse functions on surfaces”;
- I. Protasov “Topological dynamic and combinatorics”;
- M. Zarichnyi "Algebraic topology”;
- R. I. Grigorchuk “Basilica group”;
- O. Gutik “Topological inverse semigroups”;

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• W. Suszczyński “The problem of classification of locally finite simple groups”;
• M. Komarnytskyi “Model theory”;
• T. Banakh “Semifilters”;
• O. Lopushansky “Functional representations of dual symmetric Fock spaces associated with compact groups”;
• V. Andriychuk “Locally-global principle in the fields theory and its application”.

During the second Summer school two workshops were organized: "Asymptotic topology" (I. Protasov and M. Zarichnyi) and "Topological Semigroup Theory" (T. Banakh and O. Gutik).

The material below contains texts/programs of the lecture courses and the abstracts of research talks.
Invited Lectures

Local-Global Principle in Algebraic Geometry

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The author hopes to explain the notions and results whose expositions in these notes are too fragmentary. Also the author wish to thank U. Derenthal and D. Eriksson whose papers placed in the Internet were used in preparing of these notes.
1. Fields with valuations and their completions

Let \( K \) be a field. A map \( K \to \mathbb{R}, x \mapsto |x| \) is called a valuation of \( K \) if it satisfies the following properties:

1) \( |x| \geq 0, |x| = 0 \iff x = 0; \)

2) \( |xy| = |x||y| \forall x, y \in K; \)

3) \( |x + y| \leq |x| + |y|. \)

If this map satisfies the stronger condition

3') \( |x + y| \leq \max\{|x|, |y|\} \) than it is called nonarchimedean, otherwise it is archimedean.

It is known (Ostrovski's theorem (see [ant])) that every valuation of rational number field \( \mathbb{Q} \) is equivalent to either the usual absolute value or to a \( p \)-adic valuation \( | \cdot |_p \) defined by \( |p^m/m|_p = 1/p^m \) where \( p \) is a prime number, and \( m, n \) are nonzero integers, \( p \nmid mn \). Similarly, if \( K = k(t) \) is a rational function field over a field \( k \), then one can define a valuation \( | \cdot |_{p(t)} \) by \( |p(t)^a a(t)/b(t)|_{p(t)} = c^e \) where \( c \in \mathbb{R}, 0 < c < 1, a(t), b(t) \) are nonzero polynomials, \( p(t) \) is an irreducible polynomial, \( p(t) \nmid a(t) b(t) \). There is one another valuation \( | \cdot |_{\infty} \) of \( k(t) \) for which \( a(t)/b(t) = e^{\deg a(t) - \deg b(t)} \). The described valuations of the rational function field \( k(t) \) are the only ones which are trivial on the constant field \( k \).

The above valuations on \( \mathbb{Q} \) and on \( k(t) \) can be extended to any algebraic extension of these fields. In this way one obtains the valuations of number fields and of algebraic function fields in one variable, in particular, the valuations of global fields (i.e. finite extension of \( \mathbb{Q} \) and of \( k(t) \) with finite constant field \( k \)).

Then as usual one can pass to the completions: for each valuation \( v \in V^K \) we consider the ring of the corresponding fundamental sequences. The sequences converging to zero form the maximal ideal in it, and the quotient is said to be the completion of \( K \) at \( v \) and is denoted by \( K_v \). When \( K = \mathbb{Q} \), the corresponding to a \( p \)-adic valuation \( | \cdot |_p \) completion \( \mathbb{Q}_p \) is called the \( p \)-adic field.

Let \( X \) be an algebraic variety defined over a field \( K \) endowed with a set of valuations \( V^K \). I will try to explain the relationships of properties of \( X \) (especially when \( X \) is an algebraic group) defined over the field \( K \) with the corresponding properties of \( X \) regarded as a variety over the completions \( K_v \) for \( v \in V^K \) by focusing attention on the most interesting cases where \( K \)
is a global field (in particular \( \mathbb{Q} \)). Also I will consider another class of fields which I call the pseudoglobal fields, and which share with global fields the property of existence of a version of class field theory. By a pseudoglobal field I mean the finite extension of \( k(l) \) where \( k \) is a pseudofinite (see [Ax]) field \( k \), i.e. a perfect field having a unique extension of degree \( n \) for any \( n \in \mathbb{N} \) and such that every absolutely irreducible \( k \)-variety has a \( k \)-rational point.

2. Hasse Principle

There are two principal questions illustrating the local-global principle (or the Hasse principle).

1) Given a system of algebraic equations with coefficient in \( K \), when an information about its solutions over all the completions \( K_v \) of \( K \) allows us to say something about its global solutions, i.e. solutions over \( K \).

2) Let \( X/K \) be a projective variety defined over a field \( K \) with a set of valuations \( V^K \). Embedding \( K \) in its various completions \( K_v, v \in V^K \) we may consider \( X/K \) as \( X/K_v \) defined over \( K_v \). Is the only or several (up to isomorphism over \( K \)) "global" varieties \( X/K \) having the same data \( \{ X/K_v \mid v \in V^K \} \)?

The most known example of question 1) is the Hasse-Minkowski theorem which asserts that a quadratic form with coefficients in \( \mathbb{Q} \) admits a nontrivial zero over \( \mathbb{Q} \) if and only if it does so over \( \mathbb{Q}_p \) for all prime numbers \( p \) and over \( \mathbb{R} \). The quadratic forms provide an example to question 2): namely, two quadratic forms with coefficients in \( \mathbb{Q} \) which are equivalent over \( \mathbb{Q}_p \) for all prime numbers \( p \) and over \( \mathbb{R} \) are equivalent over \( \mathbb{Q} \).

Not all varieties satisfy the Hasse principle. It was shown by Selmer (1951) that the equation \( 3x^3 + 4y^3 + 5z^3 = 0 \) possesses nontrivial solutions over \( \mathbb{Q}_p \) for all prime numbers \( p \) and over \( \mathbb{R} \), but it possesses no nontrivial solutions over \( \mathbb{Q} \). Concerning question 2) it follows from the results of Rubin and Kolyvagin (1988–1991) that for Selmer’s curve \( S : 3x^3 + 4y^3 + 5z^3 = 0 \) there exist exactly five curves

\[
S : \quad 3x^3 + 4y^3 + 5z^3 = 0, \\
S_2 : \quad 12x^3 + y^3 + 5z^3 = 0, \\
S_3 : \quad 15x^3 + 4y^3 + z^3 = 0, \\
S_4 : \quad 3x^3 + 20y^3 + z^3 = 0, \\
S_5 : \quad 60x^3 + y^3 + z^3 = 0, 
\]
which are isomorphic over \(\mathbb{Q}_p\) for all prime numbers \(p\) and over \(\mathbb{R}\), but not isomorphic over \(\mathbb{Q}\).

To discuss more concretely these and other questions we will need some information about Galois cohomology groups and their interpretations.

3. Galois cohomology

Let \(G\) be a group and \(A\) be a \(G\)-module. Consider the group

\[
A^G := \{ a \in A \mid \sigma a = a \ \forall \sigma \in G \}.
\]

Define the cohomology groups \(H^r(G, A)\) by using “standard cochain complex” (see [ant], Chapter IV). Let \(C^r(G, A) := \text{Maps}(G^r, A)\). Thus the elements of \(C^r(G, A)\) are the functions (called \(r\)-cochains) \(f(\sigma_1, \ldots, \sigma_r)\) of \(r\) variables in \(G\) with values in \(A\). There is a sequence

\[
\cdots \to 0 \to C^0(G, A) \to C^1(G, A) \to C^2(G, A) \to \cdots,
\]

where \(C^0(G, A) := A\), an element \(f_0 \in C^0(G, A)\) is identified with an element \(a \in A\).

\[
\begin{align*}
\delta_0(a)(\sigma) &= \sigma(a) - a, \\
\delta_1(f_1)(\sigma, \tau) &= \sigma f_1(\tau) - f_1(\sigma \tau) + f_1(\sigma), \\
\delta_2(f_2)(\sigma, \tau, \rho) &= \sigma f_2(\tau, \rho) - f_2(\sigma \tau, \rho) + f_2(\sigma, \tau \rho) - f_2(\sigma, \tau), \\
&\vdots \\
\delta_r(f_r)(\sigma_1, \sigma_2, \ldots, \sigma_{n+1}) &= \sigma_1 f_r(\sigma_2, \ldots, \sigma_{n+1}) + \sum_{i=1}^{n} (-1)^{i+1} f_r(\sigma_1, \ldots, \sigma_{i-1}, \sigma_i \sigma_{i+1}, \ldots, \sigma_{n+1}) + (-1)^{n+1} f_r(\sigma_1, \ldots, \sigma_n).
\end{align*}
\]

The direct calculation shows that \(\delta_{r-1} \circ \delta_r = 0\). It is clear that all \(\delta_r\) are homomorphisms of corresponding groups. The group

\[H^r(G, A) = \text{Ker} \delta_r / \text{Im} \delta_{r-1},\]

is called the \(r\)th cohomology group of \(G\) with coefficients in \(A\).

The elements from \(\text{Ker} \delta_r\) and \(\text{Im} \delta_{r-1}\) are called \(r\)-cocycles and \(r\)-coboundaries respectively. 1-cocycles are the maps \(f : G \to A\) satisfying \(f(\sigma \tau) = \sigma(f(\tau)) + f(\sigma)\) (crossed homomorphisms) and 1-coboundaries are the 1-cycles of the form \(f(\sigma) = \sigma(a) - a, a \in A\). 2-cocycles are also
called factor systems. Thus we have in particular

\[ H^0(G, A) = A^G, \]
\[ H^1(G, A) = \{ \text{crossed homomorphisms} \}, \]
\[ H^2(G, A) = \{ \text{principal crossed homomorphisms} \}, \]
\[ = \{ f : G \times G \to A \mid \sigma(f(\tau, \rho)) + f(\sigma, \tau \rho) = f(\sigma \tau, \rho) + f(\sigma, \tau) \} \]
\[ = \{ f : G \times G \to A \mid f(\sigma, \tau) = \sigma(g(\tau)) - g(\sigma \tau) + g(\sigma) \}. \]

If \( G \) is the Galois group of a Galois extension \( L/K \) of a field \( K \), the groups \( H^*(G, A) \) are called the Galois cohomology.

4. Principal homogeneous spaces. Interpretation of \( H^1(G, A) \)

Let \( A \) be an abelian group. The right \( A \)-set

\[(w, a) \mapsto w + a : W \times A \to W\]

is called a principal homogeneous space for \( A \), if \( W \neq \varnothing \) and the map

\[(w, a) \mapsto (w, w + a) : W \times A \to W \times W\]

is bijective, i.e. for any pair \((w_1, w_2)\) there exists a unique \( a \in A \) such that \( w_1 + a = w_2 \).

For example, the group law \( A \times A \to A \) endows \( A \) with a structure of a principal homogeneous space for \( A \) called trivial.

Another example: an affine space is (by definition) a principal homogeneous space for a vector space.

A morphism \( W \to W' \) of principal homogeneous spaces is simply a morphism of \( A \)-sets.

If \( W \) and \( W' \) are principal homogeneous spaces for \( A \), then for every points \( w_0 \in W \), \( w'_0 \in W' \) there exists a unique morphism \( \varphi : W \to W' \) mapping \( w_0 \) to \( w'_0 \). Also any morphism \( W \to W' \) is an isomorphism.

It is easy to check the following facts:

(a) Let \( W \) be a principal homogeneous space for \( A \). For any point \( w_0 \in W \) there exists a unique morphism \( A \to W \) which maps \( 0 \) to \( w_0 \).

(b) Any element \( a \in A \) defines an automorphism \( w \mapsto w + a \) of \( W \), and any automorphism of \( W \) has such a form (for unique \( a \in A \)). Thus \( \text{Aut}(W) = A \).
Let $A$ be an abelian variety defined over a field $K$. A principal homogeneous space for $A$ is an abelian variety $W$ with a right action of $A$ given by regular (i.e. polynomial) maps

$$(w, P) \mapsto w + P : W \times A \to W$$

such that

$$(w, P) \mapsto (w, w + P) : W \times A \to W \times W$$

is an isomorphism of algebraic varieties. It follows that for any extension $L \supset K$, $W(L)$ is either empty set or a principal homogeneous space for the group $A(K)$. A morphism of principal homogeneous spaces is a regular map $\varphi : W \to W'$ such that the diagram

$$
\begin{array}{c}
W \times E \ar[r] & W \\
\downarrow & \downarrow \\
W' \times E \ar[r] & W'
\end{array}
$$

commutes.

The group law $A \times A \to A$ makes $A$ a principal homogeneous space, any principal homogeneous space isomorphic to $A$ is called trivial.

Let $W$ and $W'$ be principal homogeneous spaces for $A$. For any field extension $L \supset K$ and any points $w_0 \in W(L)$, $w'_0 \in W'(L)$ there exists the only morphism $\varphi : W \to W'$ over $L$, which maps $w_0$ to $w'_0$, and $\varphi$ is an isomorphism of principal homogeneous spaces over $K$.

Let $W$ be principal homogeneous space for $A$. For each point $w_0 \in W(K)$ there exists a unique homomorphism $A \to W$ (of principal homogeneous spaces) which maps $0$ to $w_0$. Thus $W$ is trivial if and only if $W(K) \neq \emptyset$. Since $W$ will have a point in some finite extension $L/K$ (this follows from Hilbert Nullstellensatz), it becomes trivial over such a field $L$.

A point $P \in A(L)$ defines an automorphism $w \mapsto w + P$ and any automorphism $W$ over $L$ is of this form for some unique point $P \in A(L)$.

5. Classification of principal homogeneous spaces

Let $K$ be a field, $\overline{K}$ be an algebraic closure of $K$. Let $W$ be a principal homogeneous space for $A$, choose a point $w_0 \in W(\overline{K})$. For any $\sigma \in \text{Gal}(\overline{K}/K)$, $\sigma w_0 \in \text{Gal}(\overline{K}/K)$, so $\sigma w_0 = w_0 + f(\sigma)$ for a unique $f(\sigma) \in E(\overline{K})$. Note that

$$(\sigma \tau)w_0 = \sigma(\tau w_0) = \sigma(w_0 + f(\tau)) = \sigma w_0 + \sigma f(\tau) = w_0 + f(\sigma) + \sigma f(\tau),$$
hence
\[ f(\sigma \tau) = f(\sigma) + \sigma f(\tau). \]

Thus \( f \) is a crossed homomorphism \( \text{Gal}(\overline{K}/K) \to \Lambda(\overline{K}). \) Since \( w_0 \) has coordinates in a finite Galois extension \( L/K, \) we see that \( f \) is continuous. Another point \( w_1 \in W(\overline{K}) \) defines another crossed homomorphism, but
\[ \sigma w_1 = \sigma(w_0 + P) = \sigma w_0 + \sigma P = w_0 + f(\sigma) + \sigma P = w_1 + f(\sigma) + \sigma P - P. \]

Thus
\[ f_1(\sigma) = f(\sigma) + \sigma P - P, \]
that is \( f \) and \( f' \) differ by a principal crossed homomorphism, so we obtain a well-defined class in \( H^1(\text{Gal}(L/K), \Lambda(L)). \)

If the cohomology class is trivial, then \( f(\sigma) = \sigma P - P \) for some point \( P \in A(\overline{K}), \) and
\[ \sigma(w_0 - P) = \sigma w_0 - \sigma P - P - \sigma P = w_0 - P. \]

It follows that \( w_0 - P \in W(K), \) so \( W \) is a trivial principal homogeneous space.

**Proposition 5.1.** One has a one-to-one correspondence
\[ \{ \text{classes of isomorphisms of principal homogeneous spaces for } A \} \leftrightarrow H^1(K, E). \] (*)

**Proof.** Let \( \varphi : W \to W' \) be an isomorphism of principal homogeneous spaces for \( A \) (over \( K \)), and let \( w_0 \in W(\overline{K}). \) One checks that \( (W, w_0) \) and \( (W', \varphi(w_0)) \) define the same crossed homomorphism, hence the map (*) is well-defined. Since \( W \) and \( W' \) define the same cohomology class, one can choose \( w_0 \) and \( w'_0 \) so that \( (W, w_0) \) and \( (W', w'_0) \) define the same crossed homomorphism. There exists the unique map \( \varphi : W \to W' \) over \( K \) which sends \( w_0 \) to \( w'_0 \). Let \( w \in W(\overline{K}), \) write \( w = w_0 + P. \) Then
\[ \varphi(\sigma w) = \varphi(\sigma(w_0 + P)) = \varphi(\sigma w_0 + \sigma P) = \sigma(w_0 + f(\sigma) + \sigma P) = w'_0 + f(\sigma) + \sigma P = \sigma w'_0 + \sigma P = \sigma \varphi(w). \]

It follows that the map \( \varphi \) is defined over \( K \) (i.e. it is given by polynomials with coefficients in \( K \)). So our map is injective.

It remains to prove that this map is surjective. Let \( L/K \) be a Galois extension, \( G = \text{Gal}(L/K) \) and let \( a : G \to A(L) \) be a 1-cocycle. Let \( x \) be
the generic point of $A$ over $L$ (that is $L(x)$ is the function field of $A/L$). Then the map $\varphi(\sigma): L(x) \rightarrow L(x)$, $x \mapsto x + a_\sigma$, $\alpha \mapsto \sigma(\alpha) \forall \alpha \in L$ defines an automorphism of $L(x)$. We have

$$\varphi(\sigma x)(x) = x + a_\sigma x,$$

$$(\varphi(\sigma) \circ \varphi(\tau))(x) = \varphi(\sigma)(x + a_\tau) = x + \sigma(a_\tau) + a_\sigma.$$

Since $a_\alpha$ is an $L$-cocycle, we see that $\varphi(\sigma \tau) = \varphi(\sigma) \varphi(\tau)$, and thus $\varphi(\sigma)$ forms an isomorphic image $G'$ of the group $\text{Gal}(L/K)$. The invariant under $G'$ subfield of $L(x)$ contains $K$, and so is of the form $K(y)$, where $y = (y_1, \ldots, y_n)$, $y_1, \ldots, y_n \in L(x)$. Consider the ideal

$$I = \{f(Y_1, \ldots, Y_n) \mid f(y_1, \ldots, y_n) = 0\}$$

in the polynomial ring $L[Y_1, \ldots, Y_n]$. If $f \in I$, then $\varphi(\sigma)(f(y_1, \ldots, y_n)) = 0$, and since $\varphi(\sigma)(y_i) = y_i$, we have $\sigma(f)(y_1, \ldots, y_n) = 0$ ($\sigma(f)$ denotes the polynomial obtained from $f$ by applying $\sigma$ to all coefficients of $f$).

Writing $f(Y_1, \ldots, Y_n) = \sum \omega_j g_j(Y_1, \ldots, Y_n)$ where $\omega_1, \ldots, \omega_n$ is a basis $L/K$, and $g_j(Y_1, \ldots, Y_n) \in k[Y_1, \ldots, Y_n]$, the above argument implies $g_j(y_1, \ldots, y_n) = 0$ for all $j$. It follows that the ideal $I$ can be generated by polynomials in $K[Y_1, \ldots, Y_n]$, hence it defines a principal homogeneous space for $A$. \hfill \Box

6. Brauer groups. Interpretation of $H^2(G, A)$

Central simple algebras.

**Definition 6.1.** A $K$-algebra $A$ is a $K$-vector space $V$ and an element $x \in V \otimes_k V^* \otimes_K V^*$ describing multiplication in $A$: given $V$ and $x = \sum x_i \otimes \phi_i \otimes \psi_i$, then $A = V$, where multiplication of elements $a, b \in V$ is defined by: $a \cdot b = \sum \phi_i(a) \cdot \psi_i(b) \cdot x_i$.

The ring $A$ is called a $K$-algebra if it contains the field $K$ in its center, and is a finite dimensional $K$-vector space. If $A$ is a subalgebra of a $K$-algebra $E$, then the centralizer $C_E(A)$ of $A$ in $E$ is the set of elements in $E$ commuting with all elements in $A$. $Z(A) := C_A(A)$ is called center of $A$.

The opposite algebra $A^{opp}$ is $A$ with inverse multiplication: $a \ast b \overset{\text{def}}{=} ba$.

We consider only finitely generated left $A$-modules.

**Definition 6.2.**

(i) An $A$-module $V$ is called simple, if it is nonzero, and has no $A$-submodules except 0 and $V$. 
(ii) A $K$-algebra $A$ is called simple, if $\{0\}$ and $A$ are only its ideals.

(iii) A $K$-algebra $A$ is called a division algebra, if all its nonzero elements are units.

**Example 6.1.** Let $M_n(A)$ denote an algebra of $n \times n$-matrices over an algebra $A$. If $D$ is a division algebra, then the algebra $M_n(D)$ is simple.

**Theorem 6.1 (Vedderberm).** Let $A$ be a simple $K$-algebra. Then $A$ is isomorphic to the algebra $M_n(D)$ for some $n$ and some division $K$-algebra $D$.

**Proof.** Let $S$ be a simple $A$-module (e.g., a minimal left ideal of $A$). The left multiplication defines an injective homomorphism $A \to E := \text{End}_K(S)$: because $A$ is simple, its kernel, being a two sided ideal in $A$, must be $\{0\}$ or $A$, but it $\neq A$ since $1 \mapsto 1$.

Let now $C_E(A) = \text{End}_A(S)$. It is a division algebra by Shur lemma, $A = C_E(C_E(A)) = \text{End}_{C_E(A)}(S)$ by the double centralizer theorem, and $\text{End}_{C_E(A)}(S)$ is isomorphic to a matrix algebra over $C_E(A)^{\text{opp}}$. $\square$

**Definition 6.3.** A $K$-algebra $A$ is called central if its center is $K$. It is called central simple, if it is both central and simple.

**Lemma 6.1.** A $K$-algebra $A$ is central simple if and only if it is isomorphic to $M_n(D)$ for some skew field $D$ with center $K$.

**Proof.** Since $A$ is simple, Theorem 6.1 it is isomorphic to some matrix algebra $M_n(D)$ with center $K$. On the other hand, the center of $M_n(D)$ is the set of all $dE_n$, where $E_n$ is the identity matrix and $d$ lies in the center of $D$. Conversely, $M_n(D)$ is a central simple algebra, because the matrix algebras are simple. $\square$

**Theorem 6.2.** The tensor product of two central simple $K$-algebras is a central simple $K$-algebra.

**Proof.** See, e.g. [Dr]. $\square$

Let $A$ and $B$ be central simple $K$-algebras. They are called similar ($A \sim B$), if $A \otimes_K M_m(K) \sim B \otimes_K M_m(K)$ for some $m$ and $n$. Let $\text{Br}(K)$ be the set of central simple $K$-algebras modulo similarity relation $\sim$. It is an equivalence relation. Denote by $[A] \in \text{Br}(K)$ the equivalence class containing $A$. By Theorem 6.2 $A \otimes_K B$ is again a central simple algebra, and if $A \sim A_0$ and $B \sim B_0$, then $A \otimes_K B \sim A_0 \otimes_K B_0$, so we have a well-defined multiplication on $\text{Br}(K)$. Since $A \otimes_K M_n(K) \sim A$, $[M_n(K)]$ is the identity element for any $n$, and since $A \otimes_K A^{\text{opp}} \sim M_n(K)$, $\text{Br}(K)$ is a group.
Definition 6.3. The group Br(K) of similarity classes of central simple K-algebras with just defined multiplication is called a **Brauer group of K**.

Remark 6.1. There is a bijection between elements in Br(K) and algebras D with center K, which sends D to the element represented by some algebra M(D).

Proof. Since \( M_n(D) \otimes_K M_m(K) \cong D \otimes_K M_n(M_m(K)) \cong D, M_{nm}(K) \cong M_{nm}(D) \), all the representatives of an element in Br(K) have the same basic division algebra D. Thus M_n(D) and M_m(D) represent the same element, since M_n(D) \( \otimes_K M_m(K) \cong M_{nm}(D) \sim M_{nm}(D) \otimes M_n(K) \).

Proposition 6.1. Let A be a central simple K-algebra, and let L/K be a field extension. Then A \( \otimes_K L \) is a central simple L-algebra.

Proof. This is true, since the tensor product of central algebras is central, and the center of tensor product is the tensor product of centers.

Let L/K be a field extension. Define the map Br(K) \( \rightarrow \) Br(L) by \( A \mapsto A \otimes_K L \). It is well-defined as \( (A \otimes_K M_n(K)) \otimes_K L \cong A \otimes_K M_n(L) \), and it is a homomorphism, because \( (A \otimes_K L) \otimes_L (A_0 \otimes_K L) \cong (A \otimes_K A_0) \otimes_K L \), taking into account that the tensor product is associative.

Definition 6.4. Let Br(L/K) be the kernel of just defined map Br(K) \( \rightarrow \) Br(L). The element \([A] \in Br(K)\) (and the K-algebra A) is called **split by L**, if it lies in Br(L/K), that is A \( \otimes_K L \) is a matrix algebra over L.

The **Brauer group and cohomology**. Let L/K be a finite Galois extension, \( n = [L : K] \), and let G = Gal(L/K). Denote \( H^2(L/K) := H^2(G, L^*) \). Let Br_n(L/K) be the set of elements in Br(L/K) represented by algebras A such that A \( \otimes_K L \cong M_n(L) \). Prove that \( H^2(L/K) \cong Br(L/K) \).

One approach to do this is an explicit construction of a central simple K-algebra A using “factor systems”. Namely, let \( a_{\sigma, \tau} \) be a 2-cocycle. Take A = \( \bigoplus_{\sigma \in G} L e_{\sigma} \), where the multiplication is defined as follows: e_{\sigma} e_{\tau} = a_{\sigma, \tau} e_{\sigma} e_{\tau} \forall \sigma, \tau \in G \) and \( e_{\sigma} a = a(a) e_{\sigma} \forall a \in A, \forall \sigma \in L \). See ([P], Chapter 14) for the details. Note that the condition that \( a_{\sigma, \tau} \) is a 2-cocycle is equivalent to the condition that the multiplication just defined is associative.

We adopt here another way of proving that \( H^2(L/K) \cong Br(L/K) \). One has Br(L/K) = \( \bigcup_{n \in \mathbb{Z}} Br_n(L/K) \). Consider the algebras representing the elements from Br_n(L/K) as pairs \((V, x)\) where V is an \( n^2 \)-dimensional K-vector space, and \( x \in V \otimes_K V^* \otimes_K V^* \) as in Definition 6.1.

\((V, x)\) and \((V', x')\) are called K-isomorphic, if there exists an isomorphism of spaces \( f: V \rightarrow V' \) such that \( f(x) = x' \), where \( f(x) = \sum x_i \otimes \phi_i \otimes f^{-1} \otimes \psi_i \). Such an isomorphism f exists if and only if the corresponding K-algebras are isomorphic.
\[ A \otimes_K L \cong M_n(L) \] corresponds to \( L \)-isomorphism between \((V \otimes_K L, x \otimes 1)\) and \((M_n(L), x_0)\), where \(x_0\) describes the standard multiplication of matrices. Next, \((V, x) \sim (V', x')\) if and only if they are \(K\)-isomorphic.

Let \(C_n(L) = \text{Aut}_L(M_n(L))\) be the group of automorphisms \(M_n(L)\) regarded as an \(L\)-algebra. Then \(G\) acts componentwise on \(M_n(L)\), and for \(g \in G, \phi : M_n(L) \to M_n(L)\) we set \(g(\phi) = g \circ \phi \circ g^{-1}\). This is an \(L\)-linear map, thus we get an action of \(G\) on \(C_n(L)\).

For \(g \in G, [(V; x)] \in \text{Br}_n(L/K)\) and \(f : M_n(L) \rightarrow V \otimes_K L\) we have an \(L\)-isomorphism \((V \otimes_K L, x \otimes 1) \rightarrow (M_n(L), x_0)\). Let \(\theta_f(x) = f^{-1} \circ g \circ f \circ g^{-1}\). Then \(\theta_f(x) : M_n(L) \rightarrow M_n(L)\) is an isomorphism. Since \(f\) is \(L\)-linear, we have \(\theta_f(x) \in C_n(L)\). Now \(\theta_f(x) : G \to C_n(L)\) is a 1-cocycle: \(\theta_f(\phi g) = f^{-1} \circ g \circ \phi \circ g^{-1} = (f^{-1} \circ g \circ f \circ g^{-1}) \circ (g \circ f) \circ g^{-1} = \theta_f(g) \circ g(\theta_f(f)) \circ g\).

If we choose another \(f'\) instead of \(f\), \(f' = \phi \circ f\) for some \(\phi \in C_n(L)\), thus \(\theta_{f'}(g) = (\phi f)^{-1} \circ g \circ (\phi f) \circ g^{-1} = (f^{-1} \phi^{-1}) - (f^{-1} \phi^{-1}) \circ g \circ f \circ g^{-1} = (f^{-1} \phi f) \circ g^{-1} = \theta_{f'}(g) \circ g(f^{-1} \phi f)\), so \(\theta_{f'} - \theta_f\) is a 1-coboundary.

If we choose a pair \((V', x')\), representing the same element in \(\text{Br}_n(L/K)\) as \((V, x)\), then \(f' = \phi \circ f\) for some \(\phi : V \to V'\), and the computation like above shows that \(\theta_f - \theta_{f'}\) is a 1-coboundary.

Thus we have a well defined map

\[ \theta : \text{Br}_n(L/K) \to H^1(G, C_n(L)) \]

where \((V, x) \mapsto \theta_f\).

**Proposition 6.2.** Just defined map

\[ \theta : \text{Br}_n(L/K) \to H^1(G, C_n(L)) \]

is bijective.

**Proof.** Let \((V, x)\) with \(f\) and \((V', x')\) with \(f'\) give \(\theta_f = \theta_{f'}\). Then \(f^{-1} \circ g \circ f \circ g^{-1} = f'^{-1} \circ g \circ f' \circ g'^{-1}\), so \(g^{-1} \circ f' \circ f \circ g^{-1} = f'^{-1} \circ f\), that is \(g(f'f^{-1}) = f'f^{-1}\), and thus the \(L\)-isomorphism \(f'f^{-1}\) is a \(K\)-isomorphism, so that \([[(V, x)] = [(V', x')]] \in \text{Br}_n(L/K)\). Therefore, \(\theta\) is an injective homomorphism. If an element, represented by \(M_n(L)\) maps to 0, then since in this case \(f\) is the identity, \(\theta_f = 0\).

Now let \(\phi : G \to C_n(L)\) be any cocycle. Since \(H^1(G, \mathbb{G}_m(L))\) is trivial (see [Se]), and \(C_n(L) \subset \mathbb{G}_m(L)\), there exists an \(L\)-automorphism \(f\) of the algebra \(M_n(L)\) such that \(g(f) = f^{-1} \circ g(f)\) for all \(g \in G\). Let \(x' = f(x_0)\). Then \(g(x') = g(f(x_0)) = g(f)(g(x_0)) = g(f)(x_0) = (f \circ g)(x_0) = f(\phi(g)(x_0)) = f(x_0) = x'\) where \(x_0\) corresponds to the standard matrix multiplication and is independent of choices of basis and action of \(G\). Hence \(x'\) is
defined over $K$, and $[(M_n(K), x')] \in \text{Br}_n(L/K)$ maps to $[\phi] \in H^1(G, C_n(L))$, so $\theta$ is surjective.

Since every automorphism of $M_n(L)$ is inner, the map $\mathbb{G}_m(L) \to C_n(L)$, which sends $\phi \in \mathbb{G}_m(L)$ to the conjugate of $\phi$, is surjective, and since the center of the algebra $\mathbb{G}_m(L)$ equals $\{L^* \cdot E_n\}$ where $E_n$ is the identity matrix in $M_n(L)$, the sequence $1 \to L^* \to \mathbb{G}_m(L) \to C_n(L) \to 1$ is exact. From the long exact cohomology sequence we get the maps $\Delta_n : H^1(G, C_n(L)) \to H^2(G, L^*)$, and $\delta_n = \Delta_n \circ \theta : \text{Br}_n(L/K) \to H^2(G, L^*)$. The different $\delta_n$ are agreed: for $C \in \text{Br}_n(L/K)$, $\delta_n(C) = 0$ if and only if $\Delta_n(C) = 0$, since the map $\theta$ is bijective, and it is true if and only if $C$ represented by a matrix algebra. Since $\theta$ is a bijection which sends such an element to zero, $\Delta_n$ is injective. The preceding term of the long exact sequence is $H^1(G, \mathbb{G}_m(L)) = 0$ (see [Se]). Next, an easy calculation shows that $\delta_n(C) = \delta_n'(C') = \delta_{n+1}(C \otimes_K C')$. Hence $\delta_n$ gives the injective homomorphism $\delta : \text{Br}(L/K) \to H^2(L/K)$.

**Theorem 6.3.** The map $\delta : \text{Br}(L/K) \to H^2(L/K)$ is an isomorphism.

**Proof.** Since $\delta$ is injective, by Proposition 6.2, it suffices to prove that $\Delta_n$ is surjective for $n = [L : K]$. Let $a : G \times G \to L^* \subset \mathbb{G}_m(L)$ be any cocycle. Let $V$ be the $L$-vector space with basis $\{e_h, h \in G\}$, and let $p_g$ be the automorphism of $V$ defined as follows: $p_g(e_h) = a(g, h) \cdot e_{gh}$. Then $p_g(s(p_t)(e_u)) = a(s, tu) \cdot s(a(t, u)) \cdot (e_{stu})$ and $a(s, t) \cdot p_{st}(e_u) = a(s, t) \cdot a(st, u) \cdot e_{stu}$, and since $a(s, t) \cdot a(st, u) = a(s, tu) \cdot s(a(t, u))$, and $a(s, t) = p_s(s(p_t)(p_{st}^{-1}))$, we obtain that a lies in the kernel of $\Delta_n$. \hfill \Box

**Proposition 6.4.** A $K$-algebra $A$ is central simple if and only if $A \otimes_K \overline{K} = M_n(\overline{K})$ for some $n$, where $\overline{K}$ is an algebraic closure of $K$.

**Proof.** See Bourbaki, Algebra, Chapter VIII. Proposition 3.4. \hfill \Box

**Proposition 6.4.** A $K$-algebra $A$ is central simple if and only if $A \otimes_K \overline{L} = M_n(\overline{L})$ for some $n$ and some finite Galois extension $\overline{L}/L$.

**Proof.** Choose a basis $e_1, \ldots, e_n$ in $A$. Let $A_1, \ldots, A_n$ be the images $e_i \otimes K$ in $M_n(\overline{K})$, by using the isomorphism from Proposition 6.3. Let $\overline{L}$ be a finite Galois extension which contains the inverse images $A_i$. Then the above isomorphism induces an isomorphism $A \otimes_K \overline{L} \to M_n(\overline{L})$. \hfill \Box

**Theorem 6.4.** $\text{Br}(K) \cong H^2(\overline{K}/K)$.

**Proof.** The isomorphisms $\text{Br}(L/K) \to H^2(L/K)$ agrees with the map $\text{Br}(L/K) \to \text{Br}(L'/k)$ and $H^2(L/K) \to H^2(L'/K)$ where $L'/L$ is finite Galois extension. Since by Proposition 6.3, $\text{Br}(\overline{K})$ is trivial, the sequence $0 \to \text{Br}(\overline{K}/K) \to \text{Br}(K) \to \text{Br}(K)$ implies $\text{Br}(K) \cong \text{Br}(\overline{K}/k)$; the latter group is the limit of $\text{Br}(L/K)$, where $L$ runs over finite Galois extension of $K$. By Proposition 6.4 $H^2(\overline{K}/K)$ is the limit of $H^2(L/K)$.

\hfill \Box
Example 6.1. If $K$ is a local field, then $\text{Br}(K) \xrightarrow{\text{inv}_K} \mathbb{Q}/\mathbb{Z}$ (see [ant], Chapter VI, §1).

7. Brauer group in class field theory

In this section $K$ denotes a number field. First, we prove that an element of Brauer group of $K$ splits in some cyclic cyclotomic extension of $K$. For this purpose we will need two lemmas.

Lemma 7.1. The map $\text{Br}(K) \to \bigoplus_{v \in V_K} \text{Br}(K_v)$ is injective.

Proof. One should do as at the beginning of proof of Theorem 7.1. □

Lemma 7.2. Let $S$ be a finite set of valuations of $K$. Then for any $m \in \mathbb{N}$, there exists a cyclic cyclotomic extension $L/K$ such that $m[L_v : K_v]$ for all $v \in S$.

Proof. It suffices to find an extension of $\mathbb{Q}$ with $m \cdot [K : \mathbb{Q}]$ instead of $m$. Thus we set $K = \mathbb{Q}$. Let $q$ be a prime number, and let $\zeta$ be a primitive $q^r$-th root of 1. Then $\text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) \cong (\mathbb{Z}/q^r\mathbb{Z})^*$; the latter group has a quotient group of order $q^s$ with $s \geq r - 3$, hence $s$ becomes arbitrarily large for large $r$. Let $L(q^r)$ be the subextension of $\mathbb{Q}(\zeta)/\mathbb{Q}$ with this Galois group. It is a cyclic cyclotomic extension of $\mathbb{Q}$. Using the properties of cyclic cyclotomic extensions (see, e.g., [ant]), we see that $[\mathbb{Q}_p(\zeta) : \mathbb{Q}] \to \infty$, if $r \to \infty$. Hence $[L(q^r)_p : \mathbb{Q}_p]$ becomes arbitrarily large power of $q$ for large $r$.

