A UNIFORM APPROACH TO PRODUCING MODEL SPACES OF INFINITE-DIMENSIONAL TOPOLOGY

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Abstract. Given an ordinal α and a pointed topological space X, we endow $X^{<\alpha} = \cup\{X^\beta : \beta < \alpha\}$ with the strongest topology that coincides with the product topology on every subset X^β of $X^{<\alpha}$, $\beta < \alpha$. It turns out that many important model spaces of infinite-dimensional topology (including the topology of nonmetrizable manifolds) can be obtained as spaces of the form $X^{<\alpha}$ for $X = I, \mathbb{R}$. The paper deals with some topological properties of spaces $X^{<\alpha}$. Some new classification and characterization theorems are proved for these spaces.

1. Introduction

A considerable part of the classical infinite-dimensional topology deals with manifolds modeled on some nice model infinite-dimensional spaces. Among the most important model spaces let us mention the Hilbert cube $Q = [-1, 1]^\omega$, the countable product of lines $\mathbb{R}^\omega$, the Tychonov cube I^τ, the uncountable power of the line $\mathbb{R}^\tau$, the direct limit $\mathbb{R}^\infty$ of Euclidean spaces and the direct limit Q^∞ of Hilbert cubes. The topological characterizations of these model spaces can be found in [To1], [To2], [Chi], [FC], [S], [Sa] and are among the most prominent achievements of the classical infinite-dimensional topology.

It turns out that all these model spaces are particular examples of one fairly general topological construction we are going to describe now.

We shall identify cardinals with initial ordinals of a given size. Each ordinal α will be identified with the set of all ordinals $< \alpha$. By a pointed space we understand a topological space X with some distinguished point $*$ of X. In the sequel we shall consider the real line $\mathbb{R}$ and the interval $I = [-1, 1]$ as pointed spaces whose distinguished point is zero. The distinguished point of a Tychonov cube I^τ is the constant zero function.

Given two ordinals $\beta < \alpha$ and a pointed topological space X with a distinguished point $*$ identify the power X^β with the subset $\{(x_i)_{i\in\alpha} \in X^\alpha : x_i = * \text{ for all } i \geq \beta\}$. Let

$$X^{<\alpha} = \bigcup_{\beta<\alpha} X^\beta$$

and endow the space $X^{<\alpha}$ with the strongest topology inducing the product topology on each subset $X^\beta \subset X^{<\alpha}$, $\beta < \alpha$. We shall refer to this topology on $X^{<\alpha}$ as the *strong topology* in contrast to the *product topology*. In infinite-dimensional topology the spaces of the form $X^{<\omega}$ usually are denoted by X^∞. For some special pointed spaces X like the closed interval $I = [-1, 1]$, the real line $\mathbb{R}$, Hilbert cube $Q = I^\omega$ or the Hilbert space l_2, the spaces $X^\infty = X^{<\omega}$ were topologically characterized in [Sa] and [Pe].

For such particular X the spaces $X^{<\alpha}$ yield us almost all known model spaces of the classical infinite-dimensional topology. Namely, the space $I^{<\alpha}$ coincides with

- the n-dimensional cube if $\alpha = n$;
- the direct limit $I^\infty = \varinjlim I^n$ of finite-dimensional cubes (homeomorphic to $\mathbb{R}^\infty$) if $\alpha = \omega$;

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- the Hilbert cube Q if $\alpha = \omega + 1$;
- the direct limit $Q^\infty = \varinjlim Q^n$ of Hilbert cubes if $\alpha = \omega \cdot \omega$;
- a non-metrizable Tychonov cube I^τ if $\alpha = \tau + 1$ is uncountable successor ordinal;
- the direct limit $(I^\tau)^\infty = \varinjlim (I^\tau)^n$ of Tychonov cubes if $\alpha = \tau \cdot \omega$;
- the Σ-product $\Sigma(I) = \{f \in I^{\omega_1} : |\{\alpha \in \omega_1 : f(\alpha) \neq 0\}| \le \omega\} \subset I^{\omega_1}$ of intervals if $\alpha = \omega_1$ (as we shall see in Coincidence Theorem 2.2, the strong topology on $I^{<\omega_1}$ coincides with the product topology).