Let $q_1, \ldots, q_s$ be the distinct primes dividing $m$, and let $L = L(q_1^{r_1}) \cdots L(q_s^{r_s})$. Then $L$ is a cyclic extension, since its Galois group is a product of cyclic groups, if one choose the $r_i$ large enough, one gets $m[L_v : \mathbb{Q}_p]$ for all $p \in S$. □

Theorem 7.1. Let $K$ be a number field. Then each element $\alpha \in \text{Br}(K)$ maps to $0 \in \text{Br}(L)$ for some cyclic cyclotomic extension $L/K$ (depending on $\alpha$).

Proof. Let $\{\alpha_v\}$ be the image of $\alpha$ in $\bigoplus \text{Br}(K_v)$. Then almost all $\alpha_v$ are 0. Let $m$ be the common denominator of nonzero $\text{inv}_v(\alpha_v) \in \mathbb{Q}/\mathbb{Z}$.

Let $L$ be the field from the preceding lemma. Then $\text{inv}_v(\alpha_v) = [L_w : K_v] \cdot \text{inv}_v(\alpha_v) = 0 \in \mathbb{Q}/\mathbb{Z}$ for all valuations $w$ of $L$, that is the image of $\alpha$ in $\text{Br}(L_w)$ is trivial for all $w$. Since $\text{Br}(L)$ injects in $\bigoplus \text{Br}(L_w)$, the image $\alpha$ in $\text{Br}(L)$ is trivial. □

Theorem 7.1 [The main exact sequence]. The sequence

$$0 \to \text{Br}(K) \to \bigoplus_v \text{Br}(K_v) \to \mathbb{Q}/\mathbb{Z} \to 0$$
is exact.

**Proof.** Let $L/K$ be a finite Galois extension with Galois group $G$. In [ant] the idele class group was defined: $C_L = I_L/L^*$, where $I_L$ is the idele group of $L$. There is a long exact sequence

$$H^1(G, C_L) \to H^2(G, L^*) \to H^2(G, I_L) \to H^2(G, C_L),$$

where $H^1(G, C_L) = 0$, $H^2(G, L^*) = Br(L/K)$, and

$$H^2(G, I_L) = \bigoplus H^2(G_v, L_v^*) = \bigoplus Br(L_v/K_v)$$

(see [ant]).

Hence the sequence

$$0 \to Br(L/K) \to \bigoplus Br(L_v/K_v) \to H^2(G, C_L)$$

is exact. Let $H^2(G, C_L)'$ be the image of the last map. Passing to the limit for all $L$, the first terms of this sequence give Lemma 7.1.

The local invariant map gives the isomorphism $\text{inv}_v : Br(L_v/K_v) \to \frac{1}{n_v} \mathbb{Z}/\mathbb{Z}$, and summing these, we get a surjective map

$$\bigoplus Br(L_v/K_v) \to \frac{1}{n_0} \mathbb{Z}/\mathbb{Z} \subseteq \frac{1}{n} \mathbb{Z}/\mathbb{Z}$$

where $n_0 = \text{lcm}(n_v)$.

By Theorem VII.8.1 in [ant], we have (not necessarily exact) complex

$$0 \to Br(L/K) \to \bigoplus Br(L_v/K_v) \to \frac{1}{n_0} \mathbb{Z}/\mathbb{Z},$$

so together with the exact sequence above, this gives a map $\phi : H^2(G, C_L)' \to \frac{1}{n_0} \mathbb{Z}/\mathbb{Z}$. Suppose that $n_0 = n$. Then $\phi$ is an isomorphism since it is a surjective map, and $|H^2(G, C_L)'| = |H^2(G, C_L)| \leq n$. So $H^2(G, C_L)' = H^2(G, C_L)$ have order $n$, and the sequence

$$0 \to Br(L/K) \to \bigoplus_v Br(L_v/K_v) \to \frac{1}{n} \mathbb{Z}/\mathbb{Z} \to 0$$

is exact.

If the extension $L/K$ is cyclic, then $n_0 = n$ can be proved by using the Artin map: Let $m$ be a formal product of places containing the infinite and ramified ones. Then the Artin map $J_R^m \to G$ maps $p = p_v \mapsto \text{Frob}_p$, which has order $f_p$, and $f_p = n_v$, since $p$ is unramified. Then the image of Artin
map has order \( n_0 = \text{lcm}(n_v) \leq n \). But \( G \) has order \( n \), since the extension \( L/K \) is cyclic, the Artin map is surjective. Hence \( n = n_0 \). Passing to the direct limit, we get

\[
0 \to \text{Br}(\mathbb{Q}^{\text{cyc}} K/K) \to \bigoplus_{v \in V^K} \text{Br}((\mathbb{Q}^{\text{cyc}} K)_v/K_v) \to \mathbb{Q}/\mathbb{Z} \to 0,
\]

and since \( \text{Br}(\mathbb{Q}^{\text{cyc}} K/K) = \text{Br}(K) \) by Theorem 7.1, and

\[
\text{Br}((\mathbb{Q}^{\text{cyc}} K)_v/K_v) = \text{Br}(K_v).
\]

This finishes the proof. \( \square \)

8. Class-field theory and Hasse-Minkowski theorem

Let \( K \) be an algebraic function field \( K \) in one variable with pseudofinite \([\Pi]\) constant field \( K \). We call such a field pseudoglobal. For pseudoglobal fields there is an analogue of global class field theory \([\text{An, Se}]\), in particular, for such a field \( K \) we have the following exact sequence

\[
0 \to \text{Br}(K) \to \bigoplus_{v \in V^K} \text{Br}(K_v) \to \mathbb{Q}/\mathbb{Z} \to 0,
\]  

\[ (1) \]

where \( V^K \) is the set of all valuations of \( K \) (trivial on the constant field \( K \)), \( \text{Br} K \) (resp. \( \text{Br} K_v \)) is the Brauer group of \( K \) (resp. of the completion \( K_v \) of \( K \) at \( v \in V^K \)).

Exact sequence (1) shows, in particular, that for a pseudoglobal field \( K \) the map

\[
\text{Res} : \text{Br}(K) \to \prod_{v \in V^K} \text{Br}(K_v)
\] 

\[ (2) \]

is injective, i.e. the Hasse principle for Brauer group holds over \( K \).

Our first application of the Hasse principle for Brauer group of a pseudoglobal field will be the analogue of the classical Hasse-Minkowski theorem which asserts that a quadratic form defined over a global field \( K \) is isotropic if and only if it is isotropic over all the completions of \( K \). This fact can be quickly proved by using the following proposition.

**Proposition 8.1.** Let \( K \) be a pseudoglobal field. Then:

(i) An element \( a \in K \) is a norm from a cyclic extension \( L/K \) if and only if it is a norm everywhere locally.
(ii) Let $S$ be a finite set of valuations of a global field $K$.

Let $m$ be a positive integer, $(p, \text{char}(K)) = 1$, and $a \in K^*$. If $a \in K_v^{m}$ for all $v \notin S$, then $a \in K^{*m}$.

**Proof.** (i) For a cyclic extension $L/K$ we get from (2) that there is an injective map $K^*/N_{L/K}L^* \to \prod_{v \in V^K} K_v^*/N_{L_w/K_v}L_w^*$, where for all $v \in V^K$ $w$ is a fixed extension of the valuation $v$ to $L$, and $L_w$ is the corresponding completion.

(ii) We follow the argument used in [Mil, pp. 82-83, 275-276]. Let $L/K$ be an abelian extension, and $G = \text{Gal}(L/K)$. First we show that if $L_w = K_v$ for almost all $v \in V^K$ then $L = K$. Suppose that $K \not= L$. Let $\sigma$ be a fixed generator of the absolute Galois group of the pseudofinite constant field $K$. Let $v \in V^K$, and let $k(v)$ and $k(w)$ be the residue fields of $K_v$ and $L_v$ respectively. Since almost all valuations of $K$ are unramified in $L$, we may assume $v$ to be unramified in $L$. Denote by $\sigma_w$ the restriction of $\sigma^{[k(w):k]}$ to the field $k(w)$. Then $\sigma_w$ is a generator of the cyclic group $\text{Gal}(k(w)/k(v)) \simeq \text{Gal}(L_v/K_v) \subset G$, note that $\sigma_w$ does not depend on the choice of extension $w/v$: if $\sigma$ is fixed, then $\sigma_w \in G$ is uniquely determined by $v$, so we denote it by $\sigma_v$.

Let $C_K$ (resp. $C_L$) be the idele class group of $K$ (resp. $L$). By using the isomorphism $C_K/N_{L/K}C_L \simeq G$ (cf. [An]) we see that for any finite set of valuations $S \subset V^K$ the group $G$ is generated by the elements $\sigma_v, v \notin S$. If there were exist only a finite set of valuations of $K$ which do not split completely in $L$, then by adding them to $S$ we would obtain that all $\sigma_v$ are trivial for $v \notin S$. This contradicts to the fact that $\sigma_v, v \notin S$, generate the group $G$. Thus $L = K$.

Let $a \in K_v^{*m}$ for all $v \notin S$. As in the classical case (cf. [Mil], p. 82-83) it is enough to consider the case where $m$ is a power of a prime number and the $m$-th roots of unity are in $K$. In that case the extension $L = K(\sqrt[n]{a})$ is a Kummer extension, and we have $L_w = K_v$ for all $v \notin S$ where $w$ is an extension of $v$ to $L$. Then the above argument shows that $L = K$, i.e. $a \in K^{*m}$. \square

**Theorem 8.1.** A non degenerate quadratic form $q$ over a pseudoglobal field $K, \text{char}K \not= 2$, is isotropic if and only if it is isotropic over all the completions $K_v$ of $K$.

**Proof.** Assume that the quadratic form $q$ is isotropic over all the completions $K_v$ of $K$. We shall argue by induction on $n = \text{rank } q$. First, we consider the cases $n = 1, 2, 3, 4$. When $n = 1$, there is nothing to prove. When $n = 2$, we may suppose that $q = X^2 - aY^2$, and use Proposition 8.1 (ii) for $m = 2$. If $n = 3$, after multiplying $q$ by a nonzero element from $K$, we may
assume that $q = X^2 - aY^2 - bZ^2$. The latter form represents zero in $K$ if and only if $b$ is a norm from the field $K(\sqrt{a})$, so for $n = 3$ Theorem 8.1 follows from Proposition 8.1 (i). Finally, let $n = 4$. In this case we may suppose that
\[ q = X^2 - bY^2 - cZ^2 + acT^2. \] (3)

Form (3) represents 0 if and only if $c$ as an element of $K(\sqrt{ab})$ is a norm from $K(\sqrt{a}, \sqrt{b})$ ([Mhl], 193–194). Thus Theorem 8.1 is established for $1 \leq n \leq 4$.

Now let $n \geq 5$. Write the form $q$ as follows
\[ q(X_1, \ldots, X_n) = a_1X_1^2 + a_2X_2^2 - r(X_3, \ldots, X_n). \] (4)
The form $r$ has rank $n - 2 \geq 3$. Similarly to the classical case of quadratic forms over global fields, the form $r$ represents 0 for almost all $v \in V^K$. It suffices to show this for quadratic forms of rank 3. Let $r = b_1Y_1^2 + b_2Y_2^2 + b_3Y_3^2$; let $S = \{v \in V^K \mid \exists i \in \{1, 2, 3\} \; v(b_i) \neq 0\}$. $S$ is a finite set, and for all $v \notin S$ we can reduce $r$ modulo $v$ to obtain a quadratic form $	ilde{r} = b_1Y_1^2 + b_2Y_2^2 + b_3Y_3^2$ of rank 3 over a pseudofinite field $K$ which represents 0 over $K$ (such statement is true over any finite field, thus it is true over a pseudofinite field $K$, because the pseudofinite fields are infinite models of finite fields). Henceforth, for all $v \notin S$ Hensel’s lemma implies that the form $r$ represents 0 in $K_v$ for all $v \notin S$.

Since the subgroup $K_v^{*2}$ is open in $K_v^{*}$ with respect to $v$-adic topology, and $r$ represents every element in the coset $c \cdot k^{*2}$ if it represents $c_v \in K_v^{*}$, it follows that $r$ represents the elements in a nonempty open subset of $K_v^{*}$.

Consider any $v \in S$. Since form (4) represents 0 in $K_v$, there exists $c_v \in K_v^{*}$ such that both forms $r$ and $a_1X_1^2 + a_2X_2^2$ represent it. So, there exist $x_1(v), \ldots, x_n(v) \in K_v^{*}$ such that
\[ a_1x_1(v)^2 + a_2x_2(v)^2 = r(x_3(v), \ldots, x_n(v)) = c_v. \]

According to weak approximation theorem, we can find elements $x_1, x_2 \notin K^{*}$ which are close enough to $x_1(v), x_2(v)$ for all $v \in S$, so that $c = a_1x_1^2 + a_2x_2^2$ is close enough to $c_v$ to be represented by the form $r$. Thus the form $cY^2 - r$ represents 0 in $K_v$ for $v \in S$. Since $r$ represents 0 in $K_v$ for $v \notin S$, it represents all elements in $K_v$ for $v \notin S$. So, $cY^2 - r$ represents 0 in $K_v$ for all $v \in V^K$. By induction, $cY^2 - r$ represents 0 in $K$. It follows that $q$ represents 0 in $K$. \qed
9. Brauer and $R$-equivalence

Let $X$ be a smooth algebraic variety defined over a field $K$. Recall that two points $x, y \in X(K)$ are $R$-equivalent if there is a sequence of points $z_i \in X(K), x = z_1, \ldots, y = z_n$, such that for each pair $z_i, z_{i+1}$ there exists a $K$-rational map $f_i: \mathbb{P}^1 \to X$, regular at 0 and 1, with $f_i(0) = z_i, f_i(1) = z_{i+1}, 1 \leq i \leq n - 1$. We shall denote the set of $R$-equivalence classes on $X(K)$ by $X(K)/R$.

Let $L/K$ be a Galois extension with Galois group $G$. Let $K(X)$ (resp. $L(X)$) be the function field of $X$ over $K$ (resp. over $L$). To any element of $K(X)$ we can attach its divisor in the group of divisors $\text{Div}(L(X))$. The actions of $G$ on $L(X)$ and on $\text{Div}(L(X))$ agree, so a natural homomorphism of cohomology groups $H^2(G, K(X)) \to H^2(G, \text{Div}(L(X)))$ and one define

$$\text{Br}(X(L)) = \text{Ker}(H^2(G, L(X)) \to H^2(G, \text{Div}(L(X))).$$

Next, any point $x \in X(K)$ defines a specialization (evaluation) homomorphism

$$\text{Br}(X(L)) \to \text{Br}(L/K), \ b \mapsto b(x).$$

Two points $x, y \in X(K)$ are said to be $\text{Br}$-equivalent if for any Galois extension $L/K$, and any $b \in \text{Br}(X(L))$, we have $b(x) = b(y)$. It turns out that the Br-equivalence is an equivalence relation and the set of Br-equivalence classes on $X(K)$ will be denoted by $X(K)/\text{Br}$.

**Proposition 9.1.** Let $X$ be a smooth variety defined over a pseudoglobal field $K$. Then the restriction map

$$X(K)/\text{Br} \to \prod_v X(K_v)/\text{Br}$$

is injective.

**Proof.** For a global field this is a well-known fact, it follows from the Hasse principle for Brauer group of global field. Since the Hasse principle for Brauer group holds also for pseudoglobal fields [An], the injectivity of the map $X(K)/\text{Br} \to \prod_v X(K_v)/\text{Br}$ can be proved exactly as in the case of varieties over global fields. Consider $x_1, x_2 \in X(K)$, and suppose that $x_1$ and $x_2$ are Brauer equivalent in $X(K_v)$ for all valuations $v$ of $K$. This means that for every $b_v \in \text{Br}X(K_v), (x_1, b_v) = (x_2, b_v)$. Consider $(x_1, b)$ and $(x_2, b)$, where $b \in \text{Br}X$. We have $\text{loc}_v(x_1, b) \in \text{Br}X$, $i = 1, 2$, where $\text{loc}_v: \text{Br}X \to \prod_v \text{Br}X_v$ denotes the localization map, and $\text{loc}_v(x_i, b) = (x_i, \text{loc}_v b)$. Since $\text{loc}_v(x_1, b) = \text{loc}_v(x_2, b)$ for all $v$, it follows that $(x_1, b) = (x_2, b)$, because the map loc is injective by the main exact sequence of the class field theory. Thus $x_2$ and $x_2$ are Brauer equivalent in $X(K)$. \qed
10. Brauer–Manin obstruction

Let $\mathcal{O}_v \subset K_v$ be the integer ring of valuation $v \in V^K$. For a finite subset $S \subset V^K$ define

$$\mathcal{H}_S = \{ x \in K | x \in \mathcal{O}_v \forall v \notin S \}.$$ 

Any algebraic variety $X$ over $K$ can be regarded over $\mathcal{H}_S$ for some $S$ (the coefficients of equations which define $X$ lie in $\mathcal{H}_S$ for some $S \subset V^K$). Let $S' \subset V^K$ be a finite subset such that, and

$$V(\mathcal{H}_{S'}) = \prod_v X(K_v) \subset V(\mathcal{O}_v) \forall v \in S' \subset \prod_v V(K_v).$$

Consider $X(\mathcal{H}_K) = \bigcup_{S' \supset S} V(\mathcal{H}_S)$ and call it the set of adelic points of $X$. It is clear that, not having any adelic points naturally implies that we have no rational points, by the obvious inclusion $X(K) \subset X(\mathcal{H}_K)$. If we can construct a set $\Phi(X)$ with $X(K) \subset \Phi(X) \subset X(\mathcal{H}_K)$ we obtain a refinement of the obstruction obtained from adelic points. If $X(\mathcal{H}_K) \neq \emptyset$, but $X(K) = \Phi(X) = \emptyset$, we will say that $X$ is a counter-example to the Hasse principle given by the $\Phi$-obstruction.

So, we have already noted that (without proof) that the Hasse principle does not always work. In [Ma], for example, Manin sketches reasons for explaining the absence of global points even though you have adelic points. Let $\text{Br}X = H^2_{et}(X, \mathbb{G}_m)$ be the cohomological Brauer group of $X$. A morphism $K_v$-point defines a morphism (specialization/evaluation) $\text{Br}X \rightarrow \text{Br}K_v \cong \mathbb{Q}/\mathbb{Z}$ (the latter isomorphism is given by local class field theory, see [aut]). Hence we obtain a pairing

$$\text{Br}X \times X(\mathcal{H}_K) \rightarrow \mathbb{Q}/\mathbb{Z}.$$ 

Also, if one has a $K$-point, then the above morphism factors through $\text{Br}K$, and the fundamental exact sequence in the global class field theory states that

$$0 \rightarrow \text{Br}K \rightarrow \bigoplus_v \text{Br}K_v \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0.$$ 

Denote by $X(\mathcal{H}_K)^{\text{Br}X}$ the set of adelic points of $X$ that are orthogonal to all of $\text{Br}X$, this says that $X(K) \subset X(\mathcal{H}_K)^{\text{Br}X} \subset X(\mathcal{H}_K)$. Hence, the emptiness of $X(\mathcal{H}_K)^{\text{Br}X}$ is an obstruction to the existence of a rational point. Using this obstruction Manin explains essentially all known counter-examples to the Hasse principle, and it was conjectured that this Brauer–Manin obstruction would be the only obstruction to the Hasse principle.
Theorem 10.1 [Manin]. If $A$ is an abelian variety, then the Brauer-Manin obstruction is the only obstruction to the Hasse principle, supposing that the Tate-Shafarevich group of $A$ is finite.

Let $A$ be an algebraic group over a number field $K$. $H^1(K, A)$ classifies all principal homogeneous spaces of $A$. A principal homogeneous space is trivial exactly when it has a $K$-rational point. Hence, the kernel of the natural map

$$H^1(K, A) \xrightarrow{\alpha} \prod_{v \in V_K} H^1(K_v, A)$$

are exactly the principal homogeneous spaces of $A$ that have a point locally everywhere. Saying that it is injective is thus the same thing as saying that the Hasse principle holds. Denoting the kernel of $\alpha$ by $\Sha(K, A)$ (Tate-Shafarevich group), its size is a measure of the failure of the Hasse principle.

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Infinite Games and their Applications

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Contents of lectures

1. Some History: Banach-Mazur Game and its connection with Baire categories.

2. Winning strategies in infinite games.

3. Determined and undetermined games.

4. Martin Theorem of determinacy of Borel Games.

5. Some examples and applications.


Almost Periodic Objects in Complex Analysis

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A continuous mapping $F$ of a tube $T_\Omega = \{ z = x + iy \mid x \in \mathbb{R}^n, y \in \Omega \}, \Omega$ being a domain in $\mathbb{R}^n$, $n \geq 1$, to a metric space $X$ is called almost periodic (a.p.), if its orbit $\{ F(z + h) \}_{h \in \mathbb{R}^n}$ is a relatively compact subset in the topology of uniform convergence on each tube $T_K$, $K$ being a compact set in $\Omega$.

Holomorphic a.p. functions and mappings into $C^m$ were studied by L. I. Ronkin in the ninetieth. In particular, he proved that the divisor of an a.p. holomorphic mapping is a.p. (in the sense of distributions) too. The converse assertion is not true (H. Tornehave, 1988).

Our approach to the problem of the full description of divisors of holomorphic a.p. functions is very close to the classical method of investigation of the Second Cousin Problem on a domain $D \subset C^m$. Let $K$ be the Bohr
compactification of \( \mathbb{R}^n \), i.e., the Pontryagin dual group for the discrete additive group \( \mathbb{R}^n \). We prove that to each almost periodic divisor \( d \) in a tube \( T_\Omega \) one can assign a cohomology class from \( H^2(K, \mathbb{Z}) \) (actually the first Chern class of the special line bundle over \( K \times \Omega \) generated by \( d \)) such that the trivial cohomology class corresponds to just all the divisors of a.p. holomorphic functions on \( T_\Omega \). We also give a simple structure formula for these cohomology classes.

These description yields various geometric conditions for an a.p. divisor to be the divisor of a holomorphic a.p. function. We give a full description of divisors of meromorphic almost periodic functions and holomorphic almost periodic curves as well.

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**Asymptotic Expanders and Ramanujan Graphs**

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We will define the notions of asymptotic expanders and asymptotic Ramanujan graphs and discuss their construction by using automaton groups (that is groups generated by finite automata).
Alexander Doniphan Wallace: The Founder of the Theory of Topological Semigroups

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A. D. Wallace Curriculum Vitae

Alexander Doniphan Wallace was born on August 21, 1905 in Hampton, Virginia. He received his B.A. degree from the University of Virginia in 1935, his M.A. degree from the University of Virginia in 1936, and earned the Ph.D. degree in 1939 under the direction of G. T. Wyburn at the University of Virginia. He began his professional career as a Lefschetz’s assistant at Princeton University, 1940-1941, and then went onto the University of Pennsylvania as an Associate Professor from 1941-1947.

Karl H. Hofmann drawing of Alexander Doniphan Wallace
In 1947 W. L. Duren, Jr., organized a small group of mathematicians to reactivate the graduate program at Tulane University. Duren wisely decides that the newly formed National Science Foundation and other U.S. governamental research-supporting agencies would make it possible to build a quality graduate program. And so it was that A. D. Wallace, B. J. Pettis, and W. L. Duren, Jr., reawakened the long dormant doctoral program at Tulane University. This move away from the mathematical center of the Northeastr required great courage and vision. But it was in keeping with Wallace's often-stated goal of building up mathematics in the South. He played a pivotal role in building up the program at Tulane.

In 1963 Wallace left Tulane University to become Professor, later Graduate Research Professor, at the University of Florida in Gainesville. With the exception of one year appointment at the University of Miami, he stayed here until retirement in 1973. Then in June 1973 Wallace retired to New Orleans. He retired in December 1974 as adjunct Professor in the Department of Mathematics, Tulane University. Tulane University has honored him with professorial status, without duties to enable him to enjoy the facilities Wallace needs to satisfy his intellectual appetites.

We record the

List of A. D. Wallace Ph.D. Students:


5. Wayman L. Strother, Continuity for Multi-Value Functions and Some Applications to Topology. Tulane University, 1952.


11. Lewis Edes Ward, Jr., *Partially Ordered Spaces*. Tulane University, 1953.


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¹This dissertation was originally directed by Professor C. B. Smith. Professor Wallace saw it to its formal completion following Professor Smith’s departure from University of Florida.


The dissertations were written in the areas of Set-Theoretic Topology, Algebraic Topology, Partially Ordered Topological Spaces, and Topological Algebra, especially Topological Semigroups. A. D. Wallace published 76 research papers in these areas, and composed and distributed several sets of unpublished Lecture Notes. He is the founder of one of the major fields of fundamental mathematical research, topological algebra.

A. D. Wallace served as a member of the Committee on the Undergraduate Program in Mathematics and the School Mathematics Study Group. He was a member of Sigma Xi, the American Mathematical Society, the Mathematical Association of America, and numerous international societies. He was also Member-at-Large of the National Research Council, Division of Mathematics from 1962-1965, elected Governor of the Mathematical Association of America, and served as Member-at-Large on the Council of the American Mathematical Society. He also served as consultant on mathematical matters for the North Atlantic Treaty Organization and the National Science Foundation. A. D. Wallace was an editor of Summa Brasilienis Mathematicae, American Journal of Mathematics, Duke Mathematical Journal, Aequationes Mathematicae and Semigroup Forum.

Genealogical Tree of Ph.D. Students of A. D. Wallace
Part 1.
Genealogical Tree of Ph.D. Students of A. D. Wallace
Part 2.
Ph.D. Genealogical Tree of Ph.D. Students of A. D. Wallace
Part 3
Genealogical Tree of Ph.D. Students of A. D. Wallace
Part 4.
Genealogical Tree of Ph.D. Students of A. D. Wallace
Part 5.
Overview of Wallace Mathematical Work

Even as a graduate student, Alexander Doniphan Wallace was a prolific writer, with some six papers resulting from his University of Virginia sojourn. These papers dealt primarily with monotone and interior mappings and related subjects.

The first paper [1] that seems to have been written by A. D. Wallace was a common work with D. H. Hall about invariants under monotone transformations. It was one of some four papers [2, 3, 4, 5] written by him while he was yet a student at the University of Virginia. The article [2] was about the formal relationships between the concept of density, nowhere-dense, boundary, and non-boundary. It is interesting, these two subjects, that is the behavior of connectivity with respect to mappings and the notion of “boundary” or “bounding” and “end points” seemed to recur over and over in Wallace work. These concepts dominated his early work to about 1945, although he drew his earlier expertise time and again as his interests moved into Algebraic Topology and later into Topological Semigroups and related work. For example, in [5], A. D. Wallace extends a result of H. Hopf [Ho], which states that there does not exist a “free” monotone map of a continuum into an arc, to the statement obtained by replacing “arc” by “dendrite”.

In the paper [7] Wallace gives a set of axioms sufficient to construct a theory of connectivity of sets in terms of a new undefined concept $X|Y$, 

\textbf{Genealogical Tree of Ph.D. Students of A. D. Wallace}

Part 6.
which may be read “X is separated from Y” and follows it in [12] with a short note extending the notion of compactness to separation spaces, and proving the theorem for separation space that every compact connected separation space S containing at least two points must contain at least two non-cutpoints of itself.

In [10], Wallace extends the notion of nonalternating maps. A map \( f: X \rightarrow Y \) is called non-alternating relative to a cover \( \mathcal{F} \) of \( Y \) by closed sets if for any \( y \in Y \), \( F \in \mathcal{F} \), \( f^{-1}(F) \) does not separate \( X \) between two points of \( f^{-1}(y) \). He shows that if \( \mathcal{F} \) is the family of all closed sets, \( f \) is non-alternating if and only if \( f \) is monotone, and he extend some results of G. E. Schweigert [Schw] and G. T. Whyburn [Why] to non-metric spaces. In [13], Wallace gives a rather thorough study of this notion, and works a theory of cyclic elements for compact Hausdorff spaces. Among the results on transformations obtained in the concluding section of the paper is an extension of the theorem of G. E. Schweigert [Schw] to the effect that \( A \)-sets are invariant under nonalternating transformations. This theme is continued again in [20], where Wallace proved that for a non-alternating map of a continuum onto a continuum \( Y \) with a cut point, there exists a proper subcontinuum of \( Y \) whose inverse image is connected.

In [17, 18], he presents a structure theory for continuous curves along homotopy lines, generalizing known results in this framework.

In 1947 he gave an invited address to the American Mathematical Society in which he introduced a significant modification of the Alexander cochain complex. This notion subsequently developed by Spanier in his dissertation (see: [Spa]), is frequently referred to Alexander–Wallace–Spanier–Kolmogoroff cohomology.

Wallace’s work thereafter shows strong interest and use the Alexander–Wallace–Spanier–Kolmogoroff theory in this papers. In [22], he introduces connectivity notions relative to cohomology, a well as the notion of endpoint, and similar generalizations of topological concepts. During this period, Wallace proved the Map Excision Theorem [25] for Alexander–Wallace–Spanier–Kolmogoroff theory and did work on the extension and reduction theorems as well.

Among the forerunners to Wallace semigroup period is his paper [21] about continuous group actions on continua by groups which are topological spaces, but not necessary topological groups (that is, multiplication in the group is not necessary continuous). Let \( Z \) be a group, which is a topological space. A subset \( A \) of a compact connected space \( X \) is called \( Z \)-invariant provided \( z(A) = A \) for each \( z \in Z \). In [21] he obtained the following results: (1) If \( Z \) is Abelian then there is a \( Z \)-invariant subcontinuum having no
cutpoint. Moreover, there exists in $X$ a $Z$-invariant prime chain. (2) If $Z$ is Abelian and no proper subcontinuum of $X$ is $Z$-invariant, then $X$ has no cutpoint. (3) If $Z$ is connected and $X$ metric, then every endpoint and every nondegenerate prime chain is invariant.

In 1952 (see: [Iwa]), Iwasawa proved that a compact topological semigroup with cancellation is a topological group (also see: Section 4 of [GKO]), and this sparked a flurry of activity in Topological Semigroups. Within a year Wallace had written about six papers on this subject, and delivered an invitation address to the American Mathematical Society in 1953 (see: [35]). In this year P. Mostert and A. Shields joined to the Tulane Faculty. In 1955 A. D. Wallace presented the first course in Topological Semigroups, and the notes provided a great stimulus to semigroup activity for several years.

In his first semigroups papers [27, 28], A. D. Wallace developed some of the properties of idempotents and groups in topological semigroups. In [29] he showed how the generalized homotopy theorem applied to give the cohomology of a compact connected monoid equal to that of its minimal ideal. Therefore a compact connected manifold without boundary is a topological group if it is a topological monoid. Around this time, A. D. Wallace raised the question as to whether a finite dimensional homogeneous topological semigroup is a topological group, which he later solved in the one-dimensional case [37], and which was later solved in general by A. L. Hudson and P. Mostert in [HuM], using techniques of proofs in Alexander-Wallace-Spanier-Kolmogoroff cohomology. In [30] he showed that an indecomposable continuum which is a topological monoid must be a topological group. An analogous result dealing with partially ordered spaces was given in [34], especially he proved that there exists no non-trivial "continuous" partial order, for which all the sets $\{y \mid y \leq x\}$ are connected, on an indecomposable continuum. In [31] Wallace showed that if a topological semigroup $S$ is contained in the $n$-dimensional Euclidean space, $n \geq 2$, then all elements with inverse lie on the boundary of $S$, especially a unit of continuum semigroup in the $n$-dimensional Euclidean space must lie on the boundary. This was later brought to fruition by K. H. Hofmann and P. Mostert in their study of intrinsic boundary in semigroups [HoM1, HoM2]. The Swelling Lemma was proved by Wallace in [31]: If $A$ is a closed subset of a compact topological semigroup $S$, $t \in S$, and if $A \subseteq tA$, then $A = tA$. The existence and properties of maximal ideals were discussed in Wallace joint paper with R. J. Koch [32]. These results have been of great importance.

Much of this material was presented in an invited address to the American Mathematical Society in 1953 [35]. By this time, much of the topo-
logical structure and all of the algebraic structure of the minimal ideal were known. In [40] A. D. Wallace settled all of this by “topologizing” the Rees-Sushchkewitsch Theorem. Around this time in [Fau1, Fau2, Fau3], W. M. Faucett characterized the standard treads as topological semigroups. In [43] a class of semigroups satisfying $S^2 = S$ was discussed. R. J. Koch and A. D. Wallace here show that if a topological semigroup $S$ is the one sphere, or if $S$ is a metric indecomposable continuum, and if $S \cdot S = S$, then either $S$ is a group or the multiplication is trivial. The problem is open for higher dimensional spheres. Also they show in general that $S \cdot S = S$ if and only if each dense ideal is connected. In [43] R. J. Koch and A. D. Wallace posed the question as to whether $S \cdot S = S$ implies $S = K$ for two-sphere, where $K$ is the minimal ideal of topological semigroup $S$. In the case $S = ESE$ the result was settled later in [CoK] by H. Cohen and R. J. Koch, and in another particular case in [McCh1, McCh2] by J. D. McCharen, but in general it has remained unsolved (see Section Problems in [CHK]). In [43], the result of W. M. Faucett was extended by showing that irreducible continuum between two points $a$ and $b$ and which support multiplications with $a = 0$ and $b = 1$ must be arc. Also involved were some previously developed theorems on $C$-sets [38] as well as the recently established one-parameter topological semigroups of P. S. Mostert and A. L. Shields [MoSh2].

In 1955, L. W. Anderson was writing a dissertation under direction of Wallace on topological lattices, and during this period A. D. Wallace wrote several articles in this area [43, 44, 45, 47]. In particular, he proved that the central elements in a compact connected lattice are contained in the cohomological boundary, the central elements of any topological lattice, with compact Hausdorff topology, are peripheral and totally disconnected. Also, it is shown that, if no maximal chain cuts a topological lattice $L$, which is homeomorphic to a closed square, into more than two components, then $L$ is isomorphic to $I \times I$. The first assumption could be replaced by the hypotheses that $L$ is compact, connected, imbeddable in the plane, and without cutpoints. Also, he proved that if a convex subset $A$ of a connected topological lattice $L$ disconnects $L$, then the complement of $A$ consists of two connected components: the elements above $A$ and those below $A$; and if $L$ is also compact, and $A$ is either open or closed, then $A$ is connected. Wallace establishes that compact connected lattices are shown to be cyclic chains, whose cyclic elements are convex sublattices.

In [41, 53] A. D. Wallace stated and applied a highly useful result concerning acyclicity, showing that if a collection of acyclic continua are fitted together well, then their union is acyclic. This result applies to questions
of acyclicity of semigroups satisfying $S = ESE$. Also he proved the following statements: (1) if a topological semigroup $S$ is an indecomposable continuum, then no maximal subgroup of $S$ cuts $S$; (2) if $S$ is compact, and every proper retract of $S$ has the fixed point property, then $S$ is a group or every element of the minimal ideal of $S$ is an idempotent; (3) If $S$ is an $n$-sphere, and if some subset of a maximal subgroup cuts $S$, then this maximal subgroup is a Lie group and irreducibly cuts $S$.

Many properties of the partially ordered set of ideals in compact topological semigroups have motivated results in compact quasi-ordered topological spaces with closed graph [33, 36, 52]. Some related work in the relation theory was done in [61, 62, 69]. Also, in [56] and [59] some extensions of the notion of ideal were developed. The main result in the paper [56] is that if $T$ is a compact connected subsemigroup of $S$ and if $I$ is a minimal $T$-ideal of $S$, then for a cutpoint $z$ of $I$ either $Tz = z$ or $zT = z$. This generalizes a result of W. M. Faucett [Fau3] that if the minimal ideal of a compact connected semigroup has a cutpoint, then every element of this ideal is a left zero or a right zero. The paper [59] is devoted to establish the Green relations, to the analogue of the topologized Schützenberger group, and to a generalization of the Rees-Sushkevitch-Wallace Theorem.

The previous material is more in the spirit of actions of semigroups on topological spaces, and the papers [63, 64, 65, 66, 67, 68, 69, 70, 72, 74, 75, 76] are all concerned with such matters. In [66] J. Aczél and A. D. Wallace proved that if $T \times X \to X$ is a continuous map such that each of $T$ and $X$ is either compact or discrete and such that $t(t'x) = t'(tx)$ for all $t, t' \in T$ and all $x \in X$, and $Ta = X$ for some $a \in X$, then one can introduce a multiplication $\circ$ in $X$ such that $X$ is a commutative semigroup with unit $a$ and such that $tx = (ta) \circ x$ for all $t \in T$ and $x \in X$. The result is also discussed and extended in [65, 67] and [70]. In [65] the type of "machine" herein considered is a compact Hausdorff space $X$ (the state space), a compact topological semigroup $T$ (the input semigroup) and a continuous act $T \times X \to X$. Of particular interest in the present paper are equivalence relations in $X$ which are preserved by $T$ and which are irreducible in a natural sense. If $X$ is totally disconnected, then it can be imbedded in the product of finite state spaces. These finite spaces are quotient spaces of $X$ and $T$ acts on them, and hence on their product, in a natural way. In [67] Wallace shows that if $T \times X \to X$ is a continuous map, and $X$ and $T$ are both discrete [compact] topological spaces, if $Ta = X$ for some $a \in X$, and if: (1) $t'a = t''a$ implies $t'x = t''x$ ($x \in X$; $t', t'' \in T$) and (2) $ta = t'(t''a)$ implies $tx = t'(t''x)$ ($x \in X$; $t, t' \in T$), then $X$ is a semigroup with respect to the operation $\circ$ defined by $tx = (ta) \circ x$ ($t \in T$;
x ∈ X). Furthermore, a is a right identity in X and every element of T acts as a left shift.

In [70] J. M. Day and A. D. Wallace consider conditions under which a compact connected semigroup may not act upon certain continua such as the closure of sin(1/x), where 0 < x ≤ 1. In particular, there is shown that if a continuous semigroup acts "nontrivially" on a continuum X which contains an open dense half-line W, then X \ W is homogeneous. This contains the well known and often proved result that the closed-up sin(1/x) curve does not admit a structure of a topological monoid. Some general results on actions of compact topological semigroups on compact topological spaces were given in [61] and [64]. For a transitive relation on a set X, a subset M of X is called P-intersective if M ∩ Px ≠ ∅ for each x ∈ X, where Px = {y | (y, x) ∈ P}. Several characterizations of minimal P-intersective sets are given, e.g., a P-intersective set is minimal if and only if it is maximal P-scattered. These results are then used to prove (i) a weak transitivity property for certain semigroups of transformations of a compact Hausdorff space, and (ii) an extension to compact spaces of a graph-theoretic theorem concerning paths in finite complete directed graphs. In [64] the "act" version of Numakura's Theorem (see: [Nu]) on the approximation of compact totally disconnected topological semigroups by finite semigroups is established.

On Wallace's Problems

In 1936 Montgomery [Mon] showed that every completely metrizable semitopological group is a topological group. In 1953 A. D. Wallace [35] asked: Is every locally compact regular semitopological group a topological group? In 1957 Robert Ellis obtained a positive answer to the Wallace question (see: [Ell1] and [Ell2]). In 1960 W. Zelazko used Montgomery's result showed that each completely metrizable semitopological group is a topological group [Zel]. Since both locally compact and completely metrizable semitopological group are Čech-complete spaces, this suggested Pfister [Pf] in 1985 to ask: Is every Čech-complete semitopological group a topological group? In 1996 A. Bouziad [Bou] and E. Reznichenko [Re2] independently answered affirmatively to the Pfister question.

In this lecture we shall discuss the following problem [35] posed by Wallace in 1953 at the Annual Meeting of the American Mathematical Society in Baltimore, Maryland. He remained that several authors had proved that a Hausdorff compact cancellative topological semigroup is a topolog-
ical group. A. D. Wallace asked: *Is every countably compact cancellative topological semigroup a topological group?*

E. Reznichenko [Re1] has proved that a Tychonoff pseudocompact group with continuous multiplication is a topological group. Pfister [Pf] proved that a countable compact regular paratopological group is a topological group. A. Yur’eva [Yu] and A. H. Tomita [To3] shown that a sequential countably compact Hausdorff cancellative topological semigroup is a topological group.

D. Robbie and S. Svetlichnyi [RS] showed recently under \( \text{CH} \) that there is a counterexample to the Wallace Problem. Tomita [To1], [To2] showed that there exists a counterexample to Wallace’s Problem under \( \text{MA}_{\text{countable}} \) (\( \text{MA}_{\text{countable}} \) is the Martin’s Axiom restricted to countable posets. This axiom is equivalent to a strong form of the Baire Category Theorem: *The circle \( \mathbb{T} \) is not the union of fewer than continuum many closed nowhere dense sets.*).

B. Bokalo and I. Guran [BG] give an answer to the question posed by Robbie and Svetlichnyi in [RS], i.e., they proved that a sequentially compact Hausdorff cancellative topological semigroup is a topological group.

Recall that a semigroup \( S \) is called *right reversible* if \( Sa \cap Sh \neq \emptyset \) for all \( a, b \in S \).

**O.Q. 1.** Is every countably compact cancellative semigroup right reversible?

**O.Q. 2.** Does there exist in \( \text{ZFC} \) an example of a countably compact cancellative semigroup which is not a topological group?

Further in this section we discuss problems posed Alexander Doniphan Wallace from early unpublished problems lists and from [50] and [55].

1. **Problem.** Does the \( n \)-cell \( S \) for \( n \geq 2 \) admit the structure of a compact topological semigroup such that \( \partial S = E(S) \)? (By \( \partial S \) we denote the boundary of \( S \).)

This is the ”Skin problem” which for \( n = 2 \) was answered in the negative by E. N. Ferguson in [Fe1] and [Fe2] (see the section on semigroups on the 2-cell in Chapter 5 of Vol. 2 of [CHK]).

2. **Problem.** Does an \( n \)-sphere \( S \) admit the structure of a compact topological semigroup such that \( S^2 = S \) and \( S \) has zero?

As indicated by A. D. Wallace, the answer is negative in the case that \( n = 1 \) [48]. P. S. Mostert and A. L. Shields in [MoSh1] show that if \( S \) is a two-sphere topological semigroup with a nontrivial connected subgroup,
then $S^2 \neq S$. Other contributions to this problem appear in [CoK] and [McCh1] (see the remarks at the end of the section on admissibility in Chapter 5 of Vol. 2 of [CHK]).

3. **Problem.** Suppose $S$ is a monoid on $\mathbb{R}^n$. Can a compact connected subgroup of the group of units of $S$ be self-linked?

   It was proved in [Cu] that if $S$ is a topological semigroup on $\mathbb{R}^n$ and $G$ is a compact subgroup of $S$, then $G$ is not self-linked. As indicated in [Mo1], it was established by P. S. Mostert in [Mo] that if $S$ is a contractible topological semigroup and $G$ is a subgroup of $S$, then $G$ is not self-linked.

4. **Problem.** If $S$ is a compact connected locally connected one-dimensional metric topological monoid, then it is known that $S$ is either a dendrite or contains exactly one simple closed curve which coincides with the minimal ideal of $S$. Is there an analogous proposition for higher dimensions?

5. **Problem.** If $S$ is a compact connected topological semilattice with identity, does $S$ have the fixed point property?

   We remark that a compact connected topological monoid without fixed point property was constructed by H. Cohen in [Co1], and a compact connected topological semilattice (without an identity) without the fixed point property was constructed by H. Cohen in [Co2].

6. **Problem.** What conditions on a compact connected topological semigroup $S$ with $S - ESE$ will insure that $S$ and $M(S)$ (the minimal ideal of $S$) have the same cohomology?

   This is always true if the semigroup $S$ has dimension 0 or 1. This is follows from the theorem that if $S$ is a compact connected topological semigroup such that $S = ESE$, then $H^1(S) = H^1(M(S))$ [CoK]. As we can see in [CHK] (Vol. 2, Theorem 1.2), if $S$ is a compact connected topological semigroup with a left (or right) identity, then $S$ and $M(S)$ have the same cohomology. An example was constructed by A. L. Hudson in [Hu] of a topological semigroup $S$ on 2-sphere with four stickers at the zero of $S$ such that $S = ESE$.

7. **Problem.** Let $S$ be a topological semigroup and let $\partial S$ denote the boundary of $S$ in some suitable sense.

   (i) If every element of $S$ has a square root, does every element of $\partial S$ have a square root in $\partial S$? (Problem of H. H. Corson.)
(ii) Under some interpretation of "boundary" it is known that if $S$ has an identity, then it lies in $\partial S$. Are there other useful interpretations of "boundary" for which this is so?

(iii) If one assumes that multiplication on $\partial S$ is commutative, are there agreeable conditions under which $S$ is commutative?

We consider now the problems posed in [55].

8. Problem. If $S$ is a nondegenerate finite dimensional compact connected topological monoid such that no retract of $S$ cuts $S$, is $S$ topologically an $n$-sphere, $n = 1$ or 3?

It is known that such an $S$ must be a group if it is homogeneous (see [HuM])

An element $x$ of a semigroup $S$ is called periodic if $x^{n+1} = x$ for some $n \in \mathbb{N}$. The least such $n$ is called the period of $x$ and is denoted as $p(x)$ in the next problem. A semigroup $S$ is called pointwise periodic if each element of $S$ id periodic.

9. Problem. Let $S$ be a pointwise periodic semigroup on an $n$-cell. Can it be that $S \setminus E(S) \neq \emptyset$ and $p(x) > 2$ for each $x \in S \setminus E(S)$ and that $p$ is constant on $S \setminus E(S)$?

10. Problem. If $S$ is a compact topological semigroup, under what conditions on $S$ will there exist a regular measure $\mu$ on the Borel sets of $S$ and a function $g: S \to \mathbb{R}$ such that $\mu(xA) = g(x)\mu(A)$ for each $x \in S$ and each Borel set $A$?

List of Publications of Alexander Doniphan Wallace


44. The center of a compact lattice is totally disconnected, Pacific J. Math. 7 (1957), 1237–1238. [MR 20,823].

45. Two theorems on topological lattices, Pacific J. Math. 7 (1957), 1239–1241. [MR 20,824].


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References


Pseudoholomorphic curves in complex and symplectic geometry

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Abstract. The techniques of pseudoholomorphic curves proved itself to be extremely useful in different field: non-squeezing theorem, [Gr]; Arnold conjecture and Floer homology, [FO]; Gromov-Witten invariants; polynomial hulls of totally real surfaces, [Gr],[AI]; envelopes of meromorphy, [IS-2]; etc. We shall discuss some of these applications in our lectures.

Lecture 1. Almost complex manifolds and pseudoholomorphic curves. pp. 59–78

2.1. Non-squeezing theorem. 2.2. Symplectic capacities. 2.3. Riemann-Roch Formula and index of \(\bar{\partial}\)-type operators. 2.4. Moduli space of pseudoholomorphic spheres. 2.5. Universal families and evaluation maps. 2.6. Gromov-Witten Invariant. 2.7. Quantum multiplication and Quantum Cohomology. 2.8. Quantum cohomology ring of \(\mathbb{C}P^n\).
Lecture 1.
Almost complex manifolds and pseudoholomorphic curves

1.1. Almost complex structures

An almost complex structure on a real manifold $X$ is a continuous section $J \in \text{End}(TX)$ which satisfies the condition $J^2 = -\text{Id}$. In other words this is a family $\{J_x\}_{x \in X}$ of endomorphisms of tangent spaces $T_xX$ continuously depending on $x$ and satisfying for every $x$ the identity above. Each tangent space comes now equipped with a structure of a complex vector space via the multiplication by $i$ defined as $i \cdot v := J_xv$ for $v \in T_xX$. A pair $(X, J)$ is called an almost complex manifold.

The first example is the standard model $\mathbb{C}^n := (\mathbb{R}^{2n}, J_x)$, where the operator $J_x$ is the standard multiplication by $i$, i.e.,

$$J_xv = i \cdot v = (-v_2, v_1, ..., -v_{2n}, v_{2n-1}),$$

where $v = (v_1, v_2, ..., v_{2n-1}, v_{2n})$.

1.2. Pseudoholomorphic curves

A Riemann surface is a pair $(\Sigma, j)$, where $\Sigma$ is an oriented surface and $j$ an almost complex structure on $\Sigma$. In this case $j$ is always integrable, i.e. $(\Sigma, j)$ is locally biholomorphic to the standard model $(\mathbb{R}^2, J_x) = \mathbb{C}$.

**Definition 1.2.1.** A (parameterized) pseudoholomorphic curve in an almost complex manifold $(X, J)$ is a holomorphic map $u : (\Sigma, j) \to (X, J)$ from some Riemann surface $(\Sigma, j)$ to $(X, J)$, i.e., $u \in L^1(\Sigma, X)$ and for a.a. $z \in \Sigma$ one has

$$du(z) \circ j(z) = J(u(z)) \circ du(z) \quad (1.2.1)$$

the differential of $u$ commutes with complex structures.

In the case $(\Sigma, j) = (X, J) = (\mathbb{R}^2, J_{\mathbb{R}})$ (1.2.1) is the standard Cauchy-Riemann equation $\frac{\partial u}{\partial \bar{z}} = 0$. The image $M = u(\Sigma)$ is called a nonparameterized $J$-complex curve, or simply a $J$-complex curve. There are many local $J$-complex curves in $X$ for any $J$. Namely, if $J$ is of Hölder class $C^\alpha$ for some $\alpha > 0$ then due to [NW] for any point $p \in X$ and any tangent vector $v \in T_pX$ there is a $J$-holomorphic map $u : \Delta_r \to X$ for some $r > 0$ such that $u(0) = p$ and $du(0)(\frac{\partial}{\partial \bar{z}}) = v$. Here $\Delta_r = \{z \in \mathbb{C} \mid |z| < r\}$.

Let us list some fundamental properties of pseudoholomorphic curves.
1.3. Compactness

We shall need the notion of a nodal complex curve. Recall that a standard node is the complex analytic set \( A_0 = \{ (z_1, z_2) \in \mathbb{C}^2 \mid z_1 \cdot z_2 = 0 \} \). A complex nodal curve \( C \) is a complex analytic space of pure dimension one with only nodal points as singularities. We shall always suppose that \( C \) is connected and has a “finite topology”, i.e., \( C \) has finitely many irreducible components, finitely many nodal points, and that \( C \) has a smooth boundary \( \partial C \) consisting of finitely many smooth circles \( \gamma_i \), such that \( \tilde{C} := C \cup \partial C \) is compact.

**Definition 1.3.1.** A real connected oriented surface with boundary \((\Sigma, \partial \Sigma)\) *parameterizes* a complex nodal curve \( C \) if there is a continuous map \( \sigma : \Sigma \to C \) such that

1. if \( a \in C \) is a nodal point, then \( \gamma_a = \sigma^{-1}(a) \) is a smooth imbedded circle in \( \Sigma \setminus \partial \Sigma \), and if \( a \neq b \) then \( \gamma_a \cap \gamma_b = \emptyset \);
2. \( \sigma : \Sigma \setminus \bigcup_{i=1}^{N} \gamma_{a_i} \to C \setminus \{ a_1, \ldots, a_N \} \) is a diffeomorphism, where \( a_1, \ldots, a_N \) are the nodes of \( C \).

![Diagram](image)

Figure.

Parametrization map \( \sigma \) on the picture is the vertical projection. Circles \( \gamma_1, \ldots, \gamma_5 \) are contracted by \( \sigma \) to nodal points \( a_1, \ldots, a_5 \). If \( j_C \) is a complex structure of \( C \) (in nodal points \( j_C \) is different on the different pieces) then the pullback \( \sigma^* j_C \) is a well defined complex structure on \( \Sigma \setminus \bigcup \gamma_{a_i} \).
When the curve $C$ is smooth we see it also as a pair $(\Sigma, j)$, where $j = \sigma^* j_C$ is a complex structure on $\Sigma$ inherited by the parametrization $\sigma$ from the complex structure $j_C$ of $C$ (in the smooth case $\sigma$ is a diffeomorphism). We can still apply this point of view in the case of a singular $C$ but then the corresponding $j$ is well defined only outside of the preimages of the nodal points.

**Definition 1.3.2.** A nodal $J$-complex curve over $X$ is a pair $(C, u)$, where $C$ is a complex nodal curve and $u: C \rightarrow X$ is a $J$-holomorphic map. We are going to define a topology on the space $\mathcal{C}_\Sigma(X)$ of nodal curves over $X$ parameterized by $\Sigma$ and prove that bounded sets in $\mathcal{C}_\Sigma(X)$ are relatively compact.

Let $C_n$ be nodal complex curves parameterized by the same surface $(\Sigma, \partial \Sigma)$.

**Definition 1.3.3.** We say that the complex structures on $C_n$ do not degenerate near the boundary, if there exists $R > 1$ such that for any $n$ and any boundary circle $\gamma_{n,i}$ of $C_n$ there exists an annulus $A_{n,i} \subset C_n$ adjacent to $\gamma_{n,i}$, such that all $A_{n,i}$ are mutually disjoint, do not contain nodal points of $C_n$, and have the same conformal radius $R$.

An annulus $A$ on a complex curve $C$ (or $(\Sigma, j)$) has conformal radius $R > 1$ if $A$ is biholomorphic to $A(1, R) := \{ z \in \mathbb{C} \mid 1 < |z| < R \}$. An annulus $A$ is said to be adjacent to a circle $\gamma \subset C$, if $\gamma$ is one of its boundary component.

Let $\{ J_n \}$ be continuous $\mathcal{C}^1$ structures on $X$ uniformly converging to $J$ and let $(C_n, u_n)$ be $J_n$-complex nodal curves all parameterized by the same real surface $\Sigma$.

**Theorem 1.3.1 (Gromov’s compactness theorem).** If the areas of $u_n(C_n)$ are uniformly bounded (with respect to some fixed Riemannian metric $h$ on $X$) and the structures $j_{C_n}$ do not degenerate near the boundary, then there exists a subsequence, still denoted as $(C_n, u_n)$, such that

1) $(C_n, j_{C_n})$ converge to some nodal curve $(C_\infty, j_{C_\infty})$ in an appropriate completion of the moduli space of Riemann surfaces of given topological type, i.e., there exists a parametrization map $\sigma_\infty: \Sigma \rightarrow C_\infty$ by the same real surface $\Sigma$;

2) one can choose a new parameterizations $\sigma_n$ of $C_n$ in such a way that $\sigma_n^* j_{C_n}$ will converge to $\sigma_\infty^* j_{C_\infty}$ in the $C^\infty$-topology on compact subsets outside of the finite set of circles on $\Sigma$, which are preimages of the nodal points of $C_\infty$ by $\sigma_\infty$.
3) the maps $u_\alpha \circ \sigma_\alpha$ converge, in the $C^0$-topology on $\Sigma$ and in the $L^{1,p}_\text{loc}$-topology (for all $p < \infty$) outside of the preimages of the nodes of $C_\infty$ to a map $u_\infty \circ \sigma_\infty : C_\infty \to X$ such that $u_\infty$ is a $(j_{C_\infty}, J_\infty)$-holomorphic.

Proof can be achieved by induction on the area, see [IS-1] for more details.

1.4. Attaching an Analytic Disk to a Lagrangian Submanifold of $\mathbb{C}^n$

Recall that a symplectic manifold is a pair $(X, \omega)$, where $X$ is a smooth manifold and $\omega$ is a closed, nondegenerate two-form on $X$, called a symplectic form. In this case the dimension of $X$ is necessarily even, say $2n$, and nondegeneracy of $\omega$ means that $\omega^n$ newer vanishes, or, equivalently the map $v \mapsto \omega(\cdot, v)$ is an isomorphism between $TX$ and $T^*X$ for any $v \in TX$. Let us consider $\mathbb{C}^n = (\mathbb{R}^{2n}, J_{st})$ together with some symplectic form $\omega$ taming the standard complex structure $J_{st}$. An $n$-dimensional submanifold $W \subset \mathbb{C}^n$ is called $\omega$-Lagrangian if $\omega |_W \equiv 0$.

Exercise 1.4.1. Prove that every Lagrangian submanifold of $\mathbb{C}^n$ is totally real, i.e., $J_{st}(T_p W) \cap T_p W = \{0\}$ for any $p \in W$

Exercise 1.4.2. Prove that for a Lagrangian manifold $W \subset \mathbb{C}^n$ and the unit circle $S^1 \subset \mathbb{C}$ the manifold $S^1 \times W$ is Lagrangian in $\mathbb{C}^{n+1}$ with respect to $\omega = \frac{1}{2} dz \wedge d\bar{z} + \omega$.

Definition 1.4.1. A holomorphic map $u : \Delta \to \mathbb{C}^n$ “sufficiently smooth” up to the boundary and such that $u(\partial \Delta) \subset W$ will be called an analytic disk attached to $W$.

Our goal in this section is to prove the following theorem of Gromov:

Theorem 1.4.1. Let $W$ be a compact Lagrangian submanifold of $\mathbb{C}^n$. Then there exists a non-constant analytic disk attached to $W$.

We shall closely follow the exposition of H. Alexander, [Al]. Fix some point $w_0 \in W$ and denote by $w_0(z) \equiv w_0$ the constant holomorphic map. Fix $p > 2$ and consider the Banach manifold $L^{2,p}(\Delta, \partial \Delta, 1; \mathbb{C}^n, W, w_0)$ of $L^{2,p}$-maps from $\Delta$ to $\mathbb{C}^n$ which map $\partial \Delta$ to $W$ and $1$ to $w_0$, and which are homotopic to the constant map $w_0 \equiv w_0$ as $\Delta$ maps from $\Delta$ to $\mathbb{C}^n, W, w_0)$ mappings. Note that due to the Sobolev imbedding $L^{2,p} \subset C^{1,1-\frac{2}{p}}$ our mappings are smooth up to the boundary.

Take $u \in L^{2,p}(\Delta, \partial \Delta, 1; \mathbb{C}^n, W, w_0)$. Denote by $E$ as usually the pull-back by $u$ of the tangent bundle of $\mathbb{C}^n$. In fact, $E \equiv \Delta \times \mathbb{C}^n \to \Delta$, the trivial bundle over $\Delta$. Denote by $F$ the pull-back $u^* TW$ of the tangent to
$W$ bundle. $F$ is a totally real subbundle of $E$ of real dimension $n$. The tangent space to $L^{2,p}(\Delta, \partial \Delta, 1; \mathbb{C}^n, W, w_0)$ at $u$ is

$$T_u L^{2,p}(\Delta, \partial \Delta, 1; \mathbb{C}^n, W, w_0) = \{ h \in L^{2,p}(\Delta, \mathbb{C}^n) \mid h|_{\partial \Delta} \in F, h(1) = 0 \}.$$

In the Cartesian product $L^{2,p}(\Delta, \partial \Delta, 1; \mathbb{C}^n, W, w_0) \times L^{1,p}(\Delta, \mathbb{C}^n)$ consider the submanifold $\mathcal{E} = \{(u, v) \mid \partial u = v\}$ with the natural projection $\pi: \mathcal{E} \to L^{1,p}(\Delta, \mathbb{C}^n)$.

**Exercise 1.4.3.** In $\mathbb{R}^{2n+2}$ consider the operator given by the matrix

$$J_v = \begin{pmatrix} 0 & -1 & \cdots & v_2 & -v_1 \\ 1 & 0 & \cdots & -v_1 & -v_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & v_{2n} & -v_{2n-1} \\ 0 & 0 & \cdots & -v_{2n-1} & -v_{2n} \\ 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

Prove that $J_v$ defines an almost complex structure in $\mathbb{R}^{2n+2}$, which for every $z \in \mathbb{R}^2$ on the vertical slice $\{z\} \times \mathbb{R}^n$ coincides with the standard structure $J_{st}$ of $\mathbb{C}^n$.

**Exercise 1.4.4.** Prove that the equation $\partial u = v$ for a $C^1$-map $u: \mathbb{C} \to \mathbb{C}^n$ is equivalent to the $J_v$-holomorphicity of the section $u: z \to (z, u(z))$ of the fibration $(\mathbb{R}^{2n+2}, J_v) \to (\mathbb{R}^2, J_{st})$, i.e., to the equation

$$\frac{\partial(z, u)}{\partial x} + J_v(z, u) \left[ \frac{\partial(z, u)}{\partial y} \right] = 0.$$

Now we shall prove the following alternative:

**Lemma 1.4.2.** If there is no nonconstant analytic disk $u \in L^{2,p}(\Delta, \partial \Delta, 1; \mathbb{C}^n, W, w_0)$, then the projection $\pi: \mathcal{E} \to L^{1,p}(\Delta, \mathbb{C}^n)$ is surjective.

**Proof.** Suppose that a nonconstant analytic disk $u \in L^{2,p}(\Delta, \partial \Delta, 1; \mathbb{C}^n, W, w_0)$ does not exists. We are going to prove that in this case $\pi$ is surjective.

**Step 1.** $\pi$ is a proper mapping, i.e., for the converging sequence $v_k \to v_0$ in $L^{1,p}(\Delta, \mathbb{C}^n)$ and for the sequence $u_k$ with $(u_k, v_k) \in \mathcal{E}$, there is a converging subsequence $u_{k_n}$.

Note that $\partial u_k = v_k$. According to Exercise 1.4.4 above this means that the sections $u_k := (z, u_k)$ are $J_{v_k}$-holomorphic with $J_{v_k}$ converging to $J_{v_0}$ in
$C^0$-sense. Note also that the boundaries of our disks are on the Lagrangian (and thus totally real) submanifold $W := S^1 \times W$ and they are homotopic to each other. From here we see that

\[
\text{area}(\tilde{u}_k(\Delta)) = \int_{\tilde{u}_k(\Delta)} \tilde{\omega} = \int_{\tilde{u}_k(\partial \Delta)} \partial \lambda,
\]

where $\lambda$ is some primitive of $\tilde{\omega}$. The second integral does not depend on the homology class of $\tilde{u}_k(\Delta)$ in $H_1(W, \mathbb{R})$, because $\tilde{\omega}|_W = \partial \lambda|_W \equiv 0$ ($W$ is $\tilde{\omega}$-Lagrangian).

So by Theorem 1.3.1 either the limit of some subsequence, still denoted as $u_k(\Delta)$, contains a nonconstant complex sphere (this is impossible in $C^n$), or the limit of $u_k(\Delta)$ contains some nonconstant analytic disk with the boundary on $W$ (this is prohibited by our assumption), or $u_k$ $C^1$-converge.

**Step 2.** $d\pi(u,v) : T_{(u,v)}\mathcal{E} \to L^{1,p}(\Delta, C^n)$ is Fredholm of index zero for every $u \in \mathcal{E}_v := \pi^{-1}(v)$.

The fact that the boundary value problem $\partial h = v, h|_{\partial \Delta} \in F$ is Fredholm is classical, see [Ga]. Our manifold $\mathcal{E}$ is connected, so the index of $d\pi(u,v)$ does not depend on $(u,v)$ and can be calculated at $(u_0, 0) \in \mathcal{E}$ where $d\pi$ is a bijection. Therefore $\text{ind}(d\pi)$ is everywhere zero.

One says that for the smooth mapping $\pi : \mathcal{E} \to L^{1,p}(\Delta, C^n)$ the point $(u,v)$ is regular if $d\pi(u,v)$ is surjective. $v$ is a regular value if it is not an image of a nonregular point.

**Step 3 (Smale’s theorem).** Let $\pi : \mathcal{E} \to M$ be a proper Fredholm map (i.e., $d\pi_x$ is Fredholm for all points $x \in \mathcal{E}$). Then the set of regular values is dense in $M$. Moreover, for every regular value $v \in M$ the set $\mathcal{E}_v := \pi^{-1}(v)$ is a manifold of dimension equal to the $\text{ind}(d\pi_x)$ at $x \in \mathcal{E}_v$. Moreover, for any two regular values $v_1$ and $v_2$ the manifolds $\mathcal{E}_{v_1}$ and $\mathcal{E}_{v_2}$ are cobordant. See [Sm].

In our case 0 is a regular value, so for a dense subset of $v$’s $\mathcal{E}_v$ is cobordant to a point, therefore is a point itself. Properness of $\pi$ implies now that $\mathcal{E}_v$ is always a point. □

It is not difficult to show that $\pi$ cannot be surjective. This will imply the existence of non constant analytic disk attached to $W$.

**Lemma 1.4.3.** The projection $\pi : \mathcal{E} \to L^{1,p}(\Delta, C^n)$ is not surjective.

**Proof.** Otherwise, for $v^C := (C, 0, ..., 0)$ find $u^C := (u^C_1, ..., u^C_n)$ with $\partial u^C = v^C$. Therefore, $\partial u^C_1 = C$. This implies that $u^C_1 = Cz - h^C$, where $h^C$ is a holomorphic function on $\Delta$. Since $u^C(\partial \Delta) \subset W$, the family $u^C$ is uniformly bounded on $\partial \Delta$ by a constant $k$ independent of $C$. Therefore
\[|\bar{z} - h^C(z)/C| \leq k/C \text{ for } z \in \partial \Delta. \] Since \(\bar{z} - h^C(z)/C\) is harmonic on \(\Delta\), the bound holds for all \(z \in \Delta\). This implies that \(\bar{z}\) can be uniformly approximated on \(\Delta\) by holomorphic functions. Contradiction.

1.5. Cusp-points

In this section we assume \(J \in C^1\). First of all let us call a \(J\)-holomorphic map \(u: \Sigma \to X\) primitive if there are no disjoint non-empty open sets \(U_1, U_2 \subset \Sigma\) with \(u(U_1) = u(U_2)\). A Riemann surface \(\Sigma\) is supposed here to be connected. We can restrict ourselves to primitive maps due to the following observation, see [IS-1] Theorem 3.4.1:

**Theorem 1.5.1.** Let \((\Sigma_1, j_1)\) and \((\Sigma_2, j_2)\) be connected Riemann surfaces and \(u_1: (\Sigma_i, j_i) \to (X, J)\) non-constant \(J\)-holomorphic maps. If there are non-empty open sets \(U_i \subset \Sigma_i\) with \(u_1(U_1) = u_2(U_2)\), then there exists a connected Riemann surface \((\Sigma, j)\) and a \(J\)-holomorphic map \(u: (\Sigma, j) \to (X, J)\) such that \(u_1(\Sigma_1) \cup u_2(\Sigma_2) = u(\Sigma)\) and \(u: \Sigma \to X\) is primitive. Moreover, the maps \(u_i: \Sigma_i \to X\) factorize through \(u: \Sigma \to X\), i.e., there exist holomorphic maps \(g_i: (\Sigma_i, j_i) \to (\Sigma, j)\) such that \(u_i = u \circ g_i\).

It can be shown that any primitive \(J\)-holomorphic map \(u: \Sigma \to X\) is a local immersion outside of a discrete subset \(A \subset \Sigma\) of cusp-points of \(u\), i.e., of points where the differential \(du\) vanishes. Moreover, for every \(a \in A\) the multiplicity of zero \(m_a\) of \(du\) at \(a\) is well defined and can take any integer value greater or equal to \(0\). One can interpret this multiplicity in the following way. Let \(u: (\Delta, 0) \to (\mathbb{C}^2, J, 0)\) be a germ of our curve at \(a = 0\) and let \(u(a) = 0\). Without loss of generality we can suppose that \(J(0) = J_0\). Taking into account that zeros of \(du\) are isolated, we can suppose that \(du\) vanishes only at zero. Furthermore, let \(w_1, w_2\) be the standard complex coordinates in \((\mathbb{C}^2, J_0)\). Then we can write our curve in the form

\[u(z) = z^{\alpha} \cdot b + o(|z|^\beta),\]

with some \(b \in \mathbb{C}^2\) and for any \(0 < \alpha < 1\). All this provides \(J \in C^1\).

In the same manner as for usual complex curves one defines the **Milnor number** \(\nu_a\) of the cusp-point \(a \in \Sigma\). Roughly speaking \(\nu_a\) is the number of self-intersections of a generic perturbation of \(u\).

1.6. Positivity of intersections

Let us first recall the notion of the intersection number (index) of two surfaces in \(\mathbb{R}^4\). Let \(M_1\) and \(M_2\) be two-dimensional, oriented, smooth surfaces in \(\mathbb{R}^4\) passing through the origin. Let \(M_i\) be small perturbations of \(M_1\) making them intersect transversally with each other and with a small
sphere $S^3$ around zero. Denote by $\gamma_1$ and $\gamma_2$ the intersections of $M_1, M_2$ with $S^3$.

The intersection number of $M_1$ and $M_2$ is defined to be the algebraic intersection number of $M_1$ and $M_2$. If $M_1, M_2$ intersect only at zero, we also say that the number just defined is the intersection index of $M_1$ and $M_2$ at zero. It will be denoted by $\delta(M_1, M_2)$ or $\delta_0$. This number is independent of the particular choice of perturbations $M_i$ and is equal to the linking number $l(\gamma_1, \gamma_2)$ of the (reducible in general) curves $\gamma_i$ on $S^3$, see [Rf].

Following result can be found in [MW], [McD] and [IS-1]:

**Theorem 1.6.1.** Let $u_i : \Delta \to (\mathbb{R}^4, J), i = 1, 2,$ be two primitive distinct $J$-complex disks such that $u_1(0) = u_2(0)$. Set $M_i = u_i(\Delta)$. Let $Q = M_1 \cap M_2$ be the intersection set of the disks. If $J$ is $C^1$-smooth, then the following holds.

1. The set $\{(z_1, z_2) \in \Delta \times \Delta | u_1(z_1) = u_2(z_2)\}$ is a discrete subset of $\Delta \times \Delta$. In particular, $u_1(\Delta) \cap u_2(\Delta)$ is a countable set;

2. The intersection index in any such point of $Q$ is strictly positive. Moreover, if $\mu_1$ and $\mu_2$ are the multiplicities of $u_1$ and $u_2$ in $z_1$ and $z_2$, respectively, with $u_1(z_1) = u_2(z_2) = p$, then the intersection number $\delta_p$ of branches of $M_1$ and $M_2$ at $z_1$ and $z_2$ is at least $\mu_1 \cdot \mu_2$;

3. $\delta_p = 1$ iff $M_1$ and $M_2$ intersect at $p$ transversally.

### 1.7. Symplectic Surfaces

Let $(X, \omega)$ be a manifold with nowhere degenerate exterior 2-form $\omega$.

**Definition 1.7.1.** An immersion $u : S \to X$ of a real surface $S$ into $X$ is called $\omega$-positive if $u^* \omega$ never vanishes.

If $\omega$ is symplectic we call such immersions symplectic.

**Definition 1.7.2.** An almost-complex structure $J$ is said to be tamed by an exterior 2-form $\omega$ if $\omega(u, J u) > 0$ for any nonzero $u \in TX$.

In the following lemma we suppose for simplicity of proof that $\dim_{\mathbb{R}} X = 4$.

**Lemma 1.7.1.** Let $M$ be a $\omega$-positive compact surface immersed into $(X, \omega)$ with only double positive local self-intersections, and let $U \subset U$ be neighborhoods of $M$. Then for any given $\omega$-tamed a.c. structure $J$ there exists a smooth family $\{J_t\}_{t \in [0, 1]}$ of almost-complex structures on $X$ such that:
a) \( J_0 \) is the given structure \( J \) on \( X \); 
b) for each \( t \in [0, 1] \) the set \( \{ x \in X \mid J_t(x) \neq J_0(x) \} \) is contained in \( U_1 \); 
c) \( M \) is \( J_1 \)-holomorphic; 
d) all \( \{ J_t \} \) are tamed by the given form \( \omega \), i.e., \( \omega(v, J_t(v)) > 0 \) for every nonzero \( v \in TX \).

In particular, given an \( \omega \)-positive compact surface \( M \subset X \) with only positive self-intersections there exists an \( \omega \)-tamed a.c. structure \( J \) on \( X \) such that \( M \) becomes \( J \)-complex. We shall use this remark in the next section.

1.8. First Chern class and the Genus Formula

Let \( L \to M \) be a complex line bundle over a real manifold \( M \). One of the possible definitions consists of taking a real rank two bundle over \( M \) with an operator \( J \in \text{End}(L) \), satisfying \( J^2 = -\text{id}_L \). One can than locally find a frame \( e_1(x), e_2(x) \) with \( J e_1(x) = e_2(x) \). This gives a covering \( \{ U_\alpha \} \) of \( M \) together with isomorphisms of complex line bundles \( \phi_\alpha : L_{|U_\alpha} \to U_\alpha \times \mathbb{C} \), i.e., a standard definition of a complex line bundle. Sometimes we shall mark as \( e_1^\alpha, e_2^\alpha, \phi_\alpha^d \) the corresponding objects to underline their dependence on \( \alpha \).

Denoting by \( \mathcal{A} = \mathcal{A}_M \) and by \( \mathcal{A}^* = \mathcal{A}_M^* \) the sheaves of complex valued (resp. complex valued nonvanishing) functions on \( M \), we observe the following exact sequence

\[
0 \to \mathbb{Z} \overset{i}{\to} \mathcal{A} \overset{\exp(2\pi i \cdot)}{\longrightarrow} \mathcal{A}^* \to 0. \tag{1.8.1}
\]

Here \( i \) is an imbedding of the sheaf of locally constant integer valued functions into \( \mathcal{A} \), and \( \exp(2\pi i \cdot) : f \to e^{2\pi i f} \). Sequence (1.8.1) gives rise to the following long exact sequence of Čech cohomologies

\[
0 = \check{H}^1(M, \mathcal{A}) \overset{\exp(2\pi i \cdot)}{\longrightarrow} \check{H}^1(M, \mathcal{A}^*) \overset{\delta}{\longrightarrow} \check{H}^2(M, \mathbb{Z}) \to \check{H}^2(M, \mathcal{A}) = 0. \tag{1.8.2}
\]

Equalities \( \check{H}^1(M, \mathcal{A}) = \check{H}^2(M, \mathcal{A}) = 0 \) follow from the fact that the sheaf \( \mathcal{A} \) admits a partition of unity.

Classes from \( \check{H}^1(M, \mathcal{A}^*) \) are the defining cocycles of complex line bundles - one more possible definition of the line bundle. In terms of local trivializations \( \{ \phi_\alpha \} \) such cocycles can be obtained as

\[
\phi_{\alpha, \beta} = \phi_\alpha \circ \phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{C} \to (U_\alpha \cap U_\beta) \times \mathbb{C},
\]
i.e., $\phi_{\alpha, \beta} \in A^1_{\omega \cap \nu_\beta}$. 