On the other hand, the spaces $\mathbb{R}^{<\alpha}$ yield us

- the Euclidean space $\mathbb{R}^n$ if $\alpha = n$;
- the direct limit $\mathbb{R}^\infty = \varinjlim \mathbb{R}^n$ of Euclidean spaces if $\alpha = \omega$;
- the countable product of lines $\mathbb{R}^\omega$ (homeomorphic to the separable Hilbert space l_2) if $\alpha = \omega + 1$;
- the direct limit $(\mathbb{R}^\omega)^\infty = \varinjlim (\mathbb{R}^\omega)^n$ if $\alpha = \omega \cdot \omega$ (the latter space is homeomorphic to the direct limit of Hilbert spaces $(l_2)^\infty$ and was studied by E.Pentsak [Pe]);
- an uncountable product of lines $\mathbb{R}^\tau$ if $\alpha = \tau + 1$ is an uncountable successor ordinal;
- the Σ-product $\Sigma(\mathbb{R}) = \{f \in \mathbb{R}^{\omega_1} : |\{\alpha \in \omega_1 : f(\alpha) \neq 0\}| \le \omega\}$ of the lines if $\alpha = \omega_1$.

Thus the spaces of the form $X^{<\alpha}$ can be considered as universal model spaces for infinite-dimensional topology. In this paper we shall be interested in three general problems concerning these spaces:

(1) Investigate topological properties of the spaces $X^{<\alpha}$ for various ordinals α.
(2) Give a topological classification of the spaces $X^{<\alpha}$.
(3) Find topological characterizations of the spaces $X^{<\alpha}$ for simple spaces X (like I or $\mathbb{R}$) and simple ordinals α.

2. Survey of principal results

We start the investigation of the spaces $X^{<\alpha}$ with calculating some of their cardinals characteristics.

By a *k-space* we understand a Hausdorff topological space X admitting a cover $\mathcal{K}$ by compact subspaces, generating the topology of X in the sense that a subset $U \subset X$ is open in X if and only if for any compactum $K \in \mathcal{K}$ the intersection $U \cap K$ is open in K, see [En]. The smallest possible size $|\mathcal{K}|$ of such a cover $\mathcal{K}$ is called the *k-ness* of X and is denoted by $\mathrm{k}(X)$, see [vD]. The k-ness of a topological space does not exceed the *compact covering number* $\mathrm{kc}(X)$ equal to the smallest size of a cover of X by compact subspaces. The *network weight* $\mathrm{nw}(X)$ of a topological space X is the smallest size $|\mathcal{N}|$ of a collection $\mathcal{N}$ of subsets of X such that for any open set $U \subset X$ and any point $x \in U$ there is an element $N \in \mathcal{N}$ with $x \in N \subset U$. For two cardinals κ, τ by $\kappa \times \tau$ we denote their product (as cardinals).

By the *cofinality* $\mathrm{cf}(\alpha)$ of an ordinal α we understand the smallest size $|C|$ of a cofinal subset $C \subset \alpha$ (the latter means that for each $x < \alpha$ there is $y \in C$ with $x \le y$).

Proposition 2.1. *For any pointed compact Hausdorff space X with $|X| > 1$ and any ordinal α the space $X^{<\alpha}$ is a k-space with* $\mathrm{kc}(X^{<\alpha}) = \mathrm{k}(X^{<\alpha}) = \mathrm{cf}(\alpha)$ *and* $\mathrm{nw}(X^{<\alpha}) = \mathrm{nw}(X) \times |\alpha|$.

Let us observe that the strong topology on $X^{<\alpha}$ coincides with the product topology if α is a successor cardinal. Surprisingly enough but the same is true also for certain limit ordinals. To characterize such ordinals we need to introduce the notion of the irreducible tail $\mathrm{tl}(\alpha)$ of an ordinal α. By definition, the *irreducible tail* $\mathrm{tl}(\alpha)$ of α is the smallest ordinal β for which there exists an ordinal $\gamma < \alpha$ such that $\alpha = \gamma + \beta$. Let us observe that $\mathrm{cf}(\alpha) \le \mathrm{tl}(\alpha) \le \alpha$; and $\mathrm{cf}(\alpha) = \mathrm{tl}(\alpha) = 1$ if and only if α is a successor ordinal.

Let us also note that $\mathrm{tl}(\alpha)=\alpha$ if and only if α is *additively indecomposable* in the sense that $\beta+\gamma<\alpha$ for any $\beta,\gamma<\alpha$. In particular, the ordinal $\mathrm{tl}(\alpha)$ is additively indecomposable.

Theorem 2.2 (Coincidence Theorem)**.** *Let X be a pointed (compact Hausdorff first countable) T_1-space with $|X|>1$. For an ordinal α the strong and product topologies on $X^{<\alpha}$ coincide (if and) only if $\mathrm{tl}(\alpha)$ is a cardinal with $\mathrm{cf}(\mathrm{tl}(\alpha))\neq\omega$.*

This theorem implies that the strong and product topologies coincide on $I^{<\omega_1}$ but differ on $(I^{\omega_1})^{<\omega_1}$. Also for any an ordinal α with $\mathrm{cf}(\alpha)=\omega$ and any pointed space X with non-isolated distinguished point the strong topology on $X^{<\omega}$ differs from the product topology. For such ordinals α the spaces $X^{<\alpha}$ occupy a special place in the whole theory and have especially nice topological properties.

A topological space X is called a *k_ω-space* if X is a k-space with $\mathrm{k}(X)\le\omega$. k_ω-Spaces often appear in topological algebra and have many nice properties, see [FST]. In particular, they are real complete. A topological space X is called *real complete* if it is homeomorphic to a closed subspace of $\mathbb{R}^\kappa$ for some cardinal κ. Real complete spaces admit also an inner description: a Tychonov space X is real complete if any point $x\in\beta X\setminus X$ in the remainder of the Stone-Čech compactification βX of X lies in a G_δ-subset of βX missing the set X, see [En, §3.11].

Let us call a topological space X an *absolute extensor for compact spaces in dimension 0* (briefly AE(0)) if any continuous map $f:B\to X$ defined on a closed subset B of a zero-dimensional compact Hausdorff space A admits a continuous extension $\bar f\colon A\to X$ onto the whole compactum A. Removing the dimensional restrictions we get the definition of an absolute extensor (briefly AE). A space X is called an *absolute retract* (briefly an AR) it is a compact Hausdorff AE. It is well known that a compact space is an AR if it is a retract of a Tychonov cube. In particular, Tychonov cubes are absolute retracts.

Now we show that many natural topological properties of the spaces $X^{<\alpha}$ are equivalent to the countable cofinality of α.

Theorem 2.3. *For an ordinal α the following conditions are equivalent:*

1. $\mathrm{cf}(\alpha)\le\omega$*;*
2. $X^{<\alpha}$ *is a k_ω-space for any pointed compact Hausdorff space X;*
3. $X^{<\alpha}$ *is real complete for any real complete pointed space X;*
4. $X^{<\alpha}$ *is real complete for some pointed T_1-space X containing more than one point.*
5. $X^{<\alpha}$ *is an AE for any pointed absolute extensor X;*
6. $X^{<\alpha}$ *is an AE(0) for some pointed T_1-space X with $|X|>1$.*

Now we shall discuss the topological classification of spaces $X^{<\alpha}$.

Theorem 2.4 (Reduction Theorem)**.** *For a pointed space X and an infinite ordinal α the space $X^{<\alpha}$ is homeomorphic to:*

$$\begin{cases} X^{|\alpha|} & \text{if } \alpha \text{ is a successor ordinal;}\\ X^{<|\alpha|} & \text{if } \alpha=|\alpha| \text{ is a cardinal;}\\ X^{<|\alpha|+\mathrm{cf}(\alpha)} & \text{if } 1<\mathrm{cf}(\alpha)=\mathrm{tl}(\alpha)<|\alpha|;\\ X^{<|\alpha|+|\mathrm{tl}(\alpha)|\cdot\mathrm{cf}(\alpha)} & \text{if } 1<\mathrm{cf}(\alpha)<\mathrm{tl}(\alpha)<|\alpha|;\\ X^{<|\alpha|\cdot\mathrm{cf}(\alpha)} & \text{if } 1<\mathrm{cf}(\alpha)<|\mathrm{tl}(\alpha)|=|\alpha|<\alpha. \end{cases}$$

This theorem can be proved using coordinate permutating homeomorphisms and is left to the reader. Observe that the set $X^{<\alpha+\beta\cdot\gamma}$ can be naturally identified with the product $X^\alpha\times(X^\beta)^{<\gamma}$. For compact Hausdorff X this identification is topological.

Proposition 2.5. *Let X be a pointed topological space and α,β be ordinals.*

(1) *The space* $X^{<\alpha\cdot\beta}$ *is naturally homomorphic to the space* $(X^\alpha)^{<\beta}$.
(2) *If* X *is compact and Hausdorff, then* $X^{\alpha+\beta}$ *is naturally homeomorphic to the product* $X^\alpha \times X^{<\beta}$.

Remark 1. It is interesting to notice that the second statement of this proposition does not hold for non-compact spaces X. In particular, the space $\mathbb{R}^{<\omega+\omega}$ is not homeomorphic to $\mathbb{R}^\omega \times \mathbb{R}^{<\omega}$ since the former space is a k-space while the latter is not, see [Ba2].

Proposition 2.5 and Reduction Theorem 2.4 allows us to reduce the study of spaces $X^{<\alpha}$ for compact spaces X to studying the particular cases when α is a cardinal.

Corollary 2.6. *For a pointed compact space* X *and an ordinal* α *the space* $X^{<\alpha}$ *is homomorphic to one of the spaces:* X^τ, $X^{<\tau}$, $(X^\tau)^{<\lambda}$, $X^\tau \times X^{<\lambda}$, $X^\tau \times (X^\kappa)^{<\lambda}$, *where* $\tau = |\alpha|$, $\lambda = \mathrm{cf}(\alpha)$, $\kappa = |\mathrm{tl}(\alpha)|$.