**Definition 1.8.1.** If $\{\phi_{\alpha, \beta}\} \in H^1(M, \mathcal{A}^*)$ is a defining cocycle of a complex line bundle $L$, then $\delta(\{\phi_{\alpha, \beta}\}) \in H^2(M, \mathbb{Z})$ is called the first Chern class of $L$ and is usually denoted as $c_1(L)$.

For the complex bundle $E$ of complex rank $r$ the first Chern class is defined as $c_1(A^r E)$. If $E = TX$, the tangent bundle to an almost-complex manifold $X$, then one simply writes $c_1(X)$ or $c_1(X, J)$ if an almost-complex structure is needed to be specified.

If an almost-complex structure $J$ on the real bundle $E$ varies continuously, then the corresponding trivializations $\{\phi_{\alpha}\}$ above (on $N^2 E$) can be obviously chosen also to vary continuously. Thus $c_1(E, J)$ varies continuously. But $c_1(E, J) \in H^2(M, \mathbb{Z})$, i.e., takes values in a discrete group. So, it does not change at all. This simple but important observation together with Corollary 1.9.2 of the next section leads to the following

**Corollary 1.8.1.** Let $\omega$ be a nondegenerate exterior two-form on the even-dimensional real manifold $X$. Then $c_1(X, J)$ does not depend on the choice of $\omega$-calibrating (and even $\omega$-compatible) almost-complex structure $J$.

Let us state now the Genus (or Adjunction) Formula for immersed symplectic surfaces. Let $u: \bigsqcup_{j=1}^d S_j \to (X, \omega)$ be a reduced compact symplectic surface (see Definition 1.7.1) immersed into a symplectic four-dimensional manifold. Let $g_j$ denote the genus of $S_j$ and $M_j = u(S_j)$. Put $M := \bigsqcup_{j=1}^d M_j$ and denote by $[M]^2$ the homological self-intersection number of $M$. Define a geometrical self-intersection number $\delta$ of $M$ in the following way. Perturb $M$ to obtain a symplectic surface $M$ with only transversal double points. Then $\delta$ will be the sum of indices of the intersection over those double points. Those indices can be equal to $1$ or $-1$.

**Exercise 1.8.1.** Let $P$ be an oriented plane in $\mathbb{R}^4$. Call $P$ symplectic if $\omega_{\text{sym}}(P) > 0$. Find two symplectic planes $P_1, P_2 \in \mathbb{R}^4$ which intersect transversally at origin with intersection index $-1$.

Let $J$ be some almost complex structure which is tamed by $\omega$, i.e., $\omega(\xi, J\xi) > 0$ for any nonzero $\xi \in TX$. Denote by $c_1(X, J)$ the first Chern class of $X$ with respect to $J$. Since, in fact, $c_1(X, J)$ does not depend on continuous changes of $J$ and since the set of $\omega$-tamed almost complex structures is contractible, we usually omit the dependence of $c_1(X)$ on $J$.

**Theorem 1.8.2.** (Genus Formula for immersed symplectic surfaces) Let $M = \bigsqcup_{j=1}^d M_j$ be a compact immersed symplectic surface in
four-dimensional symplectic manifold $X$. Then

$$
\sum_{j=1}^{d} g_j = \frac{[M]^2 - c_1(X)[M]}{2} + d - \delta. \quad (1.8.3)
$$

**Proof.** By replacing every $M_j$ by its small perturbation, we can suppose that $M_j$ has only transversal double self-intersection points. Let $N_j$ be a normal bundle to $M_j$ and let $\tilde{M}_j$ denote the zero section of $N_j$. Also let exp$_j$ be the exponential map from a neighborhood $V_j$ of $\tilde{M}_j \subset N_j$ onto the neighborhood $W_j$ of $M_j$. Lift $\omega$ and $J$ onto $V_j$. Since $\tilde{M}_j$ is embedded to $V_j$, we can apply Lemma 1.7.1 to obtain the $\omega$-tame almost complex structure $J_j$ on $V_j$ such that $\tilde{M}_j$ is $J_j$-holomorphic.

For every $j$ we now have the following exact sequence of complex bundles:

$$
0 \longrightarrow TS_j \xrightarrow{du} E_j \xrightarrow{\text{pr}} N_j \longrightarrow 0 \quad (1.8.4)
$$

Here $E_j = (u^*TX)|_{S_j}$ is endowed with complex structure given by $J_j$. Since $du$ is nowhere degenerate complex linear morphism, $N_j := E_j/du(TM_j)$ is a complex line bundle over $S_j$. From (1.8.4) we get

$$
c_1(E_j) = c_1(TS_j) + c_1(N_j). \quad (1.8.5)
$$

Observe now that $c_1(E_j) = c_1(X)[M_j]$ and that

$$
c_1(TS) = \sum_{j=1}^{d} c_1(TS_j) = \sum_{j=1}^{d} (2 - 2g_j) = 2d - 2 \sum_{j=1}^{d} g_j.
$$

Furthermore, $c_1(N_j)$ is the algebraic number of zeros of a generic smooth section of $N_j$. To compare this number with the self-intersection of $M_j$ in $X$, note that if we move $M_j$ generically to obtain $M'_j$, then the intersection number $\text{int}(M_j, M'_j)$ is equal to the algebraic number of zeros of a generic section of $N_j$ plus two times the sum of intersection numbers of $M_j$ in self-intersection points, i.e., $[M_j]^2 = c_1(N_j) + 2\delta_j$. So

$$
c_1(X)[M_j] = 2 - 2g_j + [M_j]^2 - 2\delta_j. \quad (1.8.6)
$$

Now it only remains to take the sum over $j = 1, \ldots , d$ and to remark that the intersection points of $M_i$ with $M_j$ for $i \neq j$ make a double contribution to $[M]^2$. \hfill \square
Let $M = \bigcup_{j=1}^{d} M_j$ be now a compact $J$-complex curve in an almost complex surface $(\tilde{X}, J)$ with the distinct irreducible components $\{M_j\}$ parameterized by $\Sigma_j$. Denote by $g_j$ the genera of parameter curves $\Sigma_j$, $c_1(X, J)$ is the first Chern class of $(X, J)$. All self-intersection of $M$ are now positive but $M$ can have more complicated singularities - cusps, see sect. 1.4. Let $\kappa$ be the sum of Milnor numbers over cusp-points of $M$. Assume that $J$ is of class $C^1$.

As for the classical case of complex curves one has the following formula relating global and local invariants of $M$.

**Theorem 1.8.3. (Genus Formula for $J$-complex curves)**

$$
\sum_{j=1}^{d} g_j = \frac{[M]^2 - c_1(X, J)[M]}{2} + d - \delta - \kappa,
$$

where $\delta$ is the sum of local intersection indices of $M$.

Proofs can be found in [IS-1] and [MW].

1.9. Existence of calibrating and tamed structures

**Proposition 1.9.1.** Let $X$ be a real manifold and $\omega$ a nowhere degenerate exterior 2-form on $X$, i.e., $\omega^2 \neq 0$, where $2n = \dim_{\mathbb{R}} X$. On the open subset $U \subset X$ a $\omega$-calibrating almost-complex structure $J$ is given. Then for any relatively compact open $U_1 \subset U$ there exists a $\omega$-calibrating almost-complex structure $J_1$ on the whole $X$ such that $J_1|_{U_1} = J|_{U_1}$.

**Proof.** Consider a Riemannian metric $g(u, v) := \omega(u, Jv)$ on $U$. Find a Riemannian metric $g_1$ on $X$ such that $g_1|_{U_1} = g|_{U_1}$. Since $\omega$ is not degenerate, there exists a (unique) isomorphism $A_1 : TX \to TX$ such that

$$
\omega(u, v) = g_1(A_1 u, v) \tag{1.9.1}
$$

for all $u, v \in TX$. Further, $\omega$ is antisymmetric, so

$$
g_1(A_1 u, v) = \omega(u, v) = -\omega(v, u) = -g_1(A_1 v, u) = -g_1(u, A_1 v).
$$

Thus $A_1^T = -A_1$. So $A_1^T A_1 = A_1 A_1^T = -A_1^2$ is positively definite and self-adjoint with respect to $g_1$. Let $Q_1 := \sqrt{-A_1^2}$ be a positive square root of $-A_1^2$. Put

$$
J_1 = A_1 Q_1^{-1}. \tag{1.9.2}
$$

$J_1$ is an almost-complex structure. Indeed

$$
J_1^2 = A_1 Q_1^{-1} A_1 Q_1^{-1} = A_1^2 (Q_1^{-1})^2 = A_1^2 (-A_1^{-2}) = -\text{id}.
$$
Let us check that on $U_1$ one has $J_1 = J$. In fact, on $U_1$ we have $g = g_1$ and so

$$\omega(J_1 u, J_1 v) = \omega(u, v) = g(A_1 u, v) = \omega(A_1 u, J_1 v),$$

and thus $A_1 = J$ on $U_1$. So $Q_1 = \sqrt{-A_1^2} = \text{id}$. From here we have $J_1 = A_1 Q_1^{-1} = A_1 = J$ on $U_1$.

**Remark 1.9.1.** Note that at this point of the proof we construct a correspondence $P: g \to J$, which maps a Riemannian metric $g$ to an $\omega$-calibrated a.c. structure $J$ with $g(u, v) = \omega(u, J_1 v)$. This correspondence is obviously a continuous map from the space of metrics to the space of structures. Moreover, note that if $g$ appears as $g = g_J(u, v) = \omega(u, J_1 v)$ for some $\omega$-calibrated $J$, then $P(g_J) = J$.

Let us now check that $\omega$ is $J_1$-calibrated. First:

$$\omega(J_1 u, J_1 v) = g_1(A_1 J_1 u, J_1 v) = g_1(A_1^2 Q_1^{-1} u, J_1 v) = g_1(Q_1 u, -J_1 v) =$$

$$= -g_1(Q_1 u, J_1 v) = -g_1(u, Q_1 J_1 v) = -g_1(u, A_1 v) =$$

$$= -g_1(A_1^2 u, u) = -\omega(v, u) = \omega(u, v).$$

Second:

$$\omega(u, J_1 u) = g_1(A_1 u, J_1 u) = g_1(A_1 u, A_1 Q_1^{-1} u) = g_1(u, u) > 0$$

for nonzero $u$. \qed

This proof and remark, made inside, lead to the following

**Corollary 1.9.2.** Let $(X, \omega)$ be as above. Then:

1. There exists an $\omega$-calibrating almost-complex structure on $X$.

2. The space of $\omega$-calibrating almost-complex structures is contractible.

**Proof.** The first statement of this corollary is a particular case of the proposition when $U = \varnothing$.

To prove the second one, we fix some calibrating structure $J_0$. Denote by $g_{J_0}$ the corresponding Riemannian metric on $X$, i.e., $g_{J_0}(u, v) = \omega(u, J_0 v)$.

The space of Riemannian metrics is a convex cone $\mathcal{C}$ in $\text{Hom}_B(TX, T^*X)$. Therefore, there exists a contraction $\Psi: \mathcal{C} \times [0, 1] \to \mathcal{C}$ to $g_{J_0}$, i.e., $\Psi(\cdot, 0) = \text{id}$, $\Psi(\cdot, 1) = g_{J_0}$ and $\Psi(g_{J_0}, t) = g_{J_0}$.

Consider the following map $\Phi: J_\omega \to \mathcal{C}$ from the space of $\omega$-calibrating structures to the space of metrics:

$$\Phi(J)(u, v) = g_J(u, v) = \omega(u, J_1 v).$$
In the proof of Proposition 1.9.1 we showed that

\[ P \circ \Phi = \text{id} : \mathcal{J}_\omega^c \to \mathcal{J}_\omega^c. \]

Now \( P \circ \Psi(\cdot, t) \circ \Phi \) will be a contraction of \( \mathcal{J}_\omega^c \) to \( J_0 \). \( \square \)

**Proposition 1.9.3.** For any symplectic manifold \( (X, \omega) \), the set \( \mathcal{J}_\omega \) of \( \omega \)-tame almost complex structures on \( X \) is a non-empty contractible manifold.

The proposition follows immediately from the following result from linear algebra.

**Lemma 1.9.4.** Let \( V \) be a (finite-dimensional) real vector space and \( \omega \) a linear symplectic form on \( V \). Then the set \( \mathcal{J}_\omega \) of \( \omega \)-tame linear complex structures on \( V \) is a non-empty open contractible subset in the set \( \mathcal{J} \) of all linear complex structures on \( V \).

1.10. Envelopes of meromorphy and Continuity Principle

This section is devoted to the study of envelopes of meromorphy of neighborhoods of two-spheres in complex algebraic surfaces. The original question which motivated our studies was posed by A. Vitushkin.

- Let \( M \) be a “small” perturbation of the complex line in \( \mathbb{CP}^2 \). Does there exist a nonconstant holomorphic function in the neighborhood of such an \( M \)?

It was asked as a test question on the way to searching for the solution to the Jacobian conjecture, and the answer, as expected, was negative.

Let us briefly test the possible approaches to the answer to this question. It is more or less clear that one should try to extend holomorphic (or meromorphic via expected nonexistence of holomorphic) functions onto the whole \( \mathbb{CP}^2 \) and then conclude

- If \( M \) is a complex sphere itself, then nonconstant holomorphic functions do not exist in any neighborhood of \( M \) for the following reason. First note that in this case \( \mathbb{CP}^2 \setminus M = \mathbb{C}^2 \). Let \( B(R) \) denote the closed ball of radius \( R \) in \( \mathbb{C}^2 \). Then \( V_N := \mathbb{CP}^2 \setminus B(N) \), \( N = 1, 2, 3, \ldots \), is a fundamental system of strictly pseudoconcave neighborhoods of \( M \). Any function holomorphic in \( V_N \) holomorphically extends onto the ball \( B(N) \) by Hartogs' theorem, and thus becomes holomorphic on the whole \( \mathbb{CP}^2 \), i.e., constant.
A small $C^1$-perturbation of a complex curve is symplectic, therefore we shall study envelopes of meromorphy of neighborhoods of two-spheres symplectically immersed in complex surfaces. Here a complex surface means a (Hausdorff) connected complex two-dimensional manifold $X$ countable at infinity. The idea is to perturb the complex structure in the given neighborhood of $M$ in such a way that $M$ becomes complex; see Lemma 1.7.1, where we then construct a family $M_t$ of $J_t$-complex spheres and try to extend functions along this family. In more formal language to extend "along a family" means "onto the envelope of meromorphy".

Let $U$ be a domain in a complex manifold $X$ and let $Y$ be another complex manifold or space. The envelope of meromorphy $(U_Y, \pi)$ of $U$ relative to $Y$ is the maximal domain over $X$ satisfying the following conditions:

(i) there exists a holomorphic embedding $i: U \to \hat{U}$ with $\pi \circ i = \text{id}_U$;

(ii) each meromorphic map $f: U \to Y$ extends to a meromorphic map $\hat{f}: \hat{U} \to Y$, that is, $\hat{f} \circ i = f$.

The envelope of meromorphy exists for each domain $U$. This can be proved, for example, by applying the Cartan–Thullen method to the sheaf of meromorphic functions on $X$, see [IV-1]. In the sequel we shall restrict ourselves to Kähler complex surfaces, that is, we assume that $X$ carries a strictly positive closed $(1, 1)$ form $\omega$. The same we shall suppose that $Y$ is also Kähler.

The aim of the present section is to prove the following result.

**Theorem 1.10.1.** Let $u: S^2 \to X$ be a symplectic immersion of the two-sphere $S^2$ in a disk-convex Kähler surface $X$ such that $M := u(S)$ has only positive double points. Assume that $c_1(X)[M] > 0$. Then the envelope of meromorphy $(U_Y, \pi)$ of an arbitrary neighborhood $U$ of $M$ relative to a disk-convex Kähler space $Y$ contains a rational curve $C$ with $\pi^*c_1(X)[C] > 0$.

**Definition 1.10.1.** A Hermitian complex space $Y$ is disk-convex if for any sequence $(C_n, u_n)$ of smooth curves over $Y$ parameterized by the same surface $\Sigma$, such that

1) area$(u_n(C_n))$ are uniformly bounded; and

2) $u_n C^1$-converges in the neighborhood of $C_n$;

there is a compact $K \subset Y$ which contains all $u_n(C_n)$.

This definition obviously carries over to the case where $Y$ is a symplectic manifold. In this case one should consider $(C_n, u_n)$ as $J_n$-holomorphic
curves, with $J_n$ converging to some $J$ (everything in $C^1$-topology) and all structures being tamed by a given symplectic form.

**Theorem 1.10.2 (Continuity Principle-I).** Let $X$ be a disk-convex complex surface and $Y$ a disk-convex Kähler space. Then the envelope of meromorphy $(\mathcal{U}_Y, \pi)$ of $U$ relative to $Y$ is also disk-convex with respect to the pulled-back Kähler form.

This result can be reformulated in more familiar terms as follows.

Let $\{(C_t, u_t)\}_{t \in [0, 1]}$ be a continuous (in a Gromov topology) family of complex curves over $X$ with boundaries, parameterized by a unit interval. More precisely, for each $t \in [0, 1]$ a smooth Riemann surface with boundary $(C_t, \partial C_t)$ is given together with the holomorphic mapping $u_t : C_t \rightarrow X$, which is $C^1$-smooth up to the boundary. Note that $C_1$ is not supposed to be smooth, i.e., it can be a nodal curve.

Suppose that in the neighborhood $V$ of $u_0(C_0)$ a meromorphic map $f$ into the complex space $Y$ is given.

**Definition 1.10.2.** The map $f$ meromorphically extends along the family $(C_t, u_t)$ if for every $t \in [0, 1]$ a neighborhood $V_t$ of $u_t(C_t)$ is given, and given a meromorphic map $f_t : V_t \rightarrow Y$ such that

a) $V_0 = V$ and $f_0 = f$;

b) if $V_t \cap V_{t_2} \neq \emptyset$ then $f_{t_1}|_{V_t \cap V_{t_2}} = f_{t_2}|_{V_t \cap V_{t_2}}$.

**Theorem 1.10.3 (Continuity Principle-II).** Let $U$ be a domain in the complex surface $X$. Let $\{(C_t, u_t)\}_{t \in [0, 1]}$ be a continuous family of complex curves over $X$ with boundaries in $U_1$, a relatively compact subdomain in $U$. Suppose moreover that $u_0(C_0) \subset U$ and that $C_t$ for $t \in [0, 1]$ are smooth. Then every meromorphic mapping $f$ from $U$ to the disk-convex Kähler space $Y$ extends meromorphically along the family $(C_t, u_t)$.

The discussion above leads to the following

**Corollary 1.10.4.** If we have the domain $U$ in a complex surface $X$, a Kähler space $Y$ and a family $\{(C_t, u_t)\}$ satisfying the conditions of the “continuity principle”, then the family $\{(C_t, u_t)\}$ can be lifted onto $\mathcal{U}_Y$, i.e., there exists a continuous family $\{(C_t, u_t)\}$ of complex curves over $U$ such that $\pi \circ u_t = u_t$ for each $t$.

Of course, the point here is that the mapping can be extended to the neighborhood of $u_t(C_1)$, which is a reducible curve having in general compact components. This makes our situation considerably more general than the classical one, i.e., when $X$ is supposed to be Stein, compare to [Ch-St].
Proof of the Continuity Principle follows the main lines of [Iv-2] and is given in all details in [IS-3].

**Remark 1.10.1.** Let us explain the meaning of this theorem by an example. Let $X$ be disk-convex and $U \subset X$ some domain. Furthermore, let $f: U \rightarrow Y$ be a meromorphic map and $\{(C_n, u_n)\}$ a sequence of stable complex curves over $X$, converging in the Gromov topology to $(C_\infty, u_\infty)$. Suppose that the images of the boundaries $u_n(C_n)$ and $u_\infty(\partial C_\infty)$ are contained in $U$. Suppose also that $f$ extends along every curve $u_n(C_n)$. This means that there exists a complex surface $V_n$, containing $C_n$, and a locally biholomorphic map $u_n': V_n \rightarrow X$ such that $u_n'|_{C_n} = u_n$ and $f$ meromorphically extend from $u_n'^{-1}U$ onto the whole $V_n$. The latter is equivalent to the lift of the curves $(C_n, u_n)$ into the envelope $\hat{U}$, i.e., to the existence of holomorphic mappings $u_n: C_n \rightarrow \hat{U}$ such that $\pi \circ u_n = u_n$. In other words, one can take as $V_n$ a neighborhood of the lift of $C_n$ into $\hat{U}$. One easily sees that the curves $(C_n, u_n)$ are stable over $\hat{U}$, have uniformly bounded areas and converge in the neighborhood of the boundary $\partial C_n$. By Theorem 10.1.2 and by the Gromov compactness theorem, some subsequence $(C_n, u_n)$ converges to the $\hat{U}$-stable curve $(C_\infty, u_\infty)$, and $\pi \circ u_\infty = u_\infty$. This means that $f$ extends along the curve $u_\infty(C_\infty)$. Thus, Theorem 1.10.2 is a generalization of the E. Levi continuity principle.

1.11. Construction of envelopes

We give now a short sketch of the proof of Theorem 1.10.1.

**Step 1: Deformation of a structure.** Using $\omega$-positivity of $M$ we construct a smooth family $\{J_t\}_{t \in [0, 1]}$ of almost complex structures on $X$ satisfying the following properties:

a) $J_0$ is the given integrable structure on $X$.

b) \(\{x \in X \mid J_t(x) \neq J_0(x)\} \subset U_1 \subset U\) for each $t \in [0, 1]$, i.e., each $J_t$ is different from $J_0$ only inside some relatively compact open subset $U_1$ (the same for all $J_t$) of the given neighborhood $U$ of $M$.

c) $M$ is $J_1$-holomorphic.

d) All $\{J_t\}$ are "tamed" by our Kähler form $\omega$, i.e., $\omega(v, J_tv) > 0$ for every nonzero $v \in TX$.

We lift then the structures $J_t$ on the envelope $(U, \pi)$ of $U$ in the following way. Having natural imbedding $i: U \hookrightarrow \hat{U}$ we define liftings $\hat{J}_t$ setting $\hat{J}_t|_{i(U)} := i_{\ast} J_t$ and extend $\hat{J}_t$ outside $i(U)$ as given integrable structure $J_0$ on $\hat{U}$. Again, $\hat{J}_t$ differs from $J_0$ only in $i(U_1)$.
**Step 2: Deformation of the sphere** \( M_t \). We construct a “semi-continuous” family of (possibly reducible) surfaces \( M_t \subset \tilde{U} \) such that:

a) \( M_t \) is \( J_t \)-holomorphic.

b) \( M_0 = \tilde{M} \).

c) For all \( t \in [0, 1] \), \( M_t \) is a finite union of \( J_t \)-holomorphic spheres \( \{M_t^j\} \) with

\[
\sum (c_1(X)[M_t^j] - \delta_j - \kappa_j) \geq p,
\]

where \( \delta_j \) denotes the geometric self-intersection number of \( M_t^j \) and \( \kappa_j \) the sum of Milnor numbers of its cusp points.

Note that by “tameness” of all \( J_t \) and closeness of \( \omega \) the area\( (M_t) \sim \int_{M_t} \omega \) is uniformly bounded for all \( t \).

**Step 3: Kontinuitätssatz.** Note that \( \Sigma_t = M_t \setminus U_t \) is a complex curve in \( \tilde{U} \) with boundary on \( \partial U_t \), and \( \Sigma_t = \emptyset \). So by the “continuity principle”, \( \Sigma_t \) and thus \( M_t \) cannot go to infinity in \( \tilde{U} \) provided \( \pi(M_t) \) do not go to the infinity in \( X \). But \( M_0 \) is holomorphic and satisfies (c) from **Step 2**. Thus the proof is complete.

### 1.12. Examples

Here we discuss a few more examples concerning envelopes of meromorphy.

**Example 1.12.1.** Let \( (X, \omega) = (\mathbb{CP}^1 \times \mathbb{CP}^1, \omega_{FS} \oplus \omega_{FS}) \), where \( \omega_{FS} \) denote the Fubini-Study metric on \( \mathbb{CP}^1 \). Note that \( c_1(X) = 2[\omega] \). Let \( J \) be an \( \omega \)-tame almost complex structure on \( X \) and \( C \) be a \( J \)-complex curve on \( X \). Denote by \( e_1 \) and \( e_2 \) the standard generators of \( H_2(X, \mathbb{Z}) = \mathbb{Z}^2 \) and write \( [C] = a \cdot e_1 + b \cdot e_2 \). Then we get \( a + b = \int_C \omega \geq 1 \) and \( c_1(X)[C] = 2(a + b) \).

Furthermore, by the genus formula \( 0 \leq g(C) + \delta + \kappa = (2ab - 2(a + b))/2 + 1 = (a - 1)(b - 1) \). Thus, we conclude that both \( a \) and \( b \) are non-negative and \( [C]^2 = 2ab \geq 0 \). So \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) is non-negative in our sense.

Let \( M \) be an imbedded symplectic sphere in \( X \). Then \( (a - 1)(b - 1) = 0 \) by the genus formula. Therefore, we can assume that \( a = 1 \) and \( b \geq 0 \). Now one concludes that the following holds:

**Corollary 1.12.1.** Let \( M \) be a symplectic sphere in \( \mathbb{CP}^1 \times \mathbb{CP}^1, \omega_{FS} \oplus \omega_{FS} \). Then the envelope of meromorphy of any neighborhood of \( M \) contains the graph of a rational map of degree \( 0 \leq d \leq b \) from \( \mathbb{CP}^1 \) to \( \mathbb{CP}^1 \).

**Example 1.12.2.** \( (X, \omega) = (\mathbb{CP}^2, \omega_{FS}) \). Let \( M \) be a symplectic surface in \( X \) of degree \( m := \int_M \omega \) with positive self-intersections. Then obviously
$c_1(X)[M] = 3m$. Note that we proceed to the construction of a family \( \{ M_t \} \) if the condition $c_1(X)[M_t^2] > \pi(M_t^2)$ is satisfied for all irreducible components $M_t^2$ of $M_t$. By the genus formula one has $\pi(C) \leq (d-1)(d-2)/2$ for every complex curve $C$ of degree $d$. So we can proceed if $3m > (m-1)(m-2)/2$, which is equivalent to $1 \leq m \leq 8$.

Thus we have the following

**Corollary 1.12.2.** Let $M$ be a symplectic surface in $\mathbb{CP}^2$ of degree $m \leq 8$ with positive self-intersections. Then the envelope of meromorphy of any neighborhood of $M$ coincides with $\mathbb{CP}^2$ itself.

**Remark 1.12.1.** Note that the examples include all imbedded symplectic surfaces in $\mathbb{CP}^2$ of genus $\leq 21$.

**Example 1.12.3.** Let $X$ be a ball in $\mathbb{C}^2$ and $w$ is a standard Euclidean form. Blow up the origin in $\mathbb{C}^2$ and denote by $E$ the exceptional curve. By $X$ denote the blow-up ball $X$. The blow-up of $\mathbb{C}^2$ is also Kähler, and we denote by $w_0$ some Kähler form there. Consider a sufficiently small $C^1$-perturbation of $E$. This will be a $w_0$-symplectic sphere in $\tilde{X}$, which is denoted by $\tilde{M}$. The Chern class of the normal bundle to $\tilde{M}$ is equal to that of $E$ and thus is $-1$. Therefore, $c_1(X)[\tilde{M}] = 1$ and the proof of Theorem 1.10.1 applies. One should only note that in the process of deformation $\tilde{M}$ cannot break into irreducible or multiply covered components in this special case. The only rational curve in $\tilde{X}$ is $E$. Thus, we see that

- the envelope of meromorphy of any neighborhood of $M$ contains $E$.

One can then blow down the picture to obtain downstairs a sphere $M_1$-image of $M$ under the blown-down map. This $M_1$ is homologous to zero, so cannot be symplectic, and for this $M_1$ our Theorem 1.10.1 cannot be applied.

**Example 1.12.4.** Chirka in [Ch] proved the following “local version” of our Theorem 1.10.1, which he called “a generalized Hartogs’ lemma”. Denote by $\Gamma$ the graph of a continuous function $f : \Delta \to \mathbb{C}$. Consider the following “generalized Hartogs’ figure” in $\mathbb{C}^2$: $H_\Gamma := \partial \Delta \times \Delta \cup \Gamma$. Chirka showed that every holomorphic function in a neighborhood of $H_\Gamma$ extends holomorphically onto the unit bidisk $\Delta^2$. This is a corollary of our Theorem 1.10.1 (as is explained in [Ch]). Indeed, denote by $(z, w)$ coordinates in $\mathbb{C}^2$. Any function $f$ holomorphic in the neighborhood of $\partial \Delta \times \Delta$ is a sum of a function holomorphic in the bidisk and a function $f_-$ holomorphic in $(\mathbb{CP}^1 \setminus \Delta) \times \Delta$. This $f$ is also holomorphic in the neighborhood of $\Gamma$. We
need only to extend $f_\cdot$. Extending $\Gamma$ "inside" $(\mathbb{C}P^1 \setminus \Delta) \times \Delta$ we obtain $\tilde{\Gamma}$, a sphere homologous to the $\{pt\} \times \mathbb{C}P^1$ in $\mathbb{C}P^1 \times \mathbb{C}P^1$. Rescaling the variable $w$ we can make $\tilde{\Gamma}$ symplectic and find ourselves in the conditions of Example 1.12.2.

The proof of Chirka is roughly the same (i.e., uses the perturbation of the structure), but somewhat simpler. It uses, instead of Gromov’s techniques, the results of Vekua on so-called generalized analytic functions. Answering the question, posed by Chirka, Rosay in [Ro] constructed an example showing that a “generalized Hartogs’ lemma” does not hold in $\mathbb{C}^3$.

**Example 1.12.5.** This example was explained to us by E. Chirka, and it shows that our Continuity Principle is not valid when the complex dimension of the manifold $X$ is more than two.

Take as $X$ the total space of the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbb{C}P^1$. Denote an affine coordinate on $\mathbb{C}P^1$ by $z$, coordinates on the fibers by $\xi_1$, $\xi_2$ and $\eta_1$, $\eta_2$ such that $\eta_1 = z\xi_1$ and $\eta_2 = z\xi_2$. Identify $\mathbb{C}P^1$ with the zero section of the bundle. Consider the meromorphic function $f = e^{z\xi_1/\xi_2}$. The set of essential singularities of $f$ is $\{\xi_1 = 0\}$, which contains the zero section $\mathbb{C}P^1$.

Consider the following sequence of analytic disks $C_n$ in $U := X \setminus \{\xi_1 = 0\}$, $C_n := \{\xi_2 = 0, |z| \leq n, \xi_1 = \frac{z}{\eta_2}\}$. Then the function $f$ is defined in a neighborhood of every $C_n$. On the other hand, the limit curve of the sequence is $C_0 = \mathbb{C}P^1 \cup \Delta_\infty$, where $\Delta_\infty := \{\eta_2 = 0, z = \infty, |\eta_1| \leq 1\}$. In particular, $f$ does not extend in a neighborhood of $C_0$. 
Lecture 2.
Non-squeezing theorem and quantum cohomology.

2.1. Gromov’s Non-squeezing theorem

A symplectomorphism from a symplectic manifold \((X_1, \omega_1)\) to a symplectic manifold \((X_2, \omega_2)\) is a map \(\phi : X_1 \rightarrow X_2\) which is a diffeomorphism onto its image and which preserves symplectic forms, i.e., \(\phi^* \omega_2 = \omega_1\).

Consider the so called standard model \((\mathbb{R}^{2n}, \omega_{st})\) where \(\omega_{st} := dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n\). By \(\mathbb{B}(R)\) denote the Euclidean ball of radius \(R\) in \(\mathbb{R}^{2n}\) and by \(\mathbb{Z}(r)\) the following infinite cylinder

\[\mathbb{Z}(r) = \{(x, y) \in \mathbb{R}^{2n} \mid x_1^2 + y_1^2 < r\}.\]

The following theorem was proved by M. Gromov in [Gr]:

**Theorem 2.1.1 (Gromov).** If there exists a symplectomorphism from \((\mathbb{B}(R), \omega_{st})\) to \((\mathbb{Z}(r), \omega_{st})\) then \(R \leq r\).

First observation which can be made from this theorem is that symplectomorphisms have very different properties if compared with volume preserving maps (note that a symplectomorphism preserves also the volume form \(\omega_{st}^n\)). It is easy to find a linear volume preserving map (i.e., with determinant \(1\)) from \((\mathbb{B}(R), \omega_{st}^n)\) to \((\mathbb{Z}(r), \omega_{st}^n)\) for any \(R\) and any \(r\).

A crucial tool for the proof of the Gromov’s theorem are pseudoholomorphic curves. The matter of fact is that any symplectic form \(\omega\) on any manifold \(X\) admits a huge quantity of so called tamed almost complex structures. An almost complex structure on a manifold \(X\) is an endomorphism \(J : TX \rightarrow TX\) such that \(J^2 = -\text{Id}\). \(J\) is said to be tamed by \(\omega\) if \(\omega(v, Jv)\) is positive definite on \(TX\). The space of tamed a.-c. structures on \((X, \omega)\) is a nonempty, contractible Banach manifold and any tamed a.-c. structure defined on an open subset \(U \subset X\) can be extended, after shrinking \(U\), onto the whole \(X\) remaining to be tamed by \(\omega\).

If now \(\phi : \mathbb{B}(R) \rightarrow \mathbb{Z}(r)\) is a symplectomorphism then one can push forward the standard complex structure \(J_{st} = \text{Id}\) onto the image \(U = \phi(\mathbb{B}(R))\) to get a \(\omega_{st}\)-tamed a.c. structure \(J\). Then after possibly shrinking extend this structure onto the whole cylinder, or, to be more precise to an appropriate compactification, which is \(\mathbb{S}^2 \times \mathbb{T}^{2n-2}\). Symplectic form \(\omega_{st}\) in the image should be also extended to \(\omega = \omega_1 + \omega_{st}\) in a standard way on the torus and in a such way on the sphere that \(\int_{\mathbb{S}^2} \omega_1 = \pi r^2 + \epsilon\). Now our mapping \(\phi\) will be a symplectomorphism and a biholomorphism at the same time.
The crucial fact is now that any a.-c. structure tamed by our extended symplectic form admits as many compact complex curves as the standard one. Therefore one can pick a $J$-complex sphere $C$ homologous to $S^2 \times \{1\}$ passing through the image of the origin $\phi(0)$. The area of $C$ is $\int_{S^2 \times \{1\}} \omega$, i.e., at most $\pi r^2 + \epsilon$. Its preimage $\phi^{-1}(C)$ is a complex curve in $\mathbb{B}(R)$ passing through the origin and therefore is at least $\pi R^2$. This gives the proof.

2.2. Symplectic capacities

To explain applications of the non-squeezing theorem let us introduce first the notion of symplectic capacity due to Ekeland and Hofer, [E-H].

**Definition 2.2.1.** A *symplectic capacity* is a function $c$ that assigns a non-negative number or plus infinity to each symplectic manifold and which satisfies the following axioms:

1. If there is a symplectomorphism $\phi: (X_1, \omega_1) \rightarrow (X_2, \omega_2)$ then $c(X_1, \omega_1) \leq c(X_2, \omega_2)$ - monotonicity;
2. $c(X, \lambda \omega) = \lambda^2 c(X, \omega)$ - conformal invariance;
3. $0 < c(\mathbb{B}(1), \omega_{st}) = c(\mathbb{B}(1), \omega_{st}) < \infty$ - nontriviality.

The last axiom ensures that symplectic capacity is a two-dimensional invariant and existence of symplectic capacities is essentially equivalent to the non-squeezing theorem. There are several known capacities, but we shall consider only one, the so called Gromov width which is defined as

$$W_G(X, \omega) := \sup \{ \pi r^2 : (\mathbb{B}(r), \omega_{st}) \text{ embeds symplectically to } (X, \omega_{st}) \}.$$ 

Again by the non-squeezing theorem $W_G(\mathbb{B}(r), \omega_{st}) = \pi r^2$.

The proof of the following not difficult statement can be found in [McD]:

**Theorem 2.2.1 (Ekeland-Hofer).** A linear map $L: \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ preserves some symplectic capacity iff it is symplectic or antisymplectic, i.e., $L^* \omega_{st} = \pm \omega_{st}$.

One more easy observation is that any capacity is continuous in Hausdorff topology on the convex subsets of $(\mathbb{R}^{2n}, \omega_{st})$.

From here one obtains the following basic fact on symplectic capacities:

**Theorem 2.2.1 (Ekeland-Hofer).** A local orientation-preserving diffeomorphism $\phi$ of $(\mathbb{R}^{2n}, \omega_{st})$ is symplectic iff there is a symplectic capacity $c$ such that $c(\phi(U)) = c(U)$ for all open $U \subseteq \mathbb{R}^{2n}$. 
Indeed, let us prove that $d\phi(p)$ is symplectic at every point $p$ from its domain of definition. We can suppose that both $p$ and $\phi(p)$ are the origin. Then $d\phi(0)$ is the uniform limit of the maps

$$\phi_t(v) = \frac{\phi(tv)}{t},$$

which are all symplectic. Therefore $d\phi(0)$ is symplectic or antisymplectic. If $n$ is odd we are done since symplecticity of $d\phi$ follows from the fact that it preserves the volume. If $n$ is even then repeat the same argument for the map $Id_{\mathbb{R}^n} \times \phi$ instead of $\phi$.

The same reasoning, i.e., passing through $\phi_t$ gives the following corollary, first proved by Y. Eliashberg in [E]:

**Corollary 2.2.2.** The group $\text{Symp}(X, \omega)$ of all symplectomorphic diffeomorphisms is $C^0$-closed in the group $\text{Diff}(X)$ of all diffeomorphisms of $X$.

In another words if a sequence $\phi_n$ of symplectomorphic diffeomorphisms converge uniformly to a diffeomorphism $\phi$ then $\phi$ is itself a symplectomorphism.

### 2.3. Riemann-Roch formula and index of $\bar{\partial}$-operators

Let $(X, \omega)$ be a symplectic manifold and $J$ an almost complex structure tamed by $\omega$, i.e., $\omega(\cdot, J\cdot)$ is a positive definite quadratic form on the tangent bundle $TX$ of $X$. To the structure $J$ we associate the first Chern class $c_1(X, J) \in H^2(X, \mathbb{Z})$. Being a discrete invariant $c_1(X, J)$ does not change under continuous perturbations of $J$. Denote by $\mathcal{J}_\omega$ the Banach manifold of almost complex structures tamed by $\omega$. Since $\mathcal{J}_\omega$ is a contractible topological space, $c_1(X, J)$ does not depend on the choice of $J \in \mathcal{J}_\omega$ and therefore is called the *first Chern class of the symplectic structure*. We shall denote it by $c_1(X, \omega)$ or just as $c_1$. For a homology class $[A] \in H_2(X, \mathbb{Z})$ represented by a surface $A$ we write $c_1(A) = \int_A c_1(X, \omega)$.

For the holomorphic bundle $E$ of complex rank $r$ over a compact Riemann surface $S$ denote by $\mathcal{O}_E$ the sheaf of its holomorphic sections. In a usual way one defines the cohomology groups $H^0(S, \mathcal{O}_E)$ and $H^1(S, \mathcal{O}_E)$. Denote $h^r = \dim_{\mathbb{C}} H^r$. By $g$ we denote the genus of $S$ and

$$c_1 = \int_S c_1(E).$$

These numbers are related by the classical

**Theorem** For a holomorphic bundle $E$ over a Riemann surface $S$ one has

$$h^0 - h^1 = c_1 + r \cdot (1 - g). \quad (2.3.1)$$
This formula can be interpreted as the formula for the index of some operators acting on the spaces of smooth sections of $E$. On the sheaf of smooth sections of a holomorphic bundle over a Riemann surface, or more generally over a complex manifold, one can define $\bar{\partial}$-operators. Those are \(\mathbb{C}\)-linear operators $\bar{\partial} : \Gamma^p(S,E) \rightarrow \Gamma^p_{0,1}(S,E)$ satisfying

$$\bar{\partial}(f \cdot \sigma) = \bar{\partial}_S f \otimes \sigma + f \cdot \bar{\partial}\sigma.$$  \hspace{1cm} (2.3.2)

Here by $\Gamma^p(S,E)$ we denote the Sobolev space of \((1,p)\)-smooth sections of $E$, and by $\Gamma^p_{0,1}(S,E)$ the space of \((0,1)\) $L^p$-integrable forms with coefficients in $E$. $\bar{\partial}_S$ is a canonical $\bar{\partial}$-operator on $S$.

If one additionally fixes some Hermitian metric on $E$, then such an operator is determined uniquely if one imposes the additional condition to preserve the scalar product, see [G-H] Ch. 0 for details.

The operator $\bar{\partial}$ being elliptic is Fredholm and its index is defined as $\text{ind} \bar{\partial} := \text{dim ker} \bar{\partial} - \text{dim Coker} \bar{\partial}$. Remark that $\text{ind} \bar{\partial} = h^0 - h^1$ and so by the Riemann-Roch formula

$$\text{ind} \bar{\partial} = c_1 + r \cdot (1 - g).$$  \hspace{1cm} (2.3.3)

**Definition 2.3.1.** An $\mathbb{R}$-linear operator $D : \bar{\partial} : \Gamma^p(S,E) \rightarrow \Gamma^p_{0,1}(S,E)$ which can be represented as $\bar{\partial} + R$ with $R \in C^0(S, \text{hom}_\mathbb{R}(E, \Lambda^{0,1} \otimes E))$ shall be called a $\bar{\partial}$-type operator.

This is again an elliptic (Fredholm) operator, which is homotopic to $\bar{\partial}$. So by the homotopy invariance of the index we have that for any $\bar{\partial}$-type operator $D$

$$\text{ind}_\mathbb{R} D = 2 \cdot (c_1 + r \cdot (1 - g)).$$  \hspace{1cm} (2.3.4)

The reader should take into account that since $D$ is real, the real dimensions in the last formula are considered. That is why the number 2 appears.

### 2.4. Moduli space of pseudoholomorphic spheres

For $(X,\omega)$ and $[A] \in H_2(X,\mathbb{Z})$ as above by $\mathcal{P}_A$ we denote the Banach manifold of all parameterized $J$-complex spheres representing the homology class $[A]$, $J \in \mathcal{J}_\omega$. More precisely $\mathcal{P}_A$ consists of all $J$-holomorphic mappings $u : \mathbb{P}^1 \rightarrow X, J \in \mathcal{J}_\omega$ and $u(\mathbb{C}P^1) \in [A]$.

$\mathcal{P}_A$ admits an obvious action of $G = PGL(2,C) : \phi \cdot u = u \circ \phi^{-1}$.

The quotient manifold $\mathcal{P}_A/G$ is denoted by $\mathcal{M}_A$ and is called the space of nonparameterized pseudoholomorphic spheres in $(X,\omega)$ representing the
homology class $[A]$. The natural projection $\pi: \mathcal{M}_A \to J_\omega$ is a Fredholm map of index

$$\text{ind}_\pi \pi = 2[c_1(A) + n - 3],$$

where $2n = \text{dim}_\mathbb{R} X$. The set $\pi^{-1}(J)$ (not always a manifold) of $J$-complex spheres for some fixed $J \in J_\omega$ representing homology class $A$ is denoted by $\mathcal{M}_{A,J}$. If index is nonnegative then for a general $J \in J_\omega$ the set $\mathcal{M}_{A,J}$ is proved to be a manifold of dimension equal to $\text{ind}_\pi \pi$. In fact in this case the differential of $\pi$ is proved to be surjective. “General”, or “generic” here means the set of second category.

The space $\mathcal{M}_{A,J}$ is not compact in general, but Gromov’s compactness theorem describes very precisely its compactification $\overline{\mathcal{M}_{A,J}}$. Namely, the elements of $\overline{\mathcal{P}_{A,J}} \setminus \mathcal{P}_{A,J}$ are so called bubbled spheres, i.e., $J$-complex curves which are finite unions of $J$-complex spheres. If the homology class $[A]$ is minimal in the sense that $\omega(A)$ is the smallest positive value of $\omega$ on spherical classes, then $\mathcal{M}_{A,J}$ is compact.

The manifold $\mathcal{M}_{A,J}$ carries a natural orientation, see [McD-S], p. 32. It comes from the fact that the tangent space $T_0 \mathcal{M}_{A,J}$ is the kernel of the so called Gromov $\bar{\partial}$-operator. The last one lies in the same connected component of Fredholm operators as the usual Cauchy-Riemann $\partial$.

2.5. Universal family and evaluation map

Define the universal family of pseudoholomorphic spheres representing the homology class $[A]$ in our symplectic manifold $(X, \omega)$ as the quotient space $\mathcal{W}_A := \mathcal{P}_A \times_G \mathbb{CP}^1$ under the natural action of $G = PGL(2, \mathbb{C})$ given by

$$\phi \cdot (u, z) = (u \circ \phi^{-1}, \phi(z)).$$  \hfill (2.5.1.)

The obvious projection of $\mathcal{W}_A$ onto $J_\omega$ will be again denoted by $\pi: \mathcal{W}_A \to J_\omega$. For general $J \in J_\omega$ the set $\mathcal{W}_{A,J} := \pi^{-1}(J)$ is a smooth manifold of real dimension $2[c_1(A) + n - 2]$ as it follows from (1). For general $J_0, J_1 \in J_\omega$ and a general path $\gamma = \{J_t \ | \ t \in [0, 1]\}$ from $J_0$ to $J_1$ the set $\mathcal{W}_{A,\gamma} := \pi^{-1}(\gamma)$ is a manifold which realizes a cobordism between $\mathcal{W}_{A,J_0}$ and $\mathcal{W}_{A,J_1}$.

**Definition 2.5.1.** The 1-fold evaluation map (or simply the evaluation map) $\text{ev}_J: \mathcal{P}_{A,J} \times \mathbb{CP}^1 \to X$ is defined as $\text{ev}_J(u, z) = u(z)$.

This map factors out through the action of $G = PGL(2, \mathbb{C})$. Therefore we obtain a map $\text{ev}_J: \mathcal{W}_{A,J} \to X$ from the universal family to $X$. The main question is how big is $\text{ev}_J(\mathcal{W}_{A,J})$ as a subset of $X$?

**Example 2.5.1.** Consider the following starting example. Let $V$ be some symplectic manifold with $\pi_2(V) = 0$ ($V = \mathbb{T}^2$ is the torus with the
standard symplectic form is a good example to have in mind). Consider $X = \mathbb{C}P^1 \times V$ with a product symplectic form and take $A = [\mathbb{C}P^1 \times pt]$. Then $[A]$ is a minimal spherical class, $c_1(A) = 2$ and therefore for general $J$ the set $W_{A,J}$ is a compact manifold of dimension $2n$. Since, moreover, the manifolds $W_{A,J}$ for different $J$-s are cobordant, the degree of $\text{ev}_1$ is independent of $J$ and is therefore equal to 1 (easy remark for $J = J_{\mathbb{CP}^1} + J_V$). As a conclusion we get that for any $J \in \mathcal{J}_o$ for every point $p \in X$ there exists a $J$-complex curve through $p$ homologous to $\mathbb{C}P^1 \times [pt]$. This is a crucial point in the Gromov’s proof of his non-squeezing theorem, see [Gr] or [iv-2].

**Definition 2.5.2.** The $p$-fold evaluation map

$$\text{ev}_{1,p} : \mathcal{P}_{A,J} \times \mathbb{C}P^1 \times \ldots \times \mathbb{C}P^1 \to X$$

is defined as $\text{ev}_{1,p}(u; z_1, \ldots, z_p) = (u(z_1), \ldots, u(z_p))$.

This map is equivariant under the natural action of $G$: $\phi \cdot (u; z_1, \ldots, z_p) = (u \circ \phi^{-1}; \phi(z_1), \ldots, \phi(z_p))$ and therefore factors to the map of the universal family $W_{A,J,p} := \mathcal{P}_{A,J} \times_G (\mathbb{C}P^1)^p$ to $X^p := \underbrace{X \times \ldots \times X}_{p}$. As before, from (2.5.1) it follows that for a general $J \in \mathcal{J}_o$, $W_{A,J,p}$ is a manifold of dimension $m := \dim W_{A,J,p} = 2[c_1(A) + n - 3 + p]$

(2.5.3) $\text{ev}_{1,p}(W_{A,J,p})$ compactifies to a pseudo-cycle, see [McD-S], in $X^p$. Its “boundary”, i.e., $\text{ev}_{1,p}(W_{A,J,p}) \setminus \text{ev}_{1,p}(W_{A,J,p})$ has dimension at most $m - 2$ and therefore the homology class $[\text{ev}_{1,p}(W_{A,J,p})] \in H_m(X^p, \mathbb{Z})$ is well defined and is independent of $J \in \mathcal{J}_o$. This class will be denoted by $[E_p]$. 

**2.6. Gromov-Witten invariant**

Take the number $d = 2np - m = \dim X^p - \dim E_p$ and take some cycles (manifolds) $\alpha_1, \ldots, \alpha_p \subset X$ such that the dimension of $\alpha := \alpha_1 \times \ldots \times \alpha_p$ is $d$. We perturb cycle $\alpha$ in $X^p$ to a general position to have only transversal intersections with $E_p$.

**Definition 2.6.1.** The value of the Gromov-Witten invariant on cycles $\alpha_1, \ldots, \alpha_p$ is the algebraic sum of intersection points of $E_p$ with $\alpha$, i.e.,

$$GW_A(\alpha_1, \ldots, \alpha_p) = [E_p] \cdot [\alpha].$$

The geometrical meaning of $GW_A(\alpha_1, \ldots, \alpha_p)$ is that it is the number of $J$-complex spheres in homology class $[A]$ intersecting each of $\alpha_1, \ldots, \alpha_p$. 
Example 2.6.1 can be interpreted now as saying that for $X = \mathbb{CP}^1 \times V$ with $\sigma_2(V) = 0$ and product symplectic structure $GW_{[\mathbb{CP}^1 \times pt]}(pt) = 1$.

**Example 2.6.1** Let now $[A] = [\mathbb{CP}^1]$ be the class of a line in $\mathbb{CP}^2$. Let us see that $GW_A(pt, pt) = 1$. We have that $c_1(A) = n + 1$, $p = 2$ and therefore $m = 4n$. So $d = 0$. General theorems, like Theorem 5.3.1 from [McD-S] imply that the standard structure $J_{st}$ of $\mathbb{CP}^2$ is general and $GW_{[\mathbb{CP}^1]}(pt, pt)$ can be computed with $J = J_{st}$, which gives the result.

Y. Ruan in [Ru] introduced the invariant $GW$ that was constructed here based on the Gromov’s theory and used it in [Ru-2] to produce two diffeomorphic but deformation nonequivalent compact Kähler 3-folds. In [Wi] E. Witten constructed slightly different invariant, which counts not only curves intersecting given cycles but also configurations of points on $\mathbb{CP}^1$ parameterizing these intersections.

2.7. Quantum multiplication and quantum cohomology

Let $(X, \omega)$ be a symplectic manifold. Denote by $H_*(X)$ and $H^*(X)$ its integral homology and cohomology rings modulo torsion. Quantum deformation of the usual cup-product on $H^*(X)$ is defined with the help of Gromov-Witten invariants.

For our symplectic manifold $(X, \omega)$ we denote by $c_1(X, \omega) = c_1$ its first Chern class. For a homology class $[A] \in H_2(X, \mathbb{Z})$ represented by a surface $A$ we write $c_1(A) = \int_A c_1(X, \omega)$. By $J_\omega$ denote the Banach manifold of almost complex structures tamed by $\omega$.

Let $\alpha$, $\beta$ and $\gamma$ be cycles (ex. compact, oriented, immersed submanifolds) in $X$ of dimensions $2n-k$, $2n-l$ and $r$ respectively, here $2n = \dim X$. In the sequel $\alpha$ and $\beta$ will be Poincaré dual to the cohomology classes of degree $k$ and $l$ correspondingly. For a general almost complex structure $J \in J_\omega$ the manifold of $J$-complex spheres in $X$ representing the homology class $[A]$ has dimension $m = 2[c_1(A) + n - 3 + 3] = 2[c_1(A) + n]$, see formula 2.5.3 in [IV]. The family of $J$-complex spheres intersecting each of $\alpha, \beta, \gamma$ has therefore dimension $(2n-k+2n-l+r)+2[c_1(A) + n] - 6n$. This set is therefore dimension zero, and in fact finite for many manifolds $X$, provided

$$r = \dim \gamma = k + l - 2c_1(A).$$ (2.7.1)

The number of points in his manifold, i.e., the number of $J$-complex spheres representing the homology class $[A]$ and intersecting each of $\alpha, \beta, \gamma$ is denoted by $GW_A(\alpha, \beta, \gamma)$ and is called the value of the Gromov-Witten invariant on the triple $(\alpha, \beta, \gamma)$. 
Let $a \in H^k(X)$ and $b \in H^l(X)$ be some integral cohomology classes and denote by $\alpha \in H_{2n-k}(X)$ and $\beta \in H_{2n-l}(X)$ their Poincaré duals. Fix a spherical homology class $[A] \in H_2(X)$ and define the class $(a \ast b)_A \in H^{k+l-2c_1(A)}(X)$ as a functional on $H_{k+l-2c_1(A)}(X)$ by setting
\[
\int_{\gamma} (a \ast b)_A = <(a \ast b)_A, \gamma> := GW_A(\alpha, \beta, \gamma) \tag{2.7.2}
\]
for classes $\gamma \in H_{k+l-2c_1(A)}(X)$.

**Example 2.7.1 (Cup-product).** Consider the case of the zero spherical class $[A] = [0]$, $r$ in this case is equal to $k+l$, i.e., our quantum product of classes $a, b$ of degrees $k$ and $l$ will be a class of degree $k+l$. Let us see that it is in fact the cup-product $a \cup b$. J-complex curves homologous to zero are constants and therefore those of them which intersect all $a, b, \gamma$ are just points of their triple intersections. In another words their algebraic sum is nothing else but intersection of $\gamma$ with the Poincaré dual to $a \wedge b$, which is the intersection $a \cup \beta$.

### 2.8. Quantum cohomology ring

The multiplication defined by (2.8.2) is only a component of the full quantum multiplication. To give the final definition we need to extend the cohomology ring $H^*(X)$ to a quantum cohomology ring
\[
QH^*(X) := H^*(X) \otimes \mathbb{Z}[q, q^{-1}] / \langle q^{r+1} = 1 \rangle, \tag{2.8.1}
\]
where $q$ is a variable of degree $2N$. Here $N$ is a minimal Chern number of $(X, \omega)$, which is defined by $c_1(\pi_2(X)) = N\mathbb{Z}$. In the sequel we suppose that $N \geq 2$. The convention that $q$ has degree $2N$ means that $\deg(a \otimes q) = \deg a + 2N$ if $a \in H^k(X)$ for some $k$. By $QH^k(X)$ we denote the submodule of our quantum ring $QH^*(X)$ of elements of degree $k$. Now we can give

**Definition 2.8.1.** The quantum product of the cohomology classes $a \in H^k(X)$ and $b \in H^l(X)$ is defined as
\[
a \ast b := \sum_A (a \ast b)_A q^{c_1(A)/N}, \tag{2.8.2}
\]
where the sum is taken over all spherical classes $[A] \in H_2(X)$ satisfying $0 \leq 2c_1(A) \leq k + l$. It extends onto the whole quantum ring $QH^*(X)$ by linearity.

This product is distributive, anticommutative and associative, the last is a difficult theorem. As it was explained in **Example 2.5.1** the constant term of 2.8.2 is the usual cup-product of cohomology classes $a$ and $b$. 
Example 2.8.1 (Quantum cohomology ring of $\mathbb{CP}^n$.) Consider the case of $(X, \omega) = (\mathbb{CP}^n, \omega_{FS})$, i.e., of the complex projective space with the Fubini-Studi form. In this case $N = n + 1$. Let $p$ be the positive generator of $H^2(\mathbb{CP}^n)$, for example one can take $p = [\omega_{FS}]$. Then $H^*(\mathbb{CP}^n) = \mathbb{Z}[p]/ < p^{n+1} = 0 >$.

Let now $[A]$ be the homology class of the line $\mathbb{CP}^1$. Then $k = \deg p = 2, l = \deg p^n = 2n$ and $c_1(A) = n + 1$. So $k + l - 2c_1(A) = 0$ and according to (2)

$$\langle (p * p^n)_A, \{pt\} \rangle = \int_{pt} (p * p^n)_A = GW_A(\mathbb{CP}^{n-1}, pt, pt) = 1 \quad (2.8.3)$$

because there exist only one complex sphere in the homology class of the line passing through given two points (and then it intersects a hyperplane automatically). For all other classes $[A]$ products $(p * p^n)_A$ vanish by the reason of dimension and therefore $p * p^n = q$ in the quantum ring $QH^*(\mathbb{CP}^n)$. So we see that $QH^*(\mathbb{CP}^n) = \mathbb{Z}[p, q]/< p^{n+1} = q, qq^{-1} = 1 >$.

In [PSS] it is proved that the quantum cohomology ring $QH^*(X)$ is isomorphic to the symplectic Floer homology ring $HF^*_\omega(X, \xi)$. 

References


Infinite-Dimensional Holomorphy, Hardy Classes of Infinitely Many Variables, Complex Symmetric Fock Spaces

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Abstract. We research the Hardy class $\mathcal{H}_D^2(\mathbb{E})$ of entire analytic functions on an infinite-dimensional complex Hilbert space $\mathbb{E}$, which is unitary equivalent to the symmetric Fock space $\mathcal{F}_q(\mathbb{E})$ associated with $\mathbb{E}$. The group of shift and the connected with it operator derivation on such Hardy class are studied. Using derivation in $\mathcal{H}_D^2(\mathbb{E})$ we introduce the subspace of exponential type functions and establish for it an analog of Berstein and Jackson inequalities on best approximation.

Introduction

Complex symmetric Fock spaces being a special case of Hilbert spaces play a role in physical as the state spaces of particle systems (see [2], [9, X.7]). We study the class of entire analytic functions naturally associated with these spaces. Such functions possess the infinitely many of complex variables. The general theory of analytical functions on Banach and locally convex spaces is advanced well enough [4, 8]. Specificity of the examined case is that the entire functions associated with symmetric Fock spaces form the Hardy classes on an infinite-dimensional Hilbert space (Theorem 2). The purpose of given lection is representation of properties connected with the group of shifts and the derivation. We established that the group of shifts on such Hardy classes is well-defined (Theorem 3) and the direction derivation is a close and dense defined linear operator generated in some natural sense by this group (Theorem 4). Explicit expression of direction derivation by an operator of multiplication for functions from Hardy classes are also calculated. Using derivation we introduce in the Hardy classes a subspace of exponential type functions. For such entire functions of exponential type we prove analogs of Berstein and Jackson inequalities on best approximation (Theorem 6).
The shift on a space of entire functions over Banach spaces has been investigated by Aron, Cole and Gamelin in [1]. The theory of Hardy classes in infinite polydisks has been advanced by Cole and Gamelin in [3]. Another approach to construction of Hardy type spaces for infinite-dimensional bounded domains using Hilbert symmetric tensor products and the symmetric Fock space has been proposed in [6, 7].

1. Preliminaries

Let \( E \) be a separable Hilbert space over \( C \) with the orthonormal basis \( \{ \xi_j \mid j \in \mathbb{N} \} \), the scalar product \( E \times E \ni (x, y) \rightarrow \langle x, y \rangle \) and the norm \( \|x\| = \sqrt{\langle x, x \rangle} \). Denote by \( S = \{ x \in E \mid \|x\| = 1 \} \) and \( B = \{ x \in E \mid \|x\| < 1 \} \) the unit sphere and open ball in \( E \) respectively. Let \( E^{\otimes k} := E \otimes \ldots \otimes E \) be the algebraic tensor product of \( k \in \mathbb{N} \) copies of \( E \) with the scalar product

\[
\langle x_1 \otimes \ldots \otimes x_k, y_1 \otimes \ldots \otimes y_k \rangle := \langle x_1, y_1 \rangle \ldots \langle x_k, y_k \rangle,
\]

\( x_1, y_1, \ldots, x_k, y_k \in E \).

It is easy to verify [2], that \( \{ \xi_{j_1} \otimes \ldots \otimes \xi_{j_k} \mid (j) := (j_1, \ldots, j_k) \in \mathbb{N}^k \} \) forms an orthonormal basis in \( E \otimes \ldots \otimes E \). Denote by \( E^{\otimes k} \) the completion of \( E \otimes \ldots \otimes E \) by the norm

\[
\|f\| = \left( \sum_{(j) \in \mathbb{N}^k} |\zeta_{(j)}|^2 \right)^{1/2}, \quad f = \sum_{(j) \in \mathbb{N}^k} \zeta_{(j)} \xi_{j_1} \otimes \ldots \otimes \xi_{j_k} \in E^{\otimes k}, \quad \zeta_{(j)} \in C.
\]

Evidently \( \|x_1 \otimes \ldots \otimes x_k\| = \|x_1\| \cdots \|x_k\| \) and as a consequence the canonical mapping

\( \otimes^k : E \times \ldots \times E \ni (x_1, \ldots, x_k) \mapsto x_1 \otimes \ldots \otimes x_k \in E^{\otimes k} \)

is continuous. More generally, for arbitrary \( k, n \in \mathbb{N}, (n \leq k) \) the completion of the tensor product \( E^{\otimes n} \otimes E^{\otimes (k-n)} \) we also denote by \( E^{\otimes k} \) and the equality \( \|f_n \otimes g_{k-n}\| = \|f_n\| \|g_{k-n}\| \) for all \( f_n \in E^{\otimes n}, g_{k-n} \in E^{\otimes (k-n)} \) is held.

Let \( S(k) \ni g : \{1, \ldots, k\} \mapsto \{g(1), \ldots, g(k)\} \) be the permutation group of \( n \) ordered elements. The continuous linear extension \( s_k : E^{\otimes k} \ni g \mapsto s_k(g) \in E^{\otimes k} \) of the mapping

\[
s_k : x_1 \otimes \cdots \otimes x_k \mapsto x_1 \cdot \ldots \cdot x_k := \frac{1}{k!} \sum_{s \in \mathfrak{S}(k)} x_{s(1)} \otimes \ldots \otimes x_{s(k)}
\]
is the orthogonal projector from $\mathbb{H}^\otimes k$ onto the closed range

$$\operatorname{Ran} g_k := \mathbb{H}^k, \quad \mathbb{H}^\otimes k = \mathbb{H}^k \bigoplus \operatorname{Ker} g_k,$$

which is called a hilbertian symmetric tensor degree of $\mathbb{H}$. In subsequently we use the denotation

$$x^k := x \cdot \ldots \cdot x = x \otimes \ldots \otimes x \in \mathbb{H}^k, \quad x \in \mathbb{H}.$$

We introduce the orthogonal projector $s_n \otimes s_{k-n} : \mathbb{H}^\otimes k \to \mathbb{H}^n \otimes \mathbb{H}^{k-n}$ and consider the restricted operator

$$g_{k/n} := g_k |_{\operatorname{Ran} \ s_n \otimes s_{k-n}} : \mathbb{H}^n \otimes \mathbb{H}^{k-n} \to \mathbb{H}^k.$$

The element

$$f_n \cdot g_{k-n} := g_k (f_n \otimes g_{k-n}) \in \mathbb{H}^k, \quad f_n \in \mathbb{H}^n, \quad g_{k-n} \in \mathbb{H}^{k-n}, \quad n \leq k,$$

is called the product of $f_n$ and $g_{k-n}$. It is easy for seeing that the projector $g_k$ supposes the natural decomposition $g_k = g_{k/n} \circ (s_n \otimes g_{k-n})$.

Let $m \in \mathbb{N}$ and $[j]$ denote the multi-index $(j_1, \ldots, j_m) \in \mathbb{N}^m$ such that $j_1 < \ldots < j_m$ and $(k)$ denote the arbitrary multi-index $(k_1, \ldots, k_m) \in \mathbb{N}^m$. We write $|(k)| := k_1 + \ldots + k_m$, $(k)! := k_1! \cdot \ldots \cdot k_m!$ and put

$$e_{[j]}^{(k)} := e_{j_1}^{k_1} \cdot \ldots \cdot e_{j_m}^{k_m} = g_k (e_{j_1}^{k_1} \otimes \ldots \otimes e_{j_m}^{k_m}).$$

Proposition 1.

(i) An orthonormal basis in the hilbertian symmetric tensor degree $\mathbb{H}^k$ is formed by the set of elements

$$e_{[j]}^{(k)} \sqrt{k!/(k)!} : \quad [j], (k) \in \mathbb{N}^m, \quad m \leq |(k)| = k.$$

(ii) The set of elements $a_{[j],(k)}^k(\theta_1, \ldots, \theta_k) = a_{[j],(k)} \otimes \ldots \otimes a_{[j],(k)}$, where

$$a_{[j],(k)} := (\theta_1 e_{j_1} + \ldots + \theta_k e_{j_k}) + \ldots + (\theta_{k_m} e_{j_m} + \ldots + \theta_{k_m} e_{j_m}) \in \mathbb{C},$$

and $[j], (k) \in \mathbb{N}^m$ is such that $m \leq |(k)| = k$, is total in the space $\mathbb{H}^k$ and possesses the property

$$\|a_{[j],(k)}^k\| = 1, \quad [j], (k) \in \mathbb{N}^m, \quad m \leq |(k)| = k.
(iii) The norm of the operator $g_{k,n}$ satisfies the inequalities

$$\sqrt{n!(k-n)!} \leq \|g_{k,n}\| \leq 1, \quad n \leq k.$$ 

The symmetric Fock space over $\mathbb{C}$ is defined as the hilbertian orthogonal sum

$$\mathcal{F}_a(\mathbb{E}) := \bigoplus_{k \in \mathbb{Z}_+} \mathbb{E}^k = \left\{ f = \bigoplus_k f_k : f_k \in \mathbb{E}^k \right\}, \quad \mathbb{E}^0 := \mathbb{C}$$

with the scalar product and the norm respectively

$$\langle f | g \rangle = \sum_{k \in \mathbb{Z}_+} \langle f_k | g_k \rangle, \quad \|f\| = \left( \sum_{k \in \mathbb{Z}_+} \|f_k\|^2 \right)^{1/2}.$$ 

Note that in view of the orthogonal property $f_k \perp f_n$ at $k \neq n$, every series $f = \bigoplus_k f_k$ is convergent in $\mathcal{F}_a(\mathbb{E})$. From Proposition 1 (i) it follows that an orthonormal basis in $\mathcal{F}_a(\mathbb{E})$ forms the system of elements $c_{[j]}^{(k)} \sqrt{k!/(k)!}$, where $[j], (k)$ run over all multi indexes in $\mathbb{N}^m$ such that $m \leq |(k)| = k$ and $k \in \mathbb{N}$.

**Proposition 2.**

(i) The system of elements on the unit sphere of $\mathcal{F}_a(\mathbb{E})$

$$\frac{1}{l_0(2)} \exp(a_{[j],(k)}) := \frac{1}{l_0(2)} \bigoplus_{n \in \mathbb{Z}_+} \frac{a_{[j],(k)}}{n!} : [j], (k) \in \mathbb{N}^m, \quad m \leq |(k)|$$

and as a consequence the subset

$$\exp(x) := \bigoplus_{k \in \mathbb{Z}_+} \frac{x^k}{k!} = \bigoplus_{k \in \mathbb{Z}_+} \sum_{k=|(k)|} \frac{1}{{k!}} \sum_{[j],[k]} c_{[j]}^{(k)} c_{[k]}^{(k)} : x = \sum_{j \in \mathbb{Z}_+} \zeta_j e_j \in \mathbb{E}$$

possesses in $\mathcal{F}_a(\mathbb{E})$ the total property. Above $c_{[j]}^{(k)} := c_{j_1}^{k_1} \cdots c_{j_m}^{k_m}$ and $l_0(2) := \sum_{k \in \mathbb{Z}_+} \zeta^{2k}/k!^2$ is the hyperbolic Bessel function, where $\zeta, \zeta_j \in \mathbb{C}$.

(ii) The system of elements in the space $\mathcal{F}_a(\mathbb{E})$

$$(1 - \zeta a_{[j](k)})^{-1} := \bigoplus_{n \in \mathbb{Z}_+} c_n a_{[j](k)}^{(n)} : [j], (k) \in \mathbb{N}^m, \quad m \leq |(k)|,$$
where \( 0 \neq \zeta \in \mathbb{D} := \{ z \in \mathbb{C} | |z| < 1 \} \), and as a consequence the subsets

\[
(1-x)^{-1} := \bigoplus_{k \in \mathbb{Z}^+} x^k = \bigoplus_{k \in \mathbb{Z}^+} k! \sum_{\{j\} \subseteq \{k\}} \frac{1}{|\{j\}|!} (\zeta_j)^{j_1} (\zeta_j)^{j_2} \cdots \zeta_j^j \in \mathbb{E}
\]

are well-defined and posses in \( \mathcal{F}_a(\mathbb{E}) \) a total property.

The dense subspaces in \( \mathcal{F}_a(\mathbb{E}) \) of elements with a finite number of nonzero addends is denoted by \( A_a(\mathbb{E}) \). It is easy to see that the subspace \( A_a(\mathbb{E}) \) is a commutative algebra with the unit \( 1 := 1 \oplus 0 \oplus \ldots \) in relative to the convolution

\[
f \ast g := \bigoplus_{k \in \mathbb{Z}^+} \sum_{n=0}^k \frac{k!}{n!(k-n)!} f_n \cdot g_{k-n}, \quad f = \bigoplus_k f_k, \quad g = \bigoplus_k g_k \in A_a(\mathbb{E}).
\]

By \( \mathbb{E}^* := \{ y^* := \langle \cdot, y \rangle \mid y \in \mathbb{E} \} \) we denote the hermitian dual space of \( \mathbb{E} \) and let \( \mathbb{E} \ni y \longmapsto y^* \in \mathbb{E}^* \) be the corresponding anti-linear isometric conjugation. For hermitian dual spaces the unitary equivalence \( (\mathbb{E}^* \otimes \mathbb{E})^* \simeq \mathbb{E}^* \otimes \mathbb{E} \) is held and for each \( u_k \in \mathbb{E}^* \otimes \mathbb{E} \) corresponds to the linear continuous forms \( u^*_k := \langle \cdot | u_k \rangle \in \mathbb{E}^* \otimes \mathbb{E} \). The adjoin symmetrization operator \( \mathcal{S}_k^* : g_1^* \otimes \ldots \otimes g_k^* \longrightarrow g_1^* \otimes \ldots \otimes g_k^* \) is the orthoprojector from \( \mathbb{E}^* \otimes \mathbb{E} \) onto \( \mathbb{E}^* \otimes \mathbb{E} \) and the unitary equivalence \( (\mathbb{E}^* \otimes \mathbb{E})^* \simeq \mathbb{E}^* \otimes \mathbb{E} \) is held.

The hermitian dual space of the symmetric Fock space \( \mathcal{F}_a(\mathbb{E}) \) is defined by the equality \( \mathcal{F}_a^*(\mathbb{E}) = \mathcal{F}_a(\mathbb{E}^*) = \bigoplus_{k \in \mathbb{Z}^+} \mathbb{E}^* \otimes \mathbb{E}^k \). The subspace in \( \mathcal{F}_a^*(\mathbb{E}) \) of elements with a finite number nonzero addends we denote by \( A_a^*(\mathbb{E}) \). The subspace \( A_a^*(\mathbb{E}) \) is a commutative algebra relative to the convolution

\[
f^* \ast g^* := \bigoplus_{k \in \mathbb{Z}^+} \sum_{n=0}^k \frac{k!}{n!(k-n)!} f^*_n \cdot g^*_{k-n},
\]

where \( f^* = \bigoplus_k f^*_k, \quad g^* = \bigoplus_k g^*_k \in A_a^*(\mathbb{E}) \) and \( f^*_k, g^*_k \in \mathbb{E}^* \otimes \mathbb{E}^k \).

2. Convolution derivation

For each element \( a \in \mathbb{E} \) we define the following gelation transformation

\[
\Lambda_{a,k} := \sum_{n=1}^k f^{(n-1)} \otimes A_a \otimes f^{(k-n)} : \mathbb{E}^* \longrightarrow \mathbb{E}^* \otimes \mathbb{E}^k
\]

acting by the rule

\[
\Lambda_{a,k}(x_1 \otimes \ldots \otimes x_k) = \sum_{n=1}^k x_1 \otimes \ldots \otimes x_{n-1} \otimes a \otimes x_{n+1} \otimes \ldots \otimes x_k
\]
and extending on the space $\mathbb{E}^{\otimes k}$ by linearity. Above, $I$ is the identical mapping on $\mathbb{E}$, $I^{\otimes n} := I \otimes \ldots \otimes I$, $I^{\otimes 0} = 1$ and

$$\Lambda_a : \mathbb{E} \ni x \mapsto \Lambda_a(x) \equiv a \in \mathbb{E}$$

is a constant mapping. For definition of the integer degree $j \in \mathbb{N}$ for the gelation transformation we use the identity

$$\frac{\Lambda_{a,k}^j}{j!} = \sum_{|j| = 1}^{j} \frac{\Lambda_{a,j}^1 \otimes \ldots \otimes \Lambda_{a,k}^j}{(j)!},$$

where $\Lambda_{a}^j(x) \equiv a$ for all $x \in \mathbb{E}$, $(j) = (j_1, \ldots, j_k) \in \mathbb{Z}_+^k$ and $\Lambda_{a}^0 = I$. Let $\Lambda_{a,k}^0 : \mathbb{E}^{\otimes k} \rightarrow \mathbb{E}^{\otimes k}$ is the identical mapping. For $\mathbb{E}^{\otimes 0} := \mathbb{C}$ we put $\Lambda_{a,0}(\zeta) \equiv 0$ for all $\zeta \in \mathbb{C}$.

**Theorem 1.**

(i) For any $a \in \mathbb{E}$ the transformation

$$\nabla_a^j := 0 \times \ldots \times 0 \times \Lambda_{a,j}^1 \times \Lambda_{a,j+1}^j \times \ldots, \nabla_a^j f = \bigoplus_{k \geq j} \Lambda_{a,k}^j f_k, j \in \mathbb{N},$$

$$\nabla_a^0 := \bigotimes_{k \in \mathbb{N}_+} \Lambda_{a,k}^0, \quad \Lambda_{a,0}^0 := 1$$

acts from the subalgebra $\mathcal{A}_a(\mathbb{E})$ into itself and is a convolution derivation in sense of the following equality

$$\frac{\nabla_a^j(f \ast g)}{j!} = \sum_{n=0}^{j} \nabla_a^n f \ast \nabla_a^{j-n} g$$

for all $f = \bigoplus_{k} f_k$, $g = \bigoplus_{k} g_k \in \mathcal{A}_a(\mathbb{E})$, where $f_k, g_k \in \mathbb{E}_k$. If $a = 0$ that $\nabla_a^0 = 0$ for all $j \in \mathbb{N}$ and $\nabla_a^0$ is the identical mapping on $\mathcal{A}_a(\mathbb{E})$.

(ii) For any $a \in \mathbb{E}$ the transformation $\nabla_a^j$ supposes an extension with the values in $\mathcal{F}_a(\mathbb{E})$ on the total subset $\{ \exp(x) \mid x \in \mathbb{E} \} \subset \mathcal{F}_a(\mathbb{E})$ which can be calculated as the direction derivatives

$$\nabla_a^j \exp(x) = \frac{d^j \exp(x + ta)}{dt^j} \bigg|_{t=0} = \bigoplus_{k \geq j} \frac{a^j}{(k-j)!} x^{k-j} \bigg|_{t=0} = \frac{a^j}{(k-j)!} x^{k-j}$$

where $\Lambda_{a,k}^j \frac{x^k}{k!} = \frac{1}{k!} \frac{d^j \exp(x + ta)}{dt^j} \bigg|_{t=0} = \frac{a^j}{(k-j)!} x^{k-j}$ for $k \geq j$. 
(iii) The shift defined for any $a \in \mathbb{B}$ on the total subset by the formula

$$T_a \exp(x) := \exp(x + a), \quad x, a \in \mathbb{E},$$

supposes a representation as the series

$$T_a \exp(x) = \exp(\nabla_a) \exp(x), \quad \exp(\nabla_a) := \sum_{j \in \mathbb{N}_+} \frac{\nabla_a^j}{j!}$$

uniformly converging in $\mathcal{F}_a(\mathbb{E})$ on every bounded subset of $\mathbb{E}$.

(iv) The equality

$$\langle \Lambda_{a,k}^j x^k | f_k \rangle = \langle x^{k-j} | \Lambda_{a,k}^{*j} f_k \rangle, \quad x \in \mathbb{E}$$

unique by defines the linear bounded operator

$$\Lambda_{a,k}^{*j} : \mathbb{E}^k \ni f_k \mapsto \Lambda_{a,k}^{*j} f_k \in \mathbb{E}^{k-j}$$

and its norm satisfies the inequality

$$\|\Lambda_{a,k}^{*j}\| \leq \frac{k!}{(k-j)!} \|a\|^j.$$

3. Hardy class, associated with symmetric Fock space

Since the canonical embedding $\otimes^k : \mathbb{E} \times \ldots \times \mathbb{E} \rightarrow \mathbb{E}^{\otimes k}$ is continuous, for every $f_k^* \in \mathbb{E}^{\otimes k}$ the composition $f_k^* \circ \otimes^k$ belongs to the space of all $k$-linear continuous forms $\mathcal{L}^k(\mathbb{E}, \mathbb{C}) := \mathcal{L}(\mathbb{E} \times \ldots \times \mathbb{E}, \mathbb{C})$. Note that in the case of infinite-dimensional space, the mapping $\mathbb{E}^{\otimes k} \ni f_k^* \mapsto f_k^* \circ \otimes^k \in \mathcal{L}^k(\mathbb{E}, \mathbb{C})$ is not a surjection. If

$$\Pi^k : \mathbb{E} \ni x \mapsto (x, \ldots, x) \in \mathbb{E} \times \ldots \times \mathbb{E}$$

denotes the cartesian degree, that the set of restrictions on the diagonal

$$\mathcal{P}^k(\mathbb{E}) := \{ g_k \circ \Pi^k \mid g_k \in \mathcal{L}^k(\mathbb{E}, \mathbb{C}) \}$$

is called the space of continuous $k$-homogeneous polynomials [4]. The subspace of the form

$$\mathcal{P}^k_0(\mathbb{E}) := \left\{ f_k := \frac{f_k^* \circ \otimes^k \circ \Pi^k}{k!} : f_k^* \in \mathbb{E}^{\otimes k}_0 \right\} \subset \mathcal{P}^k(\mathbb{E})$$
is called the space of \( k \)-homogeneous Hilbert-Schmidt polynomials

\[
\hat{f}_k : \mathbb{E} \ni x \mapsto \hat{f}_k(x) := \frac{\langle x^k \mid f_k \rangle}{k!} \in \mathbb{C}.
\]

The space of all Hilbert-Schmidt polynomials is defined as the complex linear span

\[
\mathcal{P}_h(\mathbb{E}) := \text{span} \{ \mathcal{P}_h^k(\mathbb{E}) : k \in \mathbb{Z}_+ \}, \quad \mathcal{P}_h^k(\mathbb{E}) := \mathbb{C}.
\]

According to Proposition 1 (ii) the system \( \{ x^k \mid x \in \mathbb{E} \} \) is total in \( \mathbb{E}^k \). Whence it follows that \( \text{Ker} \ f_k = \{ 0 \} \) for each \( f_k \in \mathbb{E}^k \) and there are determined the linear isomorphism and its inverse respectively

\[
\mathbb{E}^k \ni f_k \mapsto \hat{f}_k \in \mathcal{P}_h^k(\mathbb{E}), \quad \mathcal{P}_h^k(\mathbb{E}) \ni F_k \mapsto F_k^\vee \in \mathbb{E}^k. \tag{1}
\]

Subsequently, on the space \( \mathcal{P}_h^k(\mathbb{E}) \) we determinate the hilbertian norm

\[
\| \hat{f}_k \| := \| f_k^\vee \|, \quad \hat{f}_k \in \mathcal{P}_h^k(\mathbb{E}), \quad f_k \in \mathbb{E}^k.
\]

Then the linear mapping

\[
\mathbb{E}^k \ni f_k \mapsto \hat{f}_k \in \mathcal{P}_h^k(\mathbb{E})
\]

realizes the unitary isomorphism \( \mathbb{E}^k \simeq \mathcal{P}_h^k(\mathbb{E}) \).

If the complex function \( F : \mathbb{E} \to \mathbb{C} \) defined on the Hilbert space is Gâteaux-analytic and continuous, that \( F \) calls an entire analytic function. Every such an analytic function \( F \) on \( \mathbb{E} \) can be defined uniquely by its Taylor series

\[
F(x) = \sum_{k \in \mathbb{N}_+} F_k(x), \quad F_k(x) := \frac{1}{k!} d^k F(0)(x, \ldots, x), \quad x \in \mathbb{E},
\]

where \( d^k F(0) \in \mathcal{L}(\mathbb{E}, \mathbb{C}) \) is a Fréchet derivation of \( F \) at zero. We study an entire analytic functions \( F : \mathbb{E} \to \mathbb{C} \) such that every \( k \)-homogeneous polynomial \( F_k \) generating by the Taylor coefficient \( d^k F(0) \) belongs to the subspace of Hilbert-Schmidt polynomials \( \mathcal{P}_h^k(\mathbb{E}) \). For each Taylor coefficient \( d^k F(0) \) of such function there exists a unique element \( f_k \in \mathbb{E}^k \) for which

\[
F_k(x) = \hat{f}_k(x) = \frac{\langle x^k \mid f_k \rangle}{k!}, \quad x \in \mathbb{E}.
\]

The class of such entire analytic functions

\[
F : \mathbb{E} \ni x \mapsto F(x) = \sum_{k \in \mathbb{N}_+} F_k(x), \quad F_k \in \mathcal{P}_h^k(\mathbb{E}) \tag{2}
\]
for which, in addition, the inverse value of uniform convergence radius for Taylor series satisfies the condition
\[
\limsup_{k \to \infty} \sqrt[k]{\sup_{\|x\| \leq 1} |F_k(x)|} = 0,
\]
is denoted by \( \mathcal{H}_b(E) \). Following the terminology of [4, Sec 6.3] the functions from \( \mathcal{H}_b(E) \) belong to a class of entire analytic functions of bounded type. Note that for each function \( F \in \mathcal{H}_b(E) \) and for each fixed element \( a \in \mathbb{S} \) there is defined the following analytic function of scalar argument
\[
F_a : \mathbb{D} \ni \zeta \mapsto F(\zeta a) = \sum_{k \in \mathbb{Z}_+} \zeta^k F_k(a), \quad \zeta \in \mathbb{C}
\]
and differentiating it as a consequence we obtain
\[
F_k(a) = \frac{d^k}{d\zeta^k} F_a \bigg|_{\zeta = 0}, \quad a \in \mathbb{S}, \quad k \in \mathbb{Z}_+.
\]

**Definition 1.** The subspace in \( \mathcal{H}_b(E) \) of entire analytic function \( (2) \) for which the element \( F^\ast := \bigoplus_{k \in \mathbb{Z}_+} F_k^\ast \) belongs to the hermitian dual symmetric Fock space \( \mathcal{F}_s^\ast (E) \) is called the Hardy class on \( E \) and denoted by \( \mathcal{H}_b^2(E) \).

On \( \mathcal{H}_b^2(E) \) we determine the hermitian norm
\[
||F|| := ||F^\ast||, \quad F \in \mathcal{H}_b^2(E), \quad F^\ast \in \mathcal{F}_s^\ast (E).
\]

**Theorem 2.** Let \( x \in E \) and \( f = \bigoplus_{k \in \mathbb{Z}_+} f_k \in \mathcal{F}_s(E) \), where \( f_k \in E^k \).

(i) The anti-linear mapping
\[
\mathcal{F}_s(E) \ni f \mapsto \langle \exp(x) \mid f \rangle := \hat{f}(x), \quad x \in E
\]
realizes an anti-isomorphism from the Fock space \( \mathcal{F}_s(E) \) onto the Hardy class \( \mathcal{H}_b^2(E) \), uniquely defined by the relations
\[
\frac{d^k}{dx^k} \hat{f}(0) \circ \Pi^k(x) = \frac{\langle x^k \mid f_k \rangle}{k!} = \tilde{f}_k(x), \quad x \in E, \quad k \in \mathbb{Z}_+.
\]
As a consequence the linear mapping \( \mathcal{F}_s^\ast (E) \ni f^\ast \mapsto f^\ast \circ \exp(x) = \hat{f}(x) \) realizes the unitary isomorphism
\[
\mathcal{F}_s^\ast (E) \simeq \mathcal{H}_b^2(E).
\]
(ii) The restriction of isomorphism (2) on the subalgebra $A^*_h(E)$ realizes an algebraic isomorphism onto the multiplicative algebra of Hilbert-Schmidt polynomials $P_h(E)$ such that

$$\tilde{f} \ast \tilde{g}(x) = \tilde{f}(x) \tilde{g}(x), \quad f, g \in A_h(E), \quad x \in E.$$ 

(iii) The system of entire analytic functions from the unit open ball of $H^2_h(E)$

$$\langle \exp(x) \mid (1 - \zeta a[j], (k))^{-1} \rangle := \sum_{n \in \mathbb{Z}^+} \zeta^n \frac{(1 + a[j], x)^n}{n!} \langle \eta, (k) \rangle, \quad n \leq ||(k)||$$

is linear by independent and dense in $H^2_h(E)$ for arbitrary non-zero fixed $\zeta \in \mathbb{D}$.

4. Group of shifts on Hardy class

Now we shall prove that the shift determined by the formula

$$\hat{T}_a \tilde{f}(x) = \tilde{f}(x + a) = \tilde{f}(T_a x), \quad \tilde{f} \in H^2_h(E), \quad x, a \in E,$$

is well defined on the Hardy class $H^2_h(E)$. It is easy to check that the structure of multiplicative algebra in $P_h(E)$ is $\hat{T}_a$-invariant for each $a \in E$ and the shift $\hat{T}_a$ is an algebraic automorphism on $P_h(E)$.

**Theorem 3.** For any $a \in E$ every linear operator $\hat{T}_a$ is bounded on $H^2_h(E)$. The set of such operators satisfies the group property $\hat{T}_{a+b} = \hat{T}_a \circ \hat{T}_b$ for any $a, b \in E$ and $\hat{T}_0$ is the unit operator on $H^2_h(E)$. The 1-parametric group $R \ni t \mapsto \hat{T}_t$ satisfies the inequality

$$||\hat{T}_t|| \leq \sqrt{\exp(||ta||^2)}, \quad t \in \mathbb{R}.$$ 

5. Direction derivations on Hardy class

For every element $a \in E$ and integer $j \in \mathbb{Z}_+$ we determine the following linear unbounded operator on the Hardy class $H^2_h(E)$

$$\nabla^j_a \tilde{f}(x) := d^j \tilde{f}(x)(a, \ldots, a), \quad \tilde{f} \in H^2_h(E), \quad x \in E$$
with the domain

\[ \text{Dom}(\hat{\nabla}_a^j) = \{ \hat{f} \in \mathcal{H}_a^\infty(\mathbb{E}) \mid \hat{\nabla}_a^j \hat{f} \in \mathcal{H}_a^\infty(\mathbb{E}) \} . \]

The domain \( \text{Dom}(\hat{\nabla}_a^j) \) contains the subalgebra \( \mathcal{P}_a(\mathbb{E}) \) and therefore is dense in \( \mathcal{H}_a^\infty(\mathbb{E}) \).

**Theorem 4.**

(i) The operator \( \hat{\nabla}_a^j \) is closed on \( \mathcal{H}_a^\infty(\mathbb{E}) \) and is the generator of the 1-parametric group \( \mathbb{R} \ni t \mapsto \hat{T}_a \), i.e.

\[ \hat{\nabla}_a \hat{f} = \lim_{t \to 0} \frac{\hat{T}_a \hat{f} - \hat{f}}{t} = \frac{d\hat{f}(x + ta)}{dt} \bigg|_{t=0} , \quad \hat{f} \in \text{Dom}(\hat{\nabla}_a) . \]

(ii) For each \( a \in \mathbb{E} \) the formula

\[ \hat{\nabla}_a^j \hat{f}(x) = \langle \nabla_a^j \exp(x) \mid \hat{f} \rangle , \quad \hat{f} \in \text{Dom}(\hat{\nabla}_a^j) , \quad x \in \mathbb{E} \]

determines a bijection between the linear operator \( \hat{\nabla}_a^j \) and the transformation \( \nabla_a^j \). As a consequence the operator \( \nabla_a^j \) can be calculated by the formulas

\[ \hat{\nabla}_a^j \hat{f}(x) = \frac{d^j \hat{f}(x + ta)}{dt^j} \bigg|_{t=0} = \sum_{k \geq j} \frac{\langle a^j \cdot x^{k-j} \mid f_k \rangle}{(k-j)!} , \quad t \in \mathbb{R} . \]

(iii) The multiplicative subalgebra of polynomials \( \mathcal{P}_a(\mathbb{E}) \) is \( \hat{\nabla}_a \)-invariant and for each element \( a \in \mathbb{E} \) the restriction \( \hat{\nabla}_a \mid_{\mathcal{P}_a(\mathbb{E})} \) is a derivation on \( \mathcal{P}_a(\mathbb{E}) \).

**Corollary 1.** The domain of the generator \( \hat{\nabla}_a^j \) and the norm of direction derivation \( \hat{\nabla}_a^j \hat{f} \) can be respectively calculated with the help of equalities

\[ \text{Dom}(\hat{\nabla}_a^j) = \{ \hat{f} \in \mathcal{H}_a^\infty(\mathbb{E}) \mid \| \hat{\nabla}_a^j \hat{f} \| < \infty \} , \quad \| \hat{\nabla}_a^j \hat{f} \| = \left( \sum_{k \geq j} \| A_a^{j,k} f_k \|^2 \right)^{1/2} . \]
6. Entire functions of exponential type in Hardy class

Let as has been stated above, the elements \( f = \bigoplus_{k} f_k \in \mathcal{F}_a(\mathbb{E}) \), \( (f_k \in \mathbb{E}^k) \) and \( \hat{f} = \sum_{k} \hat{f}_k \in \mathcal{H}_a^2(\mathbb{E}) \), \( (\hat{f}_k \in \mathcal{P}_a^k(\mathbb{E})) \) are connected by isomorphism (2). For a function \( \hat{f} \in \bigcap_{a \in \mathbb{B}} \text{Dom}(\hat{\nabla}_a^j) \) we define the norm of its derivation \( d^j \hat{f} \) (if this number is finite) by the formula

\[
||d^j \hat{f}|| := \sup_{a \in \mathbb{B}} ||\hat{\nabla}_a^j \hat{f}||.
\]

**Definition 2.** We say that the entire analytic function

\[
\hat{f} \in \bigcap_{j \in \mathbb{B}_+} \bigcap_{a \in \mathbb{B}} \text{Dom}(\hat{\nabla}_a^j)
\]

has exponential type \( \tau \in [0, \infty) \), if the scalar function of complex variable

\[
\beta(\lambda, \hat{f}) := \sum_{j \in \mathbb{B}_+} \frac{||d^j \hat{f}||}{j!} \lambda^j, \quad \lambda \in \mathbb{C}
\]

is entire and has the exponential type \( \tau \), i.e. if it satisfies the equality

\[
\tau = \chi(\hat{f}), \quad \text{where} \quad \chi(\hat{f}) := \limsup_{r \to \infty} \left( r^{-1} \ln \max_{|\lambda|=r} |\beta(\lambda, \hat{f})| \right).
\]

The set of all functions of the exponential type \( \tau \) will be denoted by \( \mathcal{E}_a^\tau \). Denote by \( \mathcal{E}_a := \bigcup_{\tau \in [0, \infty)} \mathcal{E}_a^\tau \) the space of all entire analytic functions of exponential type.

**Proposition 3.** The following statements are equivalent:

(i) \( \hat{f} \in \mathcal{E}_a^\tau \);

(ii) the radius of convergence on infinity of the scalar power series

\[
g(\lambda, \hat{f}) := \sum_{j \in \mathbb{B}_+} \frac{||d^j \hat{f}||}{\lambda^{j+1}}, \quad \lambda \in \mathbb{C}
\]

satisfies the equality

\[
\tau(\hat{f}) = \tau, \quad \text{where} \quad \tau(\hat{f}) := \limsup_{j \to \infty} \sqrt[2j]{||d^j \hat{f}||}.
\]
Theorem 5.

(i) The subspace $E_\lambda$ contains the subalgebra $P_h(\mathbb{E})$ of the linear independent system of entire analytic functions

$$\left\langle \exp(x) \mid (1 - \zeta a_{j_1, \ldots, j_n})^{-1} \right\rangle : [j], (k) \in \mathbb{N}^m, m \leq |(k)|, \ 0 \neq \zeta \in \mathbb{D}$$

and as a consequence $E_\lambda$ is dense in $H^1_h(\mathbb{E})$.

(ii) The function

$$\|\hat{f}\| := \chi(\hat{f}) + \|\hat{f}\|, \quad \hat{f} \in E_\lambda$$

is a quasi-norm on the space $E_\lambda$ such that

$$\|\hat{f} + \hat{g}\| \leq \|\hat{f}\| + \|\hat{g}\|, \quad \hat{f}, \hat{g} \in E_\lambda.$$ 

7. Inequalities of Bernstein and Jackson types

Now we shall consider the problem of best approximation for an arbitrary function $\hat{f}$ of Hardy class $H^1_h(\mathbb{E})$ by the entire analytic functions of exponential type. For this purpose we estimate the distance

$$\delta(\tau, \hat{f}) := \inf_{\hat{g} \in E_\lambda} \|\hat{f} - \hat{g}\|, \quad \hat{f} \in H^1_h(\mathbb{E})$$

between a given $\hat{f}$ and the subspaces $E^\tau_\lambda$ with a fixed index $\tau > 0$. Using the entered above quasi-norm we determine the auxiliary functional

$$E(\tau, \hat{f}) := \inf_{\|\hat{g}\| < \tau} \|\hat{f} - \hat{g}\|, \quad \hat{g} \in E^\tau_\lambda.$$

Lemma 1. The following inequality is valid

$$\delta(\tau, \hat{f}) \leq E(\tau, \hat{f}), \quad \hat{f} \in H^1_h(\mathbb{E}), \quad \tau > 0.$$ 

It is easy to see that continuous embedding $E_\lambda \hookrightarrow H^1_h(\mathbb{E})$ is held. For the numbers $0 < \alpha < \infty$ we consider the scale of approximation spaces between the quasi-normed subspace $E_\lambda$ and the Hardy class $H^1_h(\mathbb{E})$ of the following form

$$E_\alpha (H^1_h(\mathbb{E}), E_\lambda) := \left\{ \hat{f} \in H^1_h(\mathbb{E}) \mid \|\hat{f}\|_\alpha < \infty \right\},$$
where in according to [5, Lemma 7.1.6] the function
\[ |\hat{f}|_{\alpha} := \left( \int_0^{\infty} E^{\theta}(\tau, \hat{f}) \frac{d\tau}{\tau^\theta} \right)^{1/\theta}, \quad \vartheta = \frac{1}{\alpha + 1} \]
is a quasi-norm on \( E_\alpha \left( \mathcal{H}_h^2(\mathbb{E}), \mathcal{E}_h \right) \).

**Theorem 6.** There exist constants \( C_1 \) and \( C_2 \) such that
\[ |\hat{f}|_{\alpha} \leq C_1 |\hat{f}|_{\alpha} ||\hat{f}||, \quad \hat{f} \in \mathcal{E}_h, \]
\[ \delta(\tau, \hat{f}) \leq C_2 \tau^{-\alpha} |\hat{f}|_{\alpha}, \quad \hat{f} \in E_\alpha \left( \mathcal{H}_h^2(\mathbb{E}), \mathcal{E}_h \right). \]

Inequalities (3) and (4) respectively generalizes known Bernstein and Jackson inequalities on the case of best approximations by exponential type functions in Hardy classes of infinitely many variable.

**References**


Connections between Separate and Joint Properties of Functions for Several Variables

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We shall consider various weakening of the continuity, their separate and join versions and interplay between them.

Cohomology of Semigroups

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The definition of the Eilenberg–MacLane cohomology of semigroups is the same as for groups. However, because of the presence a zero element in semigroups it is more preferably to consider a generalization of it, so-called 0-cohomology, which appears in various applied topics.

Contents of lectures:

1. Cohomology and 0-cohomology.
2. Calculation of cohomology.
3. Projective representations.
5. Partial representations of groups.
6. Semigroups and small categories.
Link Diagrams and S-Graphs

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Abstract. Malešič and Traczyk conjectured [2] that for each link diagram $D$ with the writhe $c$ and the number of Seifert circles equal to $s$ it is possible to find another link diagram of the same link with the number of Seifert circles equal to $s - \text{ind}_- (G(D)) - \text{ind}_+ (G(D))$ and the writhe equal to $c + \text{ind}_- (G(D)) - \text{ind}_+ (G(D))$, where $\text{ind}_+ (G(D))$ and $\text{ind}_- (G(D))$ denote, respectively, the positive and the negative cyclic index of the signed Seifert graph $G(D)$. Moreover it was conjectured that a reduction process on any link diagram can be performed via five types of operations, specified by them. In [2], this conjecture was translated into the one in graph theory and formulated in terms of S-graphs. In this talk, we show that the conjecture of Malešič and Traczyk about S-graphs is false, that is the five types of operations on S-graphs defined by them are not sufficient to perform a desired reduction on any S-graph. We also discuss the further relationship between link diagrams, their Seifert graphs and S-graphs subject to the Malešič and Traczyk conjectures.

Introduction

See [1] for the definition of basic notions in knot theory used here.

A classical conjecture in knot theory asserts that for every closed braid diagram $D$ which represents a link $L$ and has the minimal number of strands $b(D)$ among all closed braids representing $L$, the writhe $w(D)$ of $D$ is uniquely determined.

In view of the results of [4], this conjecture can be restated in an equivalent form [2].

Conjecture 1.1. For every diagram $D$ of a link $L$ that has the minimum possible number of Seifert circles $s(D)$ among all diagrams representing $L$ the writhe $w(D)$ of $D$ is uniquely determined.
Moreover, for any diagram $D'$ of $L$ that has $s(D) + k$ Seifert circles, it holds the following:

$$w(D) - k \leq w(D') \leq w(D) + k.$$  

Murasugi and Przytycki [3] introduced an operation on diagrams of links which reduces the number of Seifert circles on a diagram by one. Following [2], we shall call it the MP-reducing operation. The MP-reducing operations can be applied to a link diagram repeatedly. To find a controlled manner in which such a sequence of MP-operations may be performed, it is convenient to consider the Seifert graph of a given link diagram $D$. The Seifert circles of $D$ form the vertex set of $G(D)$ and to each crossing of $D$ that is common for the two Seifert circles $C_1$ and $C_2$ there corresponds an edge in $G(D)$ which joins the two vertices associated with $C_1$ and $C_2$. Any Seifert graph $G(D)$ is bipartite and planar (although is not planar in a canonical way) and can possess multiple edges. A set of single edges in $G(D)$ is called cyclically independent if in every cycle the number of edges from the given set is less than half of the length of the cycle. Here by a cycle it is understood a simple closed route in the graph. The cyclic index $\text{ind}(G(D))$ of a graph $G(D)$ is defined to be the maximum number of cyclically independent edges in $G(D)$. In [3], it was shown that in any link diagram $D$ the number of the Seifert circles in $D$ can be reduced by the cyclic index $\text{ind}(G(D))$ by performing a sequence of MP-operations.

Recall that every edge in the Seifert graph $G(D)$ has a sign, according to the sign of the corresponding crossing in the link diagram $D$. As in [2], denote by $\text{ind}_-(G(D))$ and $\text{ind}_+(G(D))$ the maximum numbers of cyclically independent single negative and positive edges, respectively, in the Seifert graph $G(D)$. The following inequality

$$\text{ind}_-(G(D)) + \text{ind}_+(G(D)) \geq \text{ind}(G(D))$$  

holds obviously for any link diagram $D$ and its Seifert graph $G(D)$ and is sharp [2]. Malešič and Traczyk [2] made the following conjecture.

**Conjecture 1.2.** Let $D$ be a diagram of an oriented link $L$. Assume that the number of Seifert circles in $D$ is $s$ and the algebraic crossing number of $D$ is $c$. Then it is possible to find another link diagram of $L$ with the number of Seifert circles equal to $s - \text{ind}_+(G(D)) - \text{ind}_-(G(D))$ and the writhe equal to $c + \text{ind}_+(G(D)) - \text{ind}_-(G(D))$.

This conjecture was proved in some particular cases [2]. There is also a connection between Conjectures 1.1 and 1.2. As shown in [2], any counterexample to Conjecture 1.2 would imply a counterexample to Conjecture 1.1.
Basing on the ideas of Murasugi and Przytycki, Malesić and Traczyk [2] introduced five operations on diagrams which were aimed to reduce any link diagram to the one with fewer number of Seifert circles.

A key role in a reduction of a link diagram $D$ via the operations of type 1) - 5) plays the Seifert graph $G(D)$ of the $D$. Malesić and Traczyk conjectured that the operations of type 1)-5) are sufficient to perform a reduction of number of Seifert circles in any link diagram $D$ by $\text{ind}_-(G(D)) + \text{ind}_+(G(D))$ and translated the problem into the one in graph theory. This is the subject of our consideration in the second part of this talk.

Reductions of S-graphs

Let $G$ be a planar bipartite graph embedded in the plane and let $E$ denote its edges. Moreover let $E_+$ and $E_-$ be the two disjoint subsets of the set $E$ consisting of positive and negative edges, respectively (that is with the signs $+$ and $-$). Assume that both $E_+$ and $E_-$ are the cyclically independent sets (so all signed edges are single). Following [2], any graph $G$ enhanced with the structure described above (including conditions of cyclical independence both the sets $E_+$ and $E_-$) is called an S-graph.

The difference between an S-graph and the Seifert graph of a diagram is the following. For any link diagram $D$, all edges in the Seifert graph $G(D)$ are naturally signed. Moreover, the set of single edges of one sign is not necessarily cyclically independent. On the other hand, in an S-graph only single edges are signed and both the sets of positive edges and negative edges are cyclically independent. It follows that any S-graph can be obtained from the Seifert graph of some link diagram by choosing independent subsets of positive and negative edges and canceling the signs of the remaining edges.

We shall say that two edges in an S-graph are neighbor edges if they have a common vertex and belong to the boundary of the same face. Malesić and Traczyk [2] have defined the following five operations on S-graphs.

1) A signed stamp may be removed from the graph. The resulting graph is always an S-graph. Note that this operation does not reduce the number of signed edges by one.

2) Let $u, e_1, v, e_2, w$ be a sequence of two neighbor edges and their vertices. Assume that $e_1$ is signed and $e_2$ neutral. Then $u$ and $w$ may be contracted into one vertex while, at the same time, the edge $e_2$ is being removed from the graph. This operation is allowed only if the resulting graph is still an S-graph. Note that this operation also reduces the number of signed edges by one.
3) Assume that $e_1$ and $e_2$ are a pair of signed edges in an $S$-graph, one negative and one positive, and so that after removing of both them the remaining graph becomes disconnected. Then both $e_1$ and $e_2$ may be contracted, each to a vertex. The case when the edges $e_1$ and $e_2$ have a (unique) common vertex is not excluded. The resulting graph is always an $S$-graph. Note that operation 3) reduces the number of signed edges by two.

4) Assume that there is a face of the embedded graph $G$ in the plane, bounded by a cycle $v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_1$. Moreover assume that $e_1$ is negative and $e_2$ is positive. Then the edges $e_3$ and $e_4$ may be removed from the graph while the vertices $v_2$ and $v_4$ are contracted into one vertex. The resulting graph is always an $S$-graph. Note that operation 4) does not reduce the number of signed edges.

5) Assume that there is a face of the embedded graph $G$ in the plane, bounded by a cycle $v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_1$. Moreover assume that $e_1$ is negative and $e_2$ is positive. Then the edges $e_2$ and $e_4$ may be removed from the graph while the vertices $v_1$ and $v_3$ are contracted into one vertex. The resulting graph is always an $S$-graph. Note that operation 5) does not reduce the number of signed edges in $G$.

The five types of operations on $S$-graphs just indicated correspond to the ones on link diagrams mentioned above (see [2]). Note that these operations are applied to the plane graphs. Malesić and Traczyk made the following conjecture.

**Conjecture 2.1.** For any $S$-graph, it is possible to perform a sequence of operations of type 1)—5) so that the final graph contains no signed edge.

As mentioned in [2], the positive solution of the conjecture on $S$-graphs would prove Conjecture 1.2 on link diagrams. Conjecture 2.1 has been proved in one particular case (see Proposition 3.2 of [2]). Our main result is the following.

**Theorem 2.1.** There are infinitely many $S$-graphs that cannot be reduced via operations of type 1)—5) to the ones which contain only neutral edges.

Finally, we indicate how the reduction operations of types 4)—5) on link diagrams and their analogues on $S$-graphs can be generalized.

Notice however that to be of any use the generalized reduction operation of type 4) may be used in a way that wouldn’t change the positive cyclic index and the negative cyclic index of the corresponding Seifert graph. On
the level of S-graphs, these operations lead, when applied, to a more simple S-graph.

**Problem 2.1.** Describe a finite number of admissible operations on diagrams of links (if possible) under which Conjecture 2.1 is true.

**Example 2.2.** We indicate a link diagram $D$ and the S-graphs $G$ and $R$ which are associated with $D$. The S-graph $R$ can be reduced to an S-graph without signed edges. On the other hand, $G$ does not admit such a reduction.

Example 2.2 and Theorem 2.1 show that when passing from the Seifert graph of a link diagram to its S-graph, we may loose, in general, an important information about the original link diagram. In particular, the following question is still open.

**Question 2.2.** Let $D$ be a diagram of an oriented link $L$. Assume that the number of Seifert circles in $D$ is $s$ and the algebraic crossing number of $D$ is $c$. Can one reduce $D$ to another link diagram of $L$ with the number of Seifert circles equal to $s - \text{ind}_+(G(D)) - \text{ind}_-(G(D))$ and the writhe equal to $c + \text{ind}_+(G(D)) - \text{ind}_-(G(D))$, by using the operations of type 1)—5) (and their generalizations)?

To avoid the situation mentioned above, we slightly change the definition of S-graph. We suggest that all the edges in any S-graph are signed and that the sets of positive and negative distinguished edges currently change during the reduction process. Then Conjecture 2.1, Question 2.2 and Problem 2.1 can be restated appropriately.

**References**


Ramsey Methods in Banach Spaces

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• Lecture 1. Finite Ramsey theorem: Let $k$ and $r$ be positive integers. Then for every $r$-colouring of the set $[N]^k$ there exists an infinite subset $X$ of $N$ such that $X^k$ is monochromatic.

Application: Theorem of Brunel and Sucheston on basic sequences. Spreading models.

• Lecture 2. Infinite Ramsey theorem. Rosenthal’s alternative: Every bounded sequence in a Banach space has a subsequence which is either weakly Cauchy or equivalent to the unit vector basis of $l_1$.

• Lecture 3. Gowers’ dichotomic: Every Banach space $X$ has a subspace $Y$ which either has an unconditional basis or is hereditarily indecomposable.

Some Lectures on Asymptology

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The basic subjects of asymptology are the balleans (= coarse structures). A ballea $B$ is a set $X$ endowed with some family of subsets of $X$ which are called the balls. The properties of the balls are postulated in such a way that a ballean can be considered as an asymptotic counterpart of a uniform topological space. The concept of balleans (with the appropriate morphisms) arise in coarse geometry, asymptotic topology and in combinatorics. It should be marked the prominent role of Gromov’s book [1] at the dawn of asymptology.

Content of lectures:

• Ball structures: uniform spaces and balleans;

• Morphisms;
- Metrizability and approximations;
- Cellularity and pseudodiscreteness;
- Graph balleans;
- Group balleans;
- Normality and slowly oscillating functions;
- Coronas of balleans;
- Combinatorial size;
- Asymptotic cardinal invariants;
- Extremal balleans.

The text of the lectures is available at
http://unicyb.kiev.ua/Site-Eng/admin/DO/Protasov.html
For some other aspects of asymptology see [2], [3], [4].

References
Mathematics in the History of Lviv University

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The Lviv University was founded in 1661 according to a deed of the king Jan Kazimier giving the rank of Academy and title of University to Jesuit college that existed since 1668 [1]. Many times during its history the university changed the name and status: Academy (University) (1661–1773), Lyceum (1773–1784), University of Jozef II (1784–1805), Lyceum (1805–1817), University of Franz I (1817–1918), University of Jan Kazimierz (1919–1939), Ivan Franko State University (1940–1999), Ivan Franko Lviv National University (since 1999).

At the time of founding the university the mathematics was taught on the philosophical faculty. This faculty initially was preparatory for studying at other faculties. In various periods studying at the faculty lasted 2–4 years. Due to the development of mathematics and natural sciences the structure of the philosophical faculty had been changed.

At the year of founding the academy (university), there was a professor of mathematics S. Małaszowski. By the tradition, the courses in logic, physics, metaphysics and mathematics were taught at the same place during 1–4 years and then were repeated in another city. Because of that during 1661–1773 there were about 50 professors lecturing mathematics at the university [2].

The amount of mathematical knowledge offered by Lviv Academy increased permanently. At the beginning of XVIII century in addition to arithmetics and geometry they taught stereoometry, logarithms, and trigonometry. Additional topics included fundamenta of mechanics, ballistics, hidraulics, geodesy. Special attention was paid to astronomic calculations [3, p.165].

The first chair of mathematics was founded in 1744 and was headed by Faustyn Grodzicki (born in 1709). He taught courses in arithmetics, geometry, statics, mechanics, hydrostatics, perspective, architecture.

In Lyceum (this was the name of the university in 1774) the mathematics was taught by Ignacy Rein (1761–1831), Józef Liesganig (1719–1799), the members of the committee for cartography in Galicia. In 1785 three years of philosophy were introduced at the university. Among other subjects there were mathematics, “applied mathematics”, geometry and technology.
In addition to the chair of pure mathematics there was a chair of applied mathematics (that is geodesy) headed by Jon Holfeld (1749–1814) that also took part in cartographing Galicia.

In 1787 on the competition basis the chair of mathematics was occupied by Franciszek Kodesch (1761–1831) who taught courses in pure and applied mathematics till 1831 (except 4 years of professorship in Cracow) [2] ([1], p.65). He wrote a two-volume textbook “Elements of pure mathematics” that was accepted as an obligatory one. In the Lyceum (when Kodesch was in Cracow) the mathematics was taught by Jan Holfeld.

Since 1787 till 1808 at the university there was Studium Ruthenium, two-year Ukrainian theological and philosophical institute where mathematics was taught in Ukrainian by Piotr Lody (1764–1829) and Jan Zemantsek.

After F. Kodesch the chair was occupied by Leopold Schulz von Straszniicki (1803–1852) who taught a modern course of high mathematics since 1834 till 1838. Leopold Schulz was a mathematician known in Europe [6, p.428]; in Lviv he had two doctorants.

Since 1840 till 1870 the mathematics was taught by Ignacy Lemoch (1802–1875). After 1848 he divided the course of mathematics into separate subjects: straightline and spherical geometry, analytic geometry, differential and integral calculus, variational calculus, courses in mechanics and astronomy. After 1848 some democratization of education process started, the philosophical faculty stopped to be preparatory. The preparation of teachers for gymnasia became its main aim.

A considerable role for further development of mathematics in Lviv belongs to Wawrzyńcow Zmurnko (1824–1884). He was born in Yavoriv, studied at Lviv University and then in Vienna. Initially Zmurnko taught at Lviv technical academy and afterwards since 1871 till 1881 at the university. He was distinguished by original methods of constructing the course of mathematics that were included in his two-volume textbook [7]. His mathematical tools for mechanical drawing of curves and also integrator were awarded at the international exhibitions in Paris and London.

Since 1884 till 1919 the principal mathematician of the university was Józef Puzyna (1856–1919). He was born at New Martyniv (currently Ivano-Frankivsk region), graduated from Lviv University, attended lectures of Weierstrass and Kronecker in Berlin. His interests concerned mainly the theory of analytic functions. At the university he taught about 30 different courses [5]. J. Puzyna organized and directed two seminars. Among participants of these seminars there were Włodzimierz Lewicki, Stanisław Rużewicz, Jan Rajewski, Otto Nikodym and other students that after-
wards became famous mathematicians [4, p.389]. The first work made by the participants of the seminar was the paper of Lewicki [12]. It was also published in the 6th volume of “Zapysky NTSh” in 1894. This was the first professional paper in mathematics written in Ukrainian.

The principal work of J. Puzyua was a two-volume book “Theory of Analytic Functions” (1898–1900). In the book he collected the newest results of Weierstrass, Cauchy, Riemann related to analytic functions. It contained also an extended introduction to Set Theory.

J. Puzyua created the second mathematical chair headed by Jan Rajewski since 1900.


Starting from the summer semester 1913/14 at the university there lectured a young talented topologist Zygmund Janiszewski (1888–1920). His ideas on development of mathematics in Poland were afterwards realized in Warsaw and Lviv mathematical schools. In 1919 in Warsaw, jointly with W. Sierpiński and S. Mazurkiewicz (1888–1945) he founded a journal “Fundamenta Mathematicae” that became the first specialized journal in Set Theory and its applications and also in Logic in the world.

During this period the principal directions of investigations of university mathematicians were the theory of analytic functions, set theory, real analysis, number theory, topology and mathematical logic. The first professional course in mathematical logic was taught by Jan Łukasiewicz (1878–1956), that lectured at the university since 1906 till 1914.

The base of a new scientific group of Lviv mathematicians was founded in 1917 by Hugo Steinhaus (1887–1972) who started to work at the university. Afterwards, some other mathematicians joined this group: in 1919 Stanisław Ruzewicz (1889–1941) and Uestachy Żyliński (1889–1954) and in 1922 the most gifted Lviv mathematician Stefan Banach (1882–1945).

It should be mentioned that the attitude of the university to Banach was informal due to his gift and scientific value of his works: he obtained the Ph. Doctor Degree (1920) and habilitation (1922) in spite of the absence of complete high education. After graduating Cracow Gymnasium in 1910 Stefan Banach studied in 1910–1914 in Lviv Polytechnics and passed the exam for two years of study.

In 1924 on the base of philosophical faculty there were created two faculties, mathematical-natural and humanitarian, which reflected the increasing role of mathematical and natural sciences in University.

The amount of classes in mathematics per week permanently increased (34 in 1924/25 and 53 in 1937/38). A special attention was paid to the organization of student seminars. Such seminars were supervised by Professors Steinhaus, Banach, Ruziewicz and Zyliński and also by some young mathematicians. Each year there were 1–2 seminars dedicated to some selected topics. A seminar in Theory of Functions and Functional Analysis functioned almost each year.

A collaboration between Banach and Steinhaus was especially fruitful. They supported young mathematicians and in 1929 founded "Studia Mathematica", the principal journal of Lviv mathematical school. The book "Théorie des Operations Lineaires" published by Banach in 1932 had great influence on the development of functional analysis in the world. The name of Banach is related also to such romantic pages of Lviv mathematics as "Scottish Cafe" and "Scottish Book" collecting open problems posed by Lviv mathematicians and their guests. "Scottish Cafe" became the place of discussions after Saturday meetings of Lviv mathematical society.

The spectrum of scientific interests of Lviv mathematicians was very wide. The principal achievements of this period were creating fundamenta of functional analysis, introducing topological methods in analysis, interpretation of probability as measure and applying probability methods to Fourier series.

A characteristic property of Lviv mathematicians was free operating by non-constructive methods, Axiom of Choice, Baire categories and Lebesgue integral. Lviv mathematical school whose indisputable leader was S.Banach made a considerable contribution in world mathematics.

In 1939 the entry of Soviet Army into western Ukraine resulted in reorganization of Lviv University. The Faculty of Physics and Mathematics was created and S.Banach became a dean of this faculty. By heads of chairs there were appointed: S. Banach (Chair of Analysis I), H.Steinhaus (Analysis II), E. Zyliński (Algebra), S. Mazur (Geometry), M. Zarycki (Probability), Ju. P. Schauder (Mechanics).
Myron Zarycki (1889-1961) graduated from Lviv University. His scientific interests were formed with the influence of lectures of Sierpiński. He obtained Ph.D. in 1930 and till 1939 he taught in Gymnasia of Galicia.

In 1940/41 many mathematicians came to the university from Poland, occupied by nazis: Stanislaw Saks (1897–1942), Bronislaw Knaster (1893–1980), Edward Szpilrajn (1907–1976), Mozes Jacob (1900–1944) and others.

Studying at the university interrupted as the war started and was renewed in 1944. Many Lviv mathematicians died during the war and the others moved to Poland in 1945–1946.

S. Banach died on the 31st of August of 1945. After his death the chair was headed by Wolodymyr Lewicki (1872–1956). W. Lewicki graduated from Lviv university. In addition to obtaining deep mathematical results he made a lot for creating Ukrainian mathematical terminology.


At the university the following main directions of scientific activity were formed: theory of probability and statistics (B. V. Gnidenko), partial derivatives differential equations (Ya. B. Lopatynskyi), theory of functions of complex variable (I. G. Volkovysky), theory of almost periodic functions and quadrability of surfaces (O. S. Kovanko), constructible function theory (I. G. Sokolov), geometry (G. L. Bujmola, V. F. Rogachenko), history of mathematics (B. V. Gnidenko, M. O. Zarytskyi, M. A. Chaikovskyi, V. F. Rogachenko), enumerative mathematics (O. M. Kostovskyi).

The results of investigations in the theory of functions, differential equations, algebra and other areas of mathematics in post-war period at Lviv university are described in [8], [9], [10], [11].

In September of 1953 the faculty of Physics and Mathematics was divided into the physical faculty and the faculty of mechanics and mathematics. In 1975 the faculty of mechanics and mathematics was divided into the faculty of mathematics, and that of applied mathematics and mechanics. In 1987 after transition of the chair of mechanics to the structure of mathematical faculty it became the faculty of mechanics and mathematics.

A substantial influence on development of mathematical investigations not only in Lviv University but also in the western region of Ukraine is due to foundation in 1991 the journal “Mathematychni Studii” and opening a Specialized Council for defense of candidate and doctor dissertations in mathematics.
The traditions of Lviv mathematics are still supported at the university. During the last decades there were created (or renewed) scientific schools in topology, algebra, logic, probability theory.

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Topology of Vector Fields
Volodymyr Sharko

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- Lecture 1. Vector fields on manifolds (general theory).
- Lecture 2. Lyapunov functions of flows and homotopy properties of manifolds.
- Lecture 3. Morse-Novikov theory of closed 1-form and application to flows.

On Some Perturbations Changing the Domain of Operator and Differential-Boundary Operators with Nonstandard Boundary Conditions
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To My Teacher Professor Władysław Elżbieta Ljantse in Occasion of His 85th Birthday

0. Introduction
Differential operators with nonstandard boundary conditions were investigated by many mathematicians. Some of these conditions are indicated in well-known reference book by E. Kamke devoted to ordinary differential equations. During the last 40-30 years the attention of many authors was directed into some perturbations of the mentioned operators. For example, A. M. Krall [1] has considered a differential operator $S_0$ generated in $L_2(a,b)$ by the expression

$$l[y] = -y'' + p(x)y$$  \hspace{1cm} (0.1)
and satisfying the conditions

\[ u_i(y) = \int_a^b y(x) \varphi_i(x) \, dx, \quad i = 1, 2. \]  

(0.2)

He showed that the adjoint operator \( S^* \) has the following form:

\[ D(S_0^*) = \{ z \in D(L_0^*) = D(L) \mid \nu_i(z) = 0, i = 1, 2 \}. \]  

(0.3)

\[ \forall z \in D(S_0^*) \quad S_0^* z = -z'' + p(x)z + \nu_0(z) \varphi_1 + \nu_4 \varphi_2. \]  

(0.4)

(here \( \nu_1, \nu_2, \nu_3, \nu_4 \) are boundary forms, \( \varphi_1, \varphi_2 \in L_2(a, b) \) and \( L, L_0 \) are respectively the maximal and minimal operators associated with the expression (0.1) in \( L_2(a, b) \).)

It should be noted that the operators of type (0.3)-(0.4) appear, for example, in the problem of descriptions of all accretive extensions for some differential operators. This was indicated by R. C. Phillips [2].

In our view it is interesting to consider a class of operators in \( L_2(a, b) \) (denote it by \( \mathbb{R} \)) satisfying the following requirements:

i) operator (0.1)-(0.2) belongs to \( \mathbb{R} \);

ii) operator (0.3)-(0.4) belongs to \( \mathbb{R} \);

iii) \( T \in \mathbb{R} \) implies \( T^* \in \mathbb{R} \).

This leads to studying so-called differential-boundary operators (in the sequel, DBO). This terminology belongs to A. M. Krall. One of the simplest DBO (with integral boundary conditions) may be presented as follows:

\[ \left\{ y \in D(L) \mid u_i(y) = \int_a^b y(x) \varphi_i(x) \, dx, \quad i = 1, 2 \right\} \]  

(0.5)

\[ \forall y \in D(S) \quad Sy = l[y] + u_3(y) \varphi_3 + u_4(y) \varphi_4. \]  

(0.6)

Note that R. C. Brown and A. V. Krall [3] have investigated differential operators with more general boundary conditions, for example,

\[ u_i(y) = \int_a^b y(x) \varphi_i(x) \, dx + \sum_{j=1}^k \gamma_j \cdot y(c_j) \quad (i = 1, 2) \]  

(0.7)

(multipoint-integral boundary conditions) or

\[ u_i(y) = \int_a^b \left[ y(x) \varphi_i(x) + y'(x) \varphi'_i(x) \right] \, dx \quad (i = 1, 2) \]  

(0.8)
A problem of constructing a corresponding abstract model arises in a natural way. One of such models has been developed by E. A. Coddington, A. Dijksma, H. S. V. de Snoo (see [4, 5] for examples), later on by V. M. Bruk, A. N. Kochubei and others (see [6-9]). This model is based on the theory of linear relations initiated by R. Arens [10].

We would like to remind some results of the mentioned authors.

Let $L_0$ be a closed symmetric operator in a Hilbert space $H$ and $L \overset{\text{def}}{=} L_0^*$ (for example: $L$ and $L_0$ such as above if $p(x) = p(x)$).

$$L_{\min} \subseteq L_0, \quad D(L_{\min}) = \{ y \in D(L_0) \mid (g/\varphi_i) = 0, i = 1, \ldots, r \}$$

\[(\varphi_1, \ldots, \varphi_r \in H).\] The following problems have been solved:

i) there was established the general form of the self-adjoint relation being the extension of $L_{\min}$;

ii) extensions-operators were selected from the self-adjoint extensions-relations of $L_{\min}$;

iii) there was proved that in the case of differential operators each self-adjoint operator extension of $L_{\min}$ is an additive bounded finite-dimensional perturbation of the operator (0.5)-(0.6) with $\varphi_1 = \varphi_3$, $\varphi_2 = \varphi_4$;

iv) there was established a general form of self-adjoint (later on of dissipative) extension for arbitrary closed linear symmetric (i.e., hermitian) relation in a Hilbert space, [11].

Another model has been proposed by W. E. Ljantse [12]. The aim of this report is to explain some aspects of his theory and to indicate possible generalizations.

**Basic notation**

We use the following denotation:

$D(T)$, $R(T)$, ker $T$ are, respectively, the domain, range, and kernel of a (linear) operator $T$;

$B(X, Y)$ is the set of linear bounded operators $A: X \to Y$ such that $D(A) = X$;

$B(X) \overset{\text{def}}{=} B(X, X)$;

$\mathbb{C}$ is a class of closed densely defined linear operators $A: X \to X$;
A ↓ E is the restriction of A onto E;

1_X is the identity of X.

If A: X → Y_i (i = 1, ..., n) are linear operators then notation A = A_1 ⊕ ... ⊕ A_n means that

∀x ∈ X \ A x = (A_1 x, ..., A_n).

The role of initial object is played by couple (L, L_0) of operators H → H (H is a fixed complex Hilbert space equipped with inner product (\cdot | \cdot) and norm || \cdot ||) such that

L, L_0 ∈ \mathbb{C}(H), \quad L_0 ∈ L, \quad M_0 \overset{def}{=} L_0^*, \quad M \overset{def}{=} L^*_0.

Under D[T] (T ∈ \mathbb{C}(H)) we understand the variety D(T) interpreted as Hilbert space with inner product:

∀y, z ∈ D(T) \quad (y | z)_T = (y | z) + (Ty | Tz)

and the corresponding graph-norm. By ⊕_T and ⊗_T we denote the symbols of orthogonal sum and orthogonal completion in D[T].

1. Abstract boundary operators

1.1. Definition [13]. Let G be an (auxiliary) Hilbert space and U ∈ B(D[L], G). The pair (G, U) is called a boundary pair for (L, L_0) if R(U) = G, ker U = D(L_0).

It was shown in the latter monograph that boundary pair exists and the following statement holds.

1.2. Theorem. Let G_1, G_2 be Hilbert spaces, U_i ∈ B(D[L], G_i) (i = 1, 2) and (G_1 ⊕ G_2, U_1 ⊕ U_2). There exist unique U_1 ∈ B(D[M], G_2), U_2 ∈ B(D[M], G_1) such that (G_2 ⊕ G_1, U_1 ⊕ U_2) is a boundary pair for (M, M_0) and

∀y ∈ D(L), ∀z ∈ D(M) \quad (L y | z) - (y | M z) = \left( U_1 y/ U_2 z \right)_{G_1} - \left( U_2 y/ U_1 z \right)_{G_2}. \quad (1.1)

Let us give two examples.

1.3. Example. Let L_0 be a symmetric operator with equal defect numbers and L = L_0. It was shown (see [6-9]) that there exists a Hilbert space \mathcal{H} and linear operators Γ_1, Γ_2: D(L) → \mathcal{H} satisfying the following conditions:
\[ R(\Gamma_1 \oplus \Gamma_2) = \mathcal{H} \oplus \mathcal{H}, \]  
\[ (\forall y, z \in D(L) \quad (Ly/z) - (y/Lz) = (\Gamma_1 y/\Gamma_2 z)_H - (\Gamma_2 y/\Gamma_1 z)_H. \]  

The triplet \((\mathcal{H}, \Gamma_1, \Gamma_2)\) was called a boundary value space (below, BVS) of \(L_0\). It follows from (\(\ast\)) and (\(\ast\ast\)) that

\[ \Gamma_1 \oplus \Gamma_2 \in B \left( D[L], \mathcal{H} \oplus \mathcal{H} \right), \quad \ker(\Gamma_1 \oplus \Gamma_2) = D(L_0). \]

Thus \((\mathcal{H} \oplus \Gamma_1 \oplus \Gamma_2)\) is a boundary pair for \((L, L_0)\).

**1.4. Example.** Let \(L_0\) be a symmetric operator with arbitrary defect numbers and \(L = L_0^\ast\). Then there exist Hilbert spaces \(G^\pm = \dim \ker(L \mp i1_H)\) and operators \(\delta_\pm \in H(D[L], G^\pm)\) satisfying the following conditions:

\[(G^+ \oplus G^-, \delta_+ \oplus \delta_-) \text{ is a boundary pair for } (L, L_0);
\]

\[ (\forall y, z \in D(L) \quad (Ly/z) - (y/Lz) = i[(\delta_+ y/\delta_- z)_{G^+} - (\delta_- y/\delta_+ z)_{G^-}]. \]

The bicouple \((G^+ \oplus G^-, \delta_+ \oplus \delta_-)\) is said to be an antisymmetric boundary value space of \(L_0\) (see [14] for the details).

Let us note that the Theorem 1.2 may be interpreted as some abstract analogy of Green formula (in other terminology, of Lagrange formula) and that is a corollary of the following statement proved by W. E. Ljantse [12] (see also [13]).

**1.5. Theorem.** Let us put

\[ H_L = D[L] \oplus_L D[L_0], \quad H_M = D[M] \oplus_M D[M_0]. \]

The following relations hold:

\[ LH_L = H_M, \quad MH_M = H_L, \]

\[ (\forall v \in H_M \quad LMv = -v, \quad \forall u \in H_L \quad MLu = -u. \]

**Remark.** In the case when \(L_0 \subseteq L_0^\ast = L\) this assertion follows from the well-known (established by J. von Neumann) decomposition

\[ D(L) = D(L_0) \oplus_L \ker(L - i1_H) \oplus_L \ker(L + i1_H). \]
2. Related operators

In this section some results indicated in [12–13] will be reviewed.

2.1. Definition. Let $T_1, T_2 \in \mathbb{C}(H)$. The operator $T_1$ is called related to the operator $T_2$ (and we write $T_1 \sim T_2$) if $T_1$ and $T_2$ have $S_0, S_1, \ldots, S_n \in \mathbb{C}(H) (S_0 = T_1, S_n = T_2)$ common finite-dimensional restriction.

2.2. Theorem. Operators $T_1, T_2 \in \mathbb{C}(H)$ are related iff each of them can be obtained from the other by finite number of finite-dimensional extensions and restrictions within the class $\mathbb{C}(H)$.

Remark. This means the following: There exist $S_0, S_1, \ldots, S_n \in \mathbb{C}(H)$ ($S_0 = T_1, S_n = T_2$) such that for each $k \in \{1, \ldots, n\}$ either $S_{k-1}$ is a finite-dimensional restriction of $S_k$ or $S_k$ is a finite-dimensional restriction of $S_{k-1}$.

2.3. Definition. Let $\dim D(L)/D(L_0) < \infty$. An operator $T \in \mathbb{C}(E)$ is called related to the couple $(L, L_0)$ (and we write $T \sim (L, L_0)$) if

\begin{enumerate}
  \item $T \sim L$ (therefore $T \sim (L, L_0)$),
  \item $D(T) \subset D(L)$,
  \item $D(T^*) \subset D(M)$, $D(L \downarrow D(D(T))^\star) \subset D(M)$.
\end{enumerate}

2.4. Theorem. An operator $T \in \mathbb{C}(H)$ is related to $(L, L_0)$ iff there exist

- a boundary pair $(G_1 \oplus G_2, U_1 \oplus U_2)$ for $(L, L_0),$
- operators $\Phi_i \in B(H, G_i)$, $i = 1, 2,$
- a finite-dimensional operator $V \in B(H)$ such that $T = S + V$, where

\begin{align}
  D(S) &= \{ y \in D(L) \mid U_1 y = \Phi_1 y \}, \\
  \forall y \in D(S) \quad S y &= L y + \Phi_2^* y.
\end{align}

In this situation (see Theorem 1.2)

\begin{align}
  D(S^\star) &= \{ z \in D(M) \mid \tilde{U}_1 z = \Phi_2 z \}, \\
  \forall z \in D(S^\star) \quad S^\star &= M z + \Phi_1^* z.
\end{align}

2.5. Corollary. Operators (2.1)–(2.2) and (2.3)–(2.4) are closed and densely defined under arbitrary $\Phi_i \in B(H, G_i)$, $i = 1, 2.$
3. Smooth restrictions and almost trivial perturbations

Taking into account the theory stated above everybody can easily verify that operator (0.5)–(0.6) is a closed densely defined operator. Moreover, it is clear that the theory of related operators can be applied to differential-boundary operators acting in $L_2(H_0; \Gamma(a,b))$, if $H_0$ is a finite-dimensional Hilbert space. A problem of constructing of finite-dimensional analog of the theory stated above arises in a natural way.

Let us consider one variant of such construction, proposed in [13].

3.1. Definition. Restriction $\hat{L}$ of $L$ is called smooth one if $\hat{L} \in \mathcal{C}(H)$, $D(\hat{L}^*) \subset D(M)$, $D(\hat{L}^*)$ is closed in $D[M]$.

3.2. Definition. Operator $Q \in \mathcal{B}(D[L], H)$ is called almost trivial (with respect to $(L, L_0)$) if $Q \perp D(L_0) = 0$.

3.3. Theorem. Assume that $S$ is an almost trivial perturbation of a smooth restriction of $L$ and $\dim G = \dim H_L$ ($G$ is a Hilbert space). Then there exists an orthogonal decomposition $G = G_1 \oplus G_2$ and

\[ U_i \in \mathcal{B}(D[L], G_i), \quad \Phi_i \in \mathcal{B}(H, G_i) \quad (i = 1, 2) \]

such that $(G_1 \oplus G_2, U_1 \oplus U_2)$ is a boundary pair for $(L, L_0)$ and

\[ R(U_1 - \Phi_1) = R(U_1) = G_1, \quad (3.1) \]

\[ D(S) = \{ y \in D(L) \mid U_1 y = \Phi_1 y \}, \quad (3.2) \]

\[ \forall y \in D(S) \quad S y = L y + \Phi_1^* U_2 y. \quad (3.3) \]

3.4. Theorem. Let $G_1$, $G_2$ be Hilbert spaces, $U_i \in \mathcal{B}(D[L], G_i)$, $\Phi_i \in \mathcal{B}(H, G_i)$ $(i = 1, 2)$ such that $(G_1 \oplus G_2, U_1 \oplus U_2)$ is a boundary pair for $(L, L_0)$, $R(U_1 - \Phi_1)$ is closed in $G_1$, $\hat{U}_1$, $\hat{U}_2$ are uniquely determined by relation (1.1), and operator $S$ by relations (3.2), (3.3). Then

i) $S$ is a densely defined and (3.1) holds;

ii) $L \perp D(S)$ is a smooth restriction of $L$;

iii) $D(S^*) = \{ z \in D(M) \mid \hat{U}_1 z = \Phi_2 z \}$, $\forall z \in D(S^*) \quad S^* z = M z$. (3.4) (3.5)
3.5. Corollary. Under the conditions of Theorem 3.4, $S^*$ is an almost trivial perturbation of a smooth (with respect to $(M, M_0)$) restriction of $M$ iff $R(\hat{U}_1 - \Phi_2) = G_2$. In this case $S \in \mathcal{C}(H)$.

It is established that the relations

$$R(U_1 - \Phi_1) = G_1, \quad R(\hat{U}_1 - \Phi_2) = G_2,$$

which yield $S \in \mathcal{C}(H)$, hold whenever one of the both requirements is fulfilled:

i) $\Phi_1, \Phi_2$ are compact operators;

ii) $U_1, U_2$ have $L$-bounds equal to zero and $\hat{U}_1, \hat{U}_2$ have $M$-bounds equal to zero.

In the sequel we assume (for the simplicity) that at least one of both hypotheses takes place.

3.6. Corollary. Assume that $L_0$ is a symmetric operator with equal defect numbers, $L \overset{\text{def}}{=} L_0^*$, $(\mathcal{H}, \Gamma_1, \Gamma_2)$ is an BVS of $L_0$, and $A \overset{\text{def}}{=} (A_{ij})_{i,j=1}^2$ is a bijection $\mathcal{H} \oplus \mathcal{H} \rightarrow \mathcal{H} \oplus \mathcal{H}$, $A^{-1} \overset{\text{def}}{=} (B_{ij})_{i,j=1}^2$.

Define an operator $S$ by the relations

$$D(S) = \{ y \in D(L) \mid A_{11} \Gamma_1 y + A_{12} \Gamma_2 y = \Phi_1 y \}$$

$$\forall y \in D(S) \quad Sy = Ly + \Phi_2^* (A_{21} \Gamma_1 y + A_{22} \Gamma_2 y).$$

In this case

$$D(S^*) = \{ z \in D(M) \mid B_{11}^* \Gamma_1 z + B_{12}^* \Gamma_2 z = \Phi_2 z \}.$$ 

$$\forall z \in D(S^*) \quad S^* = MZ + \Phi_2^* (-B_{11}^* \Gamma_1 z + B_{12}^* \Gamma_2 z).$$

3.7. Corollary. Operator $(3.7)-(3.8)$ is self-adjoint iff there exist operators $\Phi \in B(\mathcal{H}, \mathcal{H}), C = C^*, \Omega = \Omega^* \in B(\mathcal{H})$ such that

$$D(S) = \{ y \in D(L) \mid (\cos C) \Gamma_1 y - (\sin C) \Gamma_2 y = \Phi y \}.$$ 

$$\forall y \in D(S) \quad Sy = Ly + \Phi^* [(\sin C) \Gamma_1 y + (\cos C) \Gamma_2 y] + \Phi^* \Omega \Phi y.$$ 

Assume now that $V_1 \in B(D[M], G_2), V_2 \in B(D[M], G_1)$ and put

$$G = G_1 \oplus G_2, \quad \bar{G} = G_2 \oplus G_1, \quad U = U_1 \oplus U_2, V = V_1 \oplus V_2.$$
Further, let us define an (adjoint) operator $V'$ as follows:

$$\forall z \in D(M), \forall g \in \tilde{G}, \quad (V'z)_{g} = (z[V'g])_{M}.$$

Suppose that $(\tilde{G}, V)$ is a boundary pair for $(M, M_{0})$ and define operator $T$ by the relations

$$D(T) = \{ z \in D(M) \mid V_{1}z = \Phi_{2}z \},$$

$$(3.9)$$

$$\forall z \in D(T) \quad Tz = Mz + \Phi_{1}V_{2}z.$$  

(3.10)

Operators (3.2)–(3.3) and (3.9)–(3.10) are interpreted below as perturbations of operators

$$L_{1} \overset{\text{def}}{=} L \downarrow \ker U_{1}, \quad M_{1} \overset{\text{def}}{=} M \downarrow \ker V_{1}$$

(3.11)

respectively. It should be noted that each of these perturbations changes the action of the unperturbed operator as well as its domain. We mean to say that substituting $\Phi_{1} = 0$, $\Phi_{2} = 0$ in (3.2)–(3.3), (3.9)–(3.10) we obtain $S = L_{1}$, $T = M_{1}$.

3.8. Corollary. Suppose (for the simplicity) that, in additions, indicated below, $\overline{R}(\Phi_{1}) = G_{1}$, $\overline{R}(\Phi_{2}) = G_{2}$ Then $S$ and $T$ are mutually adjoint iff

i) $UMV' = \begin{pmatrix} 0 & -1_{G_{1}} \\ 1_{G_{1}} & 0 \end{pmatrix}$, where $UMV'$ is interpreted as a mapping

$G_{2} \oplus G_{1} \to G_{1} \oplus G_{2}$;

ii) $L_{1}^* = M_{1}$.

4. Criterion of maximal dissipativity for perturbed operators

Let us recall that a linear operator $T : H \to H$ is called dissipative (accumulative) if $\forall y \in D(T)$ $\text{Im}(Ty)y \geq 0$ ($\text{Im}(Ty)y \leq 0$) and maximal dissipative (maximal accumulative) if, in addition, it has no dissipative (accumulative) extensions in $H$. It is known (see [2, 15]) that $T$ is a self-adjoint operator if it is maximal dissipative and maximal accumulative simultaneously. In this section the situation is considered when $L_{0} \subset L_{1}^* = L$. Assume that $(G_{1} \oplus G_{2}, U_{1} \oplus U_{2})$ is a boundary pair for $(L, L_{0})$ and $\Phi_{i} \in B(H, G_{i})$ $(i = 1, 2)$. The purpose of this section is to establish the criterion of maximal dissipativity and maximal accumulativity for operator (3.2)–(3.3).

It may be shown that maximal dissipativity (accumulativity) of this operator implies the existence of a bijection $\tilde{C} \in B \left( \overline{R}(\Phi_{2}), \overline{R}(\Phi_{2}) \right)$ satisfying
the equality $\Phi_1 = \bar{\Phi}_2$, that is why without loss of generality we deal with an operator $S$ defined as follows:

$$D(S) = \{ y \in D(L) \mid U_1 y = C \Phi y \},$$  \hspace{1cm} (4.1)

$$\forall y \in D(S) \quad Sy = Ly + \Phi^* U_2 y,$$  \hspace{1cm} (4.2)

where $\Phi \in B(H, C_2)$, $C \in B(G_2, G_1)$.

Furthermore, let us denote by $P$ the orthoprojection $G_1 \oplus G_2 \to G_1 \oplus \overline{R(\Phi)}$. The following statement holds.

4.1. **Theorem.** Operator (4.1) – (4.2) is maximal dissipative (maximal accumulative; self-adjoint) iff

$$i) \quad P \left[ U L U^* - \begin{pmatrix} 0 & -iC \ni \end{pmatrix} \right] P \geq 0 \left( P \left[ U L U^* - \begin{pmatrix} 0 & -iC \ni \end{pmatrix} \right] P \leq 0, \right)

$$

$$P \left[ U L U^* - \begin{pmatrix} 0 & -iC \ni \end{pmatrix} \right] P = 0 \right)

$$

$$ii) \quad L_1 \overset{\text{def}}{=} L \perp \ker U_1 \text{ is a maximal dissipative (maximal accumulative; self-adjoint) operator.}$$

**Remark.** We suppose that at least one of two requirements is fulfilled:

i) $\Phi$ is a compact operator;

ii) $U_1, U_2$ have $L$-bound equal to zero.

4.2. **Corollary.** Suppose that $L_0$ has equal defect numbers, $(H, \Gamma_1, \Gamma_2)$ is a BVS of $L_0$, $\Phi \in B(H, H)$, and $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ is invertible in $B(H \oplus H)$ Define an operator $S$ by the relations

$$D(S) = \{ y \in D(L) \mid A_{11} \Gamma_1 y + A_{12} \Gamma_2 y = \Phi y \},$$  \hspace{1cm} (4.3)

$$\forall y \in D(S) \quad Sy = Ly + \Phi^* (A_{21} \Gamma_1 y + A_{22} \Gamma_2 y),$$  \hspace{1cm} (4.4)

Denote by $P$ the orthoprojection $H \oplus H \to H \oplus \overline{R(\Phi)}$ and put $\forall h_1, h_2 \in H$ $J(h_1, h_2) = (ih_1, -ih_2)$.

Operator (4.3) – (4.4) is maximal dissipative (maximal accumulative; self-adjoint) iff

$$i) \quad P(A J A^* - J) P \leq 0 \ (P(A J A^* - J) P \geq 0; \ P(A J A^* - J) P = 0)$$
ii) \( \ker(A_{11} + iA_{12}) = \{0\} \) \( \ker(A_{11} - iA_{12}) = \{0\} \) \( \ker(A_{11} \pm iA_{12}) = \{0\} \).

4.3. Corollary. Assume that \((G_2, G_1, \delta_+, \delta_-)\) is an antisymmetric BVS of \(L_0\), \(C \in B(G_2, G_1)\), \(\Phi \in B(H, G_2)\). \(P\) is the orthoprojection \(G_1 \oplus G_2 \to G_1 \oplus B(\Phi)\) and \(A = (A_{ij})_{i,j=1}^2\) is invertible in \(B(G_1 \oplus G_2)\). An operator \(S\) defined by the relations

\[
D(S) = \{ y \in D(L) \mid A_{11} \delta_- y + A_{12} \delta_+ y = C \Phi y \},
\]

\(\forall y \in D(S)\) \(Sy = Ly + \Phi^*(A_{21} \delta_- y + A_{22} \delta_+ y)\),

is maximal dissipative (maximal accumulative; self-adjoint) iff

i) \[ P \left[ \begin{array}{cc}
-1_{G_1} & 0 \\
0 & 1_{G_1}
\end{array} \right] A^* - \left[ \begin{array}{cc}
0 & 1_{G_1} \\
-1_{G_1} & 0
\end{array} \right] P \leq 0 \]

\(P \left[ \begin{array}{cc}
1_{G_1} & 0 \\
0 & 1_{G_1}
\end{array} \right] A^* - \left[ \begin{array}{cc}
0 & 1_{G_1} \\
-1_{G_1} & 0
\end{array} \right] P \geq 0\);

ii) \(L_1\) is maximal dissipative (maximal accumulative; self-adjoint).

5. Criteria of maximal positivity and accretivity for perturbated operators

In this section we suppose that \(L_0\) is positively definite operator and \(L^* \equiv L_0^*\).

5.1. Definition [6, 16]. A BVS \((\mathcal{H}, \Gamma_1, \Gamma_2)\) of \(L_0\) is said to be a positive one if

\( \ker \Gamma_1 = D(L_0) + \ker L, \quad L \downarrow \ker \Gamma_2 \gg 0. \)

Remark. In the sequel we wish that \(L = \ker \Gamma_2 = L_F, \) where \(L_F\) is the Friedrichs extension of \(L_0\).

5.2. Definition. An operator \(T : H \to H\) is called accretive if \(\forall y \in D(T)\) \(\Re(T y^* y) \geq 0\) and maximal accretive if, in addition, it has no accretive extensions.

Let \((\mathcal{H}, \Gamma_1, \Gamma_2)\) be a positive BVS of \(L_0\), \(B \in B(\mathcal{H})\) and \(\Phi \in B_{sa}(H, \mathcal{H})\) (i.e., \(\Phi : H \to \mathcal{H}\) is a compact operator). Put \(Z \overset{\text{def}}{=} (\Gamma_1, L_F^{-1})^*\) (it may be shown that \(y = Z a \iff Ly = 0, \Gamma_2 y = a\)).
5.3. **Theorem.** Operator $S_B$ defined by the relations

$$D(S_B) = \{ y \in D(L) \mid \Gamma_1 y - B \Gamma_2 y = \Phi y \}, \quad \forall y \in D(S_B) \quad S_B y = Ly + \Phi^* \Gamma_2 y,$$

is maximal accretive (maximal nonnegative, maximal positively definite) iff $\tilde{B} \overset{\Delta}{=} 2 \text{Re}(\Phi Z) - \Phi L^*_P \Phi^* + B$ is an accretive (nonnegative, positively definite) operator.

6. **Applications to differential-boundary operators in vector-function spaces**

In this section the results stated above are applied in the situation, when the operators $L$ and $L_0$, which played the role of the initial objects in the previous sections, are respectively the maximal and minimal operators generated in a Hilbert space $H \overset{\Delta}{=} L_2(H_0; (a, b))$, where $H_0$ is a separable Hilbert space equipped with inner product

$$\forall y, x \in H \quad (y | z) = \int_a^b (y(x) | z(x))_{H_0} \, dx$$

by differential expression

$$l[y] = -y'' + p(x)y \quad (x \in [a, b], -\infty < a < b < +\infty), \quad (6.1)$$

where $p(x)$ is a bounded self-adjoint operator in $H_0$, while the operator-function $x \mapsto p(x)$ is strongly continuous on $[a, b]$. Let us denote by $L$ and $L_0$ the maximal and minimal operators associated with expression (6.1), respectively. It is known (see [17] and reference therein) that the triplet $(H, \Gamma_1, \Gamma_2)$, where

$$H = H_0 \oplus H_0, \quad \Gamma_1 y = (y'(a), -y'(b)), \quad \Gamma_2 y = (y(a), y(b)),$$

is a BVS for $L_0$.

6.1. **Lemma.** The operators $\Gamma_1$, $\Gamma_2$ have $L$-bounds equal to zero.

Assume that $\Phi_{ij} \in B(H, H_0)$ $(i, j = 1, 2)$; $\alpha_{ij} \in B(H_0)$ $(i, j = 1, \ldots, 4)$, moreover, the operator $A = (\alpha_{ij})_{i,j=1}^4$ is invertible in $H_0^4 \overset{\Delta}{=} \bigoplus_{i=1}^4 H_0$. Let us put

$$u_i(y) = \alpha_{i1}y(a) + \alpha_{i2}y'(a) + \alpha_{i3}y(b) + \alpha_{i4}y'(b) \quad (y \in D(L), i = 1, \ldots, 4)$$

and define operator $S$ as follows:

$$D(S) = \{ y \in D(L) \mid u_1(y) = \Phi_{11} y, \; u_2(y) = \Phi_{12} y \}. \quad (6.2)$$
\[ \forall y \in D(S) \quad S y = L y + \Phi_{i1} u_3(y) + \Phi_{i2} u_4(y). \quad (6.3) \]

By virtue of Lemma 6.1, operator (6.2)–(6.3) is densely defined and closed.

Let us indicate some of applications. At first, note that the symbols 1 and 0 are used below for the identity and zero operators in \( H_0 \), respectively. In addition, put

\[
E = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad J = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & -1 \\
0 & -1 & 0 & 0
\end{pmatrix},
\]

\[ \hat{A} = (\hat{\alpha}_{ij})_{ij=1}^4 = -J(A^*)^{-1}E. \]

6.3. Theorem. Under the conditions mentioned above

\[ D(S^*) = \{ Z \in D(L) \mid \hat{u}_1(z) = \Phi_3(z), \hat{u}_2(z) = \Phi_4(z) \} \]

\[ \forall z \in D(S^*) \quad S^* z = L z + \Phi_1^* \hat{u}_3 z + \Phi_2^* \hat{u}_4 z, \]

where \( \hat{U}_i \) is obtained from \( u_i \) by replacing \( \alpha_{ij} \to \hat{\alpha}_{ij} \).

6.3. Theorem. Suppose that \( \Phi_3 = \Phi_1, \Phi_4 = \Phi_2 \), denote by \( P \) the ortho-

projection \( \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{R}(\Phi_1 + \Phi_2) \) and put

\[
A_1 = \begin{pmatrix}
\alpha_{12} & -\alpha_{14} \\
\alpha_{22} & -\alpha_{24}
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
\alpha_{11} & \alpha_{13} \\
\alpha_{21} & \alpha_{23}
\end{pmatrix}.
\]

Operator (6.2)–(6.3) is maximal dissipative (maximal accumulative; self-

adjoint) iff

i) \( P(i A E A^* + J)P \geq 0 \) (\( P(i A E A^* + J)P \leq 0 \), \( P(i A E A^* + J)P = 0 \))

ii) \( \ker(A_{11} + i A_{12}) = \{0\} \) (\( \ker(A_{11} - i A_{12}) = \{0\} \); \( \ker(A_{11} \pm i A_{12}) = \{0\} \)).

References


Category, Functors and Monads

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1. Categories and functors

Category theory studies mathematical objects and morphisms between them in most abstract setting. Category theory was originated about 60 years ago in order to provide a general formalism that allowed to consider, in a uniform manner, various problems arising in different parts of mathematics. The notion of category reflects a ‘sociological’ approach to mathematics: every object in considered from the point of view of its relations to the other objects rather than its inner structure.

The category theory is not only a language. Recently, the category theory found applications in the semantics of programming languages. It turned out that the Categorical viewpoint provides allows for better understanding of basic motivations of computer science.

The lectures are based on a textbook in the category theory by M. Komarnyts’kyi and the author (in preparation). These notes will help to make some impression on the topics.

1.1. Categories

A category $\mathcal{C}$ consists of the following data:

- a collection of objects (denoted $\text{Ob}\mathcal{C}$ or $|\mathcal{C}|$),

- a collection of arrows (often called morphisms) which is a disjoint union of sets $\mathcal{C}(A, B)$ of the morphisms form $A$ to $B$, $A, B \in |\mathcal{C}|$.

- an operation of composition

$$\mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C), \quad (f, g) \mapsto gf,$$

satisfying the properties:

(a) (associativity): $f(gh) = (fg)h$
(b) (identity): for any $X \in [\mathcal{C}]$ there is a morphism $1_X : X \rightarrow X$ (the identity) such that $1_X f = f$ and $g 1_Y = g$ for any morphisms $f, g$ composable with $1_X$.

That $f \in \mathcal{C}(A, B)$ is often expressed as $f : A \rightarrow B$.

### 1.2. Examples

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### 1.3. Functors

Let $\mathcal{C}, \mathcal{D}$ be categories. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is a map that sends every object $A$ of $\mathcal{C}$ to an object $F(A)$ of $\mathcal{D}$ and every morphism $f : X \rightarrow Y$ in $\mathcal{C}$ to a morphism $F(f) : F(X) \rightarrow F(Y)$ in $\mathcal{D}$ so that $F(fg) = F(f)F(g)$ and $F(1_X) = 1_{F(X)}$.

**Example.** Let $A \in [\mathcal{C}]$. The data $F(X) = \mathcal{C}(A, X)$, $F(f)(g) = gf$, determine the functor $F : \mathcal{C} \rightarrow \text{SETS}$ (the so-called covariant Hom-functor).

**Another (more or less general) example:** the forgetful functor $U : \text{GROUPS} \rightarrow \text{SETS}$: $U(G) = G$ considered as a set), $U(f) = f$ (considered as a map of sets). The functor $U$ forgets about the group structure.

A *contravariant* functor, $F$, from $\mathcal{C} \rightarrow \mathcal{D}$ is a map that sends the objects of $\mathcal{C}$ to objects of $\mathcal{D}$ and every morphism $f : X \rightarrow Y$ in $\mathcal{C}$ to a morphism $F(f) : F(Y) \rightarrow F(X)$ in $\mathcal{D}$ so that $F(1_X) = 1_{F(X)}$ and $F(gf) = F(f)F(g)$. 
Example: $F : \textbf{TOP} \rightarrow \textbf{TOP}$, $F(X) = C(X)$ (the space of continuous functions on $X$ endowed with the pointwise convergence topology), for $f : X \rightarrow Y$, the map $F(f) : C(Y) \rightarrow C(X)$ is defined by the formula: $F(f)(\varphi) = \varphi f$.

1.4. Natural transformations

The natural transformations are arrows between functors. More formally, if $F, G : \mathcal{C} \rightarrow \mathcal{D}$ are functors between categories $\mathcal{C}, \mathcal{D}$, then a natural transformation $\varphi : F \rightarrow G$ consists of a collection of morphisms $(\varphi_X : F(X) \rightarrow G(X))_{X \in \mathcal{C}}$ in $\mathcal{D}$ such that, for any morphism $f : X \rightarrow Y$ in $\mathcal{C}$ the diagram

$$
\begin{array}{ccc}
F(X) & \xrightarrow{\varphi_X} & G(X) \\
\downarrow F(f) & & \downarrow G(f) \\
F(Y) & \xrightarrow{\varphi_Y} & G(Y)
\end{array}
$$

is commutative.

Under some natural conditions, one can imagine a category whose objects are functors and whose morphisms are natural transformations.

1.5. Adjoint functors

Let $\mathcal{C}, \mathcal{D}$ be categories, $U : \mathcal{C} \rightarrow \mathcal{D}$, $F : \mathcal{D} \rightarrow \mathcal{C}$ functors. We say that $F$ is right adjoint to $U$ (and $U$ is left adjoint to $F$) if, for every $X, Y$ there exists a bijection $\varphi_{XY} : \mathcal{D}(U(X), Y) \rightarrow \mathcal{C}(X, F(Y))$ which is natural with respect to $X$ and $Y$. The naturality with respect to the first argument means that, for any $f : X \rightarrow X'$ in $\mathcal{C}$ the diagram

$$
\begin{array}{ccc}
\mathcal{D}(U(X), Y) & \xrightarrow{\varphi_{XY}} & \mathcal{C}(X, F(Y)) \\
\downarrow & & \downarrow \\
\mathcal{D}(U(X'), Y) & \xrightarrow{\varphi_{X'Y}} & \mathcal{C}(X', F(Y))
\end{array}
$$

is commutative. In a similar fashion, the naturality with respect to the second argument is defined.

The situation of adjunction arises if the functor $U$ is a forgetful functor and $F$ is the free functor. Example: $U : \textbf{GROUPS} \rightarrow \textbf{SETS}$ is a forgetful functor, $F : \textbf{SETS} \rightarrow \textbf{GROUPS}$ is the free group functor.
2. Monads

Citation: “The underlying idea of monads in computer science is the distinction between simple data-valued functions and functions that perform computations: A data-valued function is one in which the returned value is solely determined by the values of its arguments (no side-effects). A function that performs a computation can encompass ideas such as state or non-determinism. Moreover, as a consequence of its application, a function that performs a computation produce implicitly more results than the explicitly returned value”.

If $T$ is an endofunctor in a category $\mathcal{C}$ and $\eta: 1_\mathcal{C} \to T$ and $\mu: T^2 \cong TT \to T$ are natural transformations, then $\mathbb{T} = (T, \eta, \mu)$ is called a triple (monad) if and only if the following diagrams commute:

\[
\begin{array}{c}
T \xrightarrow{\eta} T^2 \\
\downarrow T \eta \quad \quad \quad \quad \downarrow 1_T \\
T^2 \xrightarrow{\mu} T
\end{array}
\quad \quad \quad
\begin{array}{c}
T^3 \xrightarrow{\mu T} T^2 \\
\downarrow T \mu \quad \quad \quad \downarrow \mu \\
T^2 \xrightarrow{\mu} T
\end{array}
\]

Then $\eta$ is called the unity and $\mu$ the multiplication of $\mathbb{T}$.

**Example:** $\mathcal{C} = \text{SETS}$, $T = 2^{(-)}$ (the power set functor), $\eta_X(x) = \{x\}$, $\mu_X(A) = \cup A$.

3. Algebras and Kleisli categories

There are two categories related to every monad, namely, the Eilenberg-Moore and Kleisli category.

3.1. Eilenberg-Moore category

For an arbitrary triple $\mathbb{T} = (\mathbb{T}, \eta, \mu)$ in $\mathcal{C}$ a pair $(X, f)$, where $f: TX \to X$ is a morphism in $\mathcal{C}$, is called a $\mathbb{T}$-algebra iff the following commute:

\[
\begin{array}{c}
X \xrightarrow{\eta_X} TX \\
\downarrow 1_X \quad \quad \quad \downarrow f \\
X \xrightarrow{f} TX \xrightarrow{\mu_X} TX
\end{array}
\]
We provide examples of monads in algebra, topology, functional analysis, and (even) applications of monads to semantics of programming languages.

An arrow \( \phi: X \to Y \) is called a map of algebras \( (X, f) \to (Y, g) \) if and only if the diagram

\[
\begin{array}{ccc}
TX & \xrightarrow{T\phi} & TY \\
\downarrow f & & \downarrow g \\
X & \xrightarrow{\phi} & Y
\end{array}
\]

commutes. The category of algebras of a monad \( T \) is denoted by \( C^T \). We obtain two adjoint functors, \( U^T: C^T \to C \) defined by \( U(X, f) = X, U\phi = \phi \), and the free \( T \)-algebra functor \( F^T: C \to C^T, F^T(X) = (TX, \mu_X) \).

### 3.2. Kleisli category

The Kleisli category of a monad \( T \) is the category \( C_T \) defined as follows: \( |C_T| = |C|, C_T(X, Y) = C(X, TY) \), and the composition \( g \circ f \) of morphisms \( f \in C_T(X, Y), g \in C_T(Y, Z) \) is given by \( g \circ f = \mu_Z \circ Tg \circ f \).

Actually, there is a pair of adjoint functors corresponding to the Kleisli category of a monad.

Define the functor \( I: C \to C_T \) by \( IX = X, X \subseteq |C| \) and \( IF = \eta_Y \circ f \) for \( f \in C(X, Y) \).

A functor \( F^T: C \to C_T \) called an extension of the functor \( F: C \to C \) on the Kleisli category \( C_T \) if \( IF = TF \).

There exists a bijective correspondence between the extensions of a contravariant functor \( F \) onto the category \( C_T \) and the natural transformations \( \xi: F \to TF \) satisfying the conditions:

(i) \( TF\eta \circ \xi = \eta F \);

(ii) \( TF\mu \circ \xi = \mu F^{T^2} \circ T\xi T \circ \xi \).

### References


Abstracts of Research Reports

Coarse Topology on Functional Spaces

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Spaces of coarse maps of metric are considered. We introduce a coarse topology on these spaces, which, in some sense, is a counterpart of the fine Whitney topology [R].

Let \( C(X, Y) \) denote the set of coarse maps (see [R1] for the definition) from a topological space \( X \) to a topological coarse space \( Y \).

Introduce the equivalence relation \( \sim \) on spaces of coarse maps:

\[ f \sim g \iff \text{there exists } C > 0 \text{ such that } \rho(f(x), g(x)) < C \text{ for all } x \in X. \]

Let \( [f(x)] = \{ g(x) \mid g(x) \sim f(x) \} \) denote the equivalence class and \( \tilde{C}(X, Y) \) denote the spaces of all equivalence classes.

A proper metric space \( (Y, g) \) we denote by \( E \) the set of continuous function \( \varepsilon : Y \to (0, \infty) \) such that \( \lim_{x \to \infty} \varepsilon(x) = \infty. \)

**Theorem 1.** Let \( f(x) \in \tilde{C}(X, Y) \), \( \varepsilon(x) \in E \). The collection

\[ O([f(x)], \varepsilon(x)) = \left\{ [g(x)] \mid \frac{\rho(f(x), g(x))}{\varepsilon(x)} \xrightarrow{\varepsilon \to \infty} 0, \ \varepsilon(x) \in E \right\}, \]

forms a basis for coarse topology.

**Proposition 2.** The space \( \tilde{C}(X, \mathbb{R}) \) is not separable.

**Theorem 2.** Let \( X \) be a proper, not compact metric space. Then \( \tilde{C}(X, \mathbb{R}) \) is not connected.
References

Some Properties of Topological Spaces of Scatteredly Continuous Maps

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Let $X$, $Y$ be topological spaces and $\mathcal{P}$ a family of nonempty subsets of $X$. A map $f: X \to Y$ is called $\mathcal{P}$-scatteredly continuous if for arbitrary $A \in \mathcal{P}$ the map $f|_A: A \to F(A)$ has a point of continuity. If the family $\mathcal{P}$ consists of all nonempty subsets of the set $X$, then the $\mathcal{P}$-scatteredly continuous maps $f: X \to Y$ are called scatteredly continuous.

In the talk we shall discuss the following interplay between the topological properties of a a Hausdorff space $X$ and properties of the space $SC(X) \subset \mathbb{R}^X$ of scatteredly continuous real-valued functions on $X$ endowed with the topology of point-wise convergence.

Pseudo-Compact Semitopological Semigroups of Matrix Units

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A topological space $S$ that is algebraically a semigroup with the separately (jointly) continuous semigroup operation is called a semitopological (topological) semigroup. A semitopological (topological) inverse semigroup
is a semitopological (topological) semigroup $S$ which is algebraically an inverse semigroup and the inversion is continuous in $S$.

We follow the terminology of [3, 4, 6]. In this paper all spaces are Hausdorff.

Let $I_\lambda$ be a set of cardinality $\lambda \geq 2$. On the set $B_\lambda = I_\lambda \times I_\lambda \cup \{0\}$ we define the semigroup operation \("\) as follows:

$$(\alpha, \beta) \cdot (\gamma, \delta) = \begin{cases} (\alpha, \delta), & \text{if } \beta = \gamma, \\ (\alpha, \beta), & \text{if } \beta \neq \gamma, \end{cases}$$

and $(\alpha, \beta) \cdot 0 = 0 \cdot (\alpha, \beta) = 0 \cdot 0 = 0$ for $\alpha, \beta, \gamma, \delta \in I_\lambda$. The semigroup $B_\lambda$ is called the semigroup of $I_\lambda \times I_\lambda$-matrix units. The infinite topological semigroup of matrix units has topological properties similar to those of the bicyclic semigroup. For example, the bicyclic semigroup admits only discrete semigroup topology [5] and any non-zero element of the infinite topological semigroup of matrix units $B_\lambda$ is an isolated point in $B_\lambda$ [7]. L. W. Anderson, R. P. Hunter and R. J. Koch in [1] proved that the bicyclic semigroup cannot be embedded into a stable semigroup, and hence into a compact topological semigroup. Also the authors proved that a compact semigroup contains no infinite semigroup of matrix units [7]. M. O. Bertman and T. T. West proved that any Hausdorff topology $\tau$ on the bicyclic semigroup $B(p, q)$ such that $(B(p, q), \tau)$ is a semitopological semigroup is discrete [2]. Also in [2] they showed that the bicyclic semigroup can be embedded into a compact semitopological semigroup. In our paper we shall prove that if the infinite semigroup of matrix units $B_\lambda$ is a semitopological semigroup, then any non-zero element of $B_\lambda$ is isolated and show that $B_\lambda$ admits a compact topology which turns $B_\lambda$ into a semitopological semigroup. Also we describe all pseudo-compact topologies $\tau$ on $B_\lambda$ such that $(B_\lambda, \tau)$ is a semitopological semigroup.

**Lemma.** Let $\tau$ be a topology on $B_\lambda$ under which it is a semitopological semigroup. Then any nonzero element of $B_\lambda$ is an isolated point of $(B_\lambda, \tau)$.

**Example.** Let $\lambda$ be an infinite cardinal. A topology $\tau_c$ on $B_\lambda$ is defined as follows:

1. all nonzero elements of $B_\lambda$ are isolated points in $B_\lambda$;
2. $B(0) = \{U \subseteq B_\lambda \mid 0 \in U$ and the set $B_\lambda \setminus U$ is finite $\}$ is the base of the topology $\tau_c$ at the zero of $B_\lambda$.

**Proposition.** $(B_\lambda, \tau_c)$ is a compact semitopological inverse semigroup.
Corollary. Let $\lambda$ be an infinite cardinal. Then there exists no other topology $\tau$ on $B_\lambda$ such that $\tau \neq \tau_c$ and $(B_\lambda, \tau)$ is a compact semitopological semigroup.

A topological space $X$ is called countably compact if any countable open cover of $X$ contains a finite subcover [6]. A topological space $X$ is called pseudo-compact (discretely pseudo-compact) if any locally finite (discrete) collection of open subsets of $X$ is finite.

Theorem. Let $\lambda$ be an infinite cardinal and $\tau$ be a topology on the semigroup of matrix units $B_\lambda$ such that it is a semitopological semigroup. Then the following statements are equivalent:

(i) $(B_\lambda, \tau)$ is a compact semitopological semigroup;

(ii) $(B_\lambda, \tau)$ is a countably compact semitopological semigroup;

(iii) $(B_\lambda, \tau)$ is a discretely pseudo-compact semitopological semigroup;

(iv) $(B_\lambda, \tau)$ is a pseudo-compact semitopological semigroup;

(v) $(B_\lambda, \tau)$ is topologically isomorphic to $(B_\lambda, \tau_c)$.

References


On Complementation of the Jacobson Group in an Adjoint Group of a Ring

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On *-Wildness of a Semidirect Product of Free Group on Two Generators and a Finite Group

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We consider the complexity of description of unitary representations of discrete groups up to the unitary equivalence. It was proved that the problem of classification of unitary representations of a semidirect product of free group with two generators and a finite is wild.

Asymptotic Rays

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A ray $R$ is a non-oriented graph with the set of vertices $\omega = \{0, 1, \ldots\}$ and the set of edges $\{(i, i+1) \mid i \in \omega\}$. An asymptotic ray is a non-oriented graph, asymptotically isomorphic to the ray. The notion of asymptotic isomorphism arouses from the following general combinatoric scheme.

A ball structure is a triple $B = (X, P, B)$ where $X, P$ are non-empty sets, and for all $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of $X$ which is called a ball of radius $\alpha$ around $x$. It is supposed that $x \in B(x, \alpha)$ for all $x \in X, \alpha \in P$.

Given any $x \in X$, $A \subseteq X$, $\alpha \in P$, we put $B^*(x, \alpha) = \{y \in X \mid x \in B(y, \alpha)\}$. A ball structure $B = (X, P, B)$ is called a ballean if
• \( \forall \alpha, \beta \in P \ \exists \alpha', \beta' \in P \) such that \( \forall x \in X \ B(x, \alpha) \subseteq B^*(x, \alpha'), \ B^*(x, \beta) \subseteq B(x, \beta') \);

• \( \forall \alpha, \beta \in P \ \exists \gamma \in P \) such that \( \forall x \in X \ B(B(x, \alpha), \beta) \subseteq B(x, \gamma) \).

The balleans appeared independently in asymptotic geometry, asymptotic topology under name coarse structures and in combinatorics [1]. Directly from the definition it follows that the balleans can be considered as asymptotic counterparts of uniform topological spaces.

Let \( B_1 = (X_1, P_1, B_1) \) and \( B_2 = (X_2, P_2, B_2) \) be balleans. A mapping \( f : X_1 \rightarrow X_2 \) is called a \( \prec \)-mapping if \( \forall \alpha \in P_1 \ \exists \beta \in P_2 \) such that \( f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta) \). A bijection \( f : X_1 \rightarrow X_2 \) is called an asymptotic isomorphism between \( B_1 \) and \( B_2 \) if \( f \) and \( f^{-1} \) are \( \prec \)-mappings.

The problem of characterization of asymptotic rays arose in combinatorics and was posed in [1, Problem 10.1]. We prove that a graph \( \Gamma \) is asymptotically isomorphic to the ray if and only if \( \Gamma \) is uniformly spherically bounded and is of bounded local degrees. In conclusion we precise this result for trees.

References


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On the Space of Exponential Type Distributions

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\[ \omega(t) = C \prod_{k=1}^{\infty} \left( 1 - \frac{t}{\nu_k} \right) \]

Dual spaces to entire functions spaces of exponential type, that fast decreasing on a real axis are investigated.

Let \( \omega(t) (-\infty < t < \infty) \) be an entire transcendental function of zero kind, which roots are on a positive imaginary semiaxis.
Where

\[ C = \text{const}, \quad 0 \leq t_1 \leq t_2 \leq t_3 \leq \ldots, \quad \sum_{k=1}^{\infty} \frac{1}{t_k} < \infty. \]

Let us denote by \( E_\omega \) the class of entire functions of exponential type that satisfy the condition

\[ M^{(m,a)}_\omega(\varphi) := \int_{-\infty}^{\infty} |t^m \omega(at) \varphi(t)| \, dt < \infty \]

for any \( m = 0, 1, 2, \ldots \) and \( a > 0 \).

The class of functions \( E_\omega \) is convolution algebra (that is \( \varphi_1 \ast \varphi_2 \in E_\omega \)
for any \( \varphi_1, \varphi_2 \in E_\omega \)).

Denote by \( E_\omega \) the vector spaces of such functions endowed with the seminorms

\[ p^{(m,a)}(\varphi) := \sup_{-\infty < t < \infty} |t^m \omega(at) \varphi(t)| \quad (m = 0, 1, 2, \ldots; a > 0). \]

The strong dual space \( E'_\omega \) to \( E_\omega \) is a locally convex linear topology space.

For any functional \( f \in E'_\omega \) and function \( \varphi(t) \in E_\omega \) we can define the convolution:

\[ (f \ast \varphi)(t) = \langle f(s) \cdot \varphi(t + s) \rangle = \langle f(s) \cdot T_{-s} \varphi(t) \rangle, \]

where \( f(s) \) denotes the action of the functional \( f \) on the function \( T_{-s} \varphi(t) \) by \( s \).

The following assertion holds.

**Theorem.** For any functional \( f \in E'_\omega \) the convolution operator

\[ F: E_\omega \ni \varphi \rightarrow f \ast \varphi \quad (1) \]

belongs to the space of continuous linear mappings \( \mathcal{L}(E_\omega) \) on the space \( E_\omega \)
and satisfies the relation

\[ FT_{s} \varphi = T_{s} F \varphi \quad (\forall \varphi \in E_\omega, \ s \in R). \quad (2) \]

On the other hand, if the operator \( F \in \mathcal{L}(E_\omega) \) satisfies condition (2), then there exists the unique \( f \in E'_\omega \) that \( F \) satisfies (1).
Analytic Functions and Symmetric Fock Space

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Let $E$ be a Hilbert space with an orthonormal basis $(e_n)$. We say that a Hilbert space $F = F(E)$ with a norm $\| \cdot \|_F$ is an (abstract) symmetric Fock space over given Hilbert space $E$ if the vectors $1, e^{(k)}_1 = e^{i_1}_{t_1} \cdots e^{i_n}_{t_n}$ form an orthogonal basis in $F$ for $n = |(k)| = 1, \ldots, \infty$, $k_j \geq 0$, where $i_1 < \cdots < i_n$, and $(e_i)_{i=1}^\infty$ is the orthonormal basis in $E$.

Evidently, the norm $\| \cdot \|_F$ is completely defined by its value on the basis vectors. Hence, setting $\| e^{(k)}_1 \|_F$ by arbitrary positive numbers, we can get various symmetric Fock space over $E$. We will use notation $F_\eta = F_\eta (E)$ for $(F, \| \cdot \|_F)$. Let $\langle \cdot | \cdot \rangle$ be the inner product in $F_\eta$.

Put $c^{(k)}_{[1]} := \| e^{(k)}_1 \|_F^{-2}$ and $c_0 = 1$. Let us consider a power series

$$\eta(x) = \sum_{k_1 + \cdots + k_n = 0}^\infty \prod_{i=1}^n \sum_{i_1 < \cdots < i_n} e^{k_1}_{i_1} \cdots e^{k_n}_{i_n} x^{k_1}_{i_1} \cdots x^{k_n}_{i_n} e^{k_1}_{i_1} \cdots e^{k_n}_{i_n}$$

for any $x = \sum_{i=1}^\infty x_i e_i \in E$.

**Theorem.** Suppose that there is a constant $S > 0$ and a sequence of positive numbers $(M_n)$ such that for every $n$, $\limsup_{n \to \infty} \sqrt[n]{M_n} = M < \infty$ and

$$0 < c^{(k)}_{[1]} = c^{k_1 \cdots k_n}_{i_1 \cdots i_n} \leq SM_2 \frac{(k_1 + \cdots + k_n)!}{k_1! \cdots k_n!} = SM_2 \frac{n!}{k_1! \cdots k_n!}$$

where $n = k_1 + \cdots + k_n$. Then there exists an open subset $U \subset E$, $U \ni 0$ such that

(i) Series (1) is convergent for every $x \in U$ and $\eta$ is an analytic map from $U$ into $F_\eta$.

(ii) For every $\phi \in F_\eta$ the map $f_\phi (x) := \eta(x) \phi$ is an analytic function on $U$.

(iii) The function $\langle \eta(x) e^{(k)}_{[1]} \rangle$ is an $n$-homogeneous Hilbert-Schmidt polynomial and $\langle \eta(x) e^{(k)}_{[1]} \rangle = x^{k_1}_{i_1} \cdots x^{k_n}_{i_n}$. 
Example. Let us denote by $\mathcal{H}_q$ the Hilbert space of analytic function $f_\phi = \langle \eta(\cdot) | \phi \rangle$ that is Hermitian dual to $\mathcal{F}_q$. Let

$$\eta(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$ 

Denote by $\mathcal{H}^2(E)$ the corresponding space $\mathcal{H}_q$. It is easy to see that $\mathcal{H}^2(E)$ consists with bounded type entire functions on $E$ and $\|r^{(k)}_q\|_q^2 = k_1! \ldots k_n!$.

Theorem. Let $E$ be a separable Hilbert space and $0 \neq a \in E$. Then the operator

$$T_a : \mathcal{H}^2(E) \rightarrow \mathcal{H}^2(E),$$

$$f \rightarrow f(x + a),$$

is hypercyclic.

References

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**On Universal Cellular Balleans**

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A ball structure is a triplet $\mathcal{B} = (X, P, B)$, where $X$ and $P$ are nonempty sets and, for any $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of $X$ which is called a ball of radius $\alpha$ around $x$. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$, $\alpha \in P$. Given any $x \in X$, $\alpha \in P$, we put $B^*(x, \alpha) = \{y \in X | x \in B(y, \alpha)\}$.

A ball structure $\mathcal{B} = (X, P, B)$ is called a ballean if

- for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,

$$B(x, \alpha) \subseteq B^*(x, \alpha'), B^*(x, \beta) \subseteq B(x, \beta);$$
for any $\alpha, \beta \in P$, there exists $\gamma \in P$ such that, for every $x \in X$,
\[ B(B(x, \alpha), \beta) \subseteq B(x, \gamma). \]

Let $B_1 = (X_1, P_1, B_1), B_2 = (X_2, P_2, B_2)$ be balleans. A mapping $f: X_1 \longrightarrow X_2$ is called a $\prec$-mapping if, for every $\alpha \in P_1$, there exists $\beta \in P_2$ such that, for every $x \in X_1$,
\[ f(B_1(x, \alpha)) \subseteq B_2(f(x), \beta). \]

If $f$ is a bijection such that $f$ and $f^{-1}$ are the $\prec$-mappings, we say that $f$ is an asymorphism between $B_1$ and $B_2$.

Let $B = (X, P, B)$ be a ballean, $Y$ be a nonempty subset of $X$. The ballean $B_Y = (Y, P, B_Y)$, where $B_Y(y, \alpha) = B(y, \alpha) \cap Y$, is called a subballean of $B$.

Given any ballean $B = (X, P, B)$, $x, y \in X$ and $\alpha \in P$, we say that $x, y$ are $\alpha$-path connected if there exists a sequence $x_0, x_1, \ldots, x_n, x_0 = x, x_n = y$ such that
\[ x_i \in B(x_{i+1}, \alpha), x_{i+1} \in B(x_i, \alpha) \]
for every $i \in \{0, 1, \ldots, n - 1\}$. For any $x \in X$ and $\alpha \in P$, we put
\[ R^\alpha(x, \alpha) = \{ y \in X \mid x, y \text{ are } \alpha\text{-path connected} \} \]

The ballean $B = (X, P, B)$ is called the cellularization of $B$. A ballean $B$ is called a cellular if $B = B$.

Let $(X, d)$ be a metric space, $\mathbb{R}^+ = \{ r \in \mathbb{R} \mid r \geq 0 \}$. Given any $x \in X$ and $r \in \mathbb{R}^+$, we put
\[ B_d(x, r) = \{ y \in X \mid d(x, y) \leq r \}. \]

The ballean $B(X, d) = (X, \mathbb{R}^+, B_d)$ is called a metric ballean. We say that a ballean $B$ is metrisable if $B$ is asymptotic $B(X, d)$ for some metric space $(X, d)$. For criterion of metrizability see [2, Theorem 9.1]. By [2, Theorem 9.3], a ballean $B = (X, P, B)$ is metrisable and cellular if and only if there exists a non-Archimedean metric $d$ on $X$ such that $B$ is asymorphic to $B(X, d)$.

Let $K$ be a class (with respect to asymorphisms) of balleans. A ballean $B \in K$ is called universal if every ballean from $K$ is asymorphic to some sub-ballean of $B$. Let $M_0$ denote the set of all positive integers whose ternimal expansion consists only from 0s and 1s. By [1, Theorem 3.11], the metric ballean defined by $M_0$ is universal in the class $K_0$ of countable metrisable
cellular balleans as follows: a balean $\mathcal{B} = (X, P, B)$ belongs to $\mathcal{K}_0$ if and only if, for every $r > 0$, there exists a natural number $c(r)$ such that $\left| B(x, r) \right| < c(r)$ for every $x \in X$.

Let $\{Z_n \mid n \in \omega\}$ be a family of nonempty sets. For every $n \in \omega$, we fix some element $e_n \in Z_n$ and say that the family $\{(Z_n, e_n) \mid n \in \omega\}$ is pointed. Let us consider the direct product $Z = \bigotimes_{n \in \omega} (Z_n, e_n)$. Every element $z \in Z$ can be written as a sequence $(z_n)_{n \in \omega}$ such that $z_n \in Z_n$, $n \in \omega$, and $z_n = e_n$ for all but finitely many $n \in \omega$. We define the metric $\rho$ on $Z$ by the rule: $\rho(z, z') = 0$ if $z = z'$, $\rho(z, z') = m + 1$, where $m$ is the smallest number such that $pr_n z \neq pr_n z'$.

**Theorem 1.** Let $\{(Z_n, e_n) \mid n \in \omega\}$ be a pointed family of countable sets. Then the balean $\mathcal{B}(Z, \rho)$ is universal in the class of all countable metrizable cellular balleans.

**Theorem 2.** For every countable cellular metrizable balean, there exists a subspace $Y$ of Hilbert space such that $\mathcal{B}$ is isomorphic to the metric balean determined by $Y$.

**Theorem 3.** Let $\{(Z_n, e_n) \mid n \in \omega\}$ be a pointed family of finite sets, $|Z_n| \geq 2$, $n \in \omega$. Then the balean $\mathcal{B}(Z, \rho)$ is universal in the class $\mathcal{K}_0$.

**Corollary.** There exists $2^{86}$ pairwise non-asymptotic universal balleans in $\mathcal{K}_0$.

**References**


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**M-Equivalence of Cylinders, Cones and Joins**

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Topological spaces $X$ and $Y$ are called $M$-equivalent ($X \overset{M}{\sim} Y$) if their free topological groups in the sense of Markov [2] $F(X)$ and $F(Y)$ are
topologically isomorphic. Two mappings \( f : X_1 \to Y_1 \) and \( g : X_2 \to Y_2 \) are called \( M \)-equivalent if there exist topological isomorphisms \( i : F(X_1) \to F(X_2) \) and \( j : F(Y_1) \to F(Y_2) \) such that \( j \circ f^* = g^* \circ i \) where \( f^* : F(X_1) \to F(Y_1) \) and \( g^* : F(X_2) \to F(Y_2) \) are the homomorphisms extending \( f \) and \( g \).

We denote by \( \text{Cyl}(f) \) and \( \text{Con}(f) \) the cylinder and the cone of a continuous mapping \( f \) (see [1, p. 22]).

**Proposition 1.** If mappings \( f_i \) are \( M \)-equivalent \( i = 1, 2 \), then \( \text{Cyl}(f_1) \overset{M}{\sim} \text{Cyl}(f_2) \) and \( \text{Con}(f_1) \overset{M}{\sim} \text{Con}(f_2) \).

We denote by \( X * Y \) the join of topological spaces \( X \) and \( Y \) (see [1, p. 22]).

**Proposition 1.** Let \( X \overset{M}{\sim} Y \) and \( Z \) be topological spaces such that the space \((X \oplus Y) \times Z\) is a \( k \)-space or \( Z \) is locally compact. Then \( X * Z \overset{M}{\sim} Y * Z \).

**References**


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**The Partitions of the Subsets of \( \mathbb{R}^n \) onto a Small Number of Homeomorphic Sets**

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The set \( X \) is parted onto parts \( X_s, s \in S \), for some set of indexes \( S \) if the family \( \{X_s\} \) is disjoint and \( X = \bigcup X_s \).

**Proposition.** Every open subset of \( \mathbb{R}^n \) can be parted onto \( m \) homeomorphic parts provided \( m \geq 2n + 2 \).

**Example.** There is a dense subset of the real line which cannot be parted onto two homeomorphic parts.
A metric space is said to be proper if every its closed ball is compact. We say that a map $f : X \to Y$ of proper metric spaces (a generic metric is denoted by $d$) is a coarse embedding (see, e.g., [1]) if there exist nondecreasing functions $\varphi_1, \varphi_2 : [0, \infty) \to [0, \infty)$ such that $\varphi_1(t), \varphi_2(t) \to \infty$ as $t \to \infty$ and

$\varphi_1(d(x, y)) \leq d(f(x), f(y)) \leq \varphi_2(d(x, y))$

for every $x, y \in X$.

By $C$ we denote the so-called anti-Cantor set on $\mathbb{R}$, i.e., the set $\bigcup_{i=1}^{\infty} C_i$, where $C_1 = [0, 1]$, $C_2 = C_1 \cup (2 \cdot 3 + C_1)$, ..., $C_n = C_{n-1} \cup (2 \cdot 3^{n-1} + C_{n-1})$.

Let $\mathcal{F}$ be either the $n$th hypersymmetric power functor $\text{exp}_n$ or the $G$-symmetric power functor, $n \geq 2$. We consider the question of coarse embeddability of the spaces of the form $F(C)$ into $C$ (the spaces $F(C)$ are endowed with the natural metric).

References

Extending Partial Ultrametrics

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The problem of extension of continuous metrics defined on a closed subspace of a metrizable space has a long history. It was Felix Hausdorff who proved that every admissible metric defined on a closed subspace of a metric space can be extended to an admissible metric on the whole space (see [1]). Since that time the result of Hausdorff was improved and rediscovered by many authors. The problem of extension of metrics which have special properties, in particular ultrametrics, was considered by E. D. Tymchatyn and M. Zarichnyj [4]. Their main result states that there exists an operator simultaneously extending continuous ultrametrics defined on closed subsets of a compact metrizable zero-dimensional space. The operator constructed by the authors preserves the maximum of two ultrametrics but fails to be homogeneous. A homogeneous operator which extends partial ultrametrics, preserves the operation of pointwise maximum and is continuous in the topology of uniform convergence is constructed in [5].

We consider a similar problem with the assumption that the set of ultrametrics is endowed with the pointwise convergence topology and all ultrametrics are defined on a fixed subspace of a separable zero-dimensional metric space. Note that the existence of an operator extending metrics which is continuous in the pointwise convergence topology was proved by C. Bessaga and T. Banakh in [3].

Recall that a metric $\rho$ on a set $Y$ is called an ultrametric if $\rho(x,y) \leq \max\{\rho(x,z),\rho(y,z)\}$ for all $x, y, z \in Y$. For a metrizable space $X$ there exists a compatible ultrametric if and only if $\dim X = 0$ (see e.g. [2]). Let $X$ be a zero-dimensional separable metrizable space and let $d$ be a compatible metric on $X$ bounded by 1. Take a nonempty closed subset $A$ of $X$ with $|A| \geq 2$ and let $\mathcal{U}M(A)$ and $\mathcal{U}M(X)$ denote the sets of all continuous ultrametrics defined on $A$ and $X$ respectively. The sets $\mathcal{U}M(A)$ and $\mathcal{U}M(X)$ are closed under the operations of pointwise maximum and multiplication by positive reals. Our main result is the following theorem.

**Theorem.** There exists an operator $v: \mathcal{U}M(A) \to \mathcal{U}M(X)$ with the following properties for every $\rho, \sigma \in \mathcal{U}M(A)$:

1) $v(\rho)$ is an extension of $\rho$;
2) \( v(\max\{\rho, \sigma\}) = \max\{v(\rho), v(\sigma)\} \) and \( v(c\rho) = cv(\rho) \) for \( c > 0 \);

3) \( v \) is continuous with respect to the pointwise convergence topology on \( \mathcal{UM}(A) \).

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