For two ordinals $\alpha \geq \beta$ by $\alpha - \beta$ we denote the unique ordinal γ such that $\alpha = \beta + \gamma$. The Reduction Theorem 2.4 allows us to prove the following

Theorem 2.7 (Classification Theorem)**.** *Let* X *be a pointed metrizable separable space containing more than one point. For two infinite ordinals* α, β *the spaces* $X^{<\alpha}$ *and* $X^{<\beta}$ *are homeomorphic if and only if* $|\alpha| = |\beta|$, $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta)$ *and* $|\mathrm{tl}(\alpha) - \mathrm{cf}(\alpha)| = |\mathrm{tl}(\beta) - \mathrm{cf}(\beta)|$.

For metrizable ARs X studying the topology of the spaces $X^{<\alpha}$ can be reduced to investigating the spaces $I^{<\alpha}$.

Theorem 2.8. *For any pointed compact metrizable absolute retract* X *and any ordinal* $\alpha > \omega$ *the space* $X^{<\alpha}$ *is homeomorphic to the space* $I^{<\alpha}$. *In its turn the space* $I^{<\alpha}$ *is homeomorphic to one of the spaces:* I^τ, $I^{<\tau}$, $(I^\tau)^{<\lambda}$, $I^\tau \times I^{<\lambda}$, *or* $I^\tau \times (I^\kappa)^{<\lambda}$, *where* $\tau = |\alpha|$, $\lambda = |\mathrm{cf}(\alpha)|$, *and* $\kappa = |\mathrm{tl}(\alpha)|$.

This theorem can be easily deduced from Corollary 2.6 and a result of H.Toruńczyk [To1] asserting that the countable power of a non-degenerate metrizable AR is homeomorphic to the Hilbert cube I^ω.

Finally we consider the problem of topological characterization of the spaces $I^{<\alpha}$. In case of countable cofinality of α this problem reduces to characterizing the spaces I^τ, $(I^\tau)^{<\omega}$, $I^{<\tau}$, $I^\tau \times I^{<\omega}$, and $I^\tau \times (I^\kappa)^{<\omega}$ for infinite cardinals $\kappa < \tau$. In fact, such characterizations are known for the first three spaces: I^τ, $(I^\tau)^\infty$ and $I^{<\tau}$.

We distinguish between countable and uncountable cardinals τ. For $\tau = \omega$ the power $I^\tau = I^\omega$ is nothing else but the Hilbert cube. The topological characterization of the Hilbert cube is one of the most brilliant achievements of infinite-dimensional topology and belongs to H.Toruńczyk [To1].

Characterization 2.9 (Toruńczyk)**.** *A topological space* X *is homeomorphic to the Hilbert cube* I^ω *if and only if* X *is a compact metrizable absolute retract satisfying the disjoint cells property in the sense that any two maps* $f, g : I^n \to X$ *from a finite-dimensional cube can be uniformly approximated by maps with disjoint images.*

A topological characterization of Tychonov cubes I^τ for uncountable cardinals τ is even shorter and belongs to E. Ščepin [S].

Characterization 2.10 (Ščepin)**.** *A topological space* X *is homeomorphic to a non-metrizable Tychonov cube* I^τ *if and only if* X *is a non-metrizable uniform-by-character compact AR of weight* τ.

A topological space X is called *uniform-by-character* if the character at each point of X equals the character of X.

To give a topological characterization of spaces $(I^\tau)^{<\omega}$ and $I^{<\tau}$ we need to recall the notion of a strongly universal space.

Definition 1. Let $\mathcal{K}$ be a class of compact Hausdorff spaces. A topological space X is defined to be

- *universal for the class* $\mathcal{K}$ if each compact subspace of X belongs to $\mathcal{K}$ and each compactum $K \in \mathcal{K}$ is homeomorphic to some compact subset of X;
- *strongly universal for the class* $\mathcal{K}$ if each compact subspace of X belongs to $\mathcal{K}$ and for any compact space $K \in \mathcal{K}$ any embedding $f : B \to X$ of a closed subset B of K can be extended to an embedding $\bar{f} : K \to X$ of the whole K;
- *strongly universal* if X is strongly universal for some class $\mathcal{K}$ of compacta.

It is easy to see that each strongly universal space X is strongly universal for the class $\mathcal{K}(X)$ of all spaces homeomorphic to compact subsets of X.

We shall say that a topological space X has the *compact unknotting property* if every homeomorphism $h : A \to B$ between compact subsets $A, B \subset X$ extends to an autohomeomorphism of X. It is easy to see that each space with compact unknotting property is strongly universal. The converse is true for k_ω-spaces.

Theorem 2.11 (Unknotting Theorem)**.** *A k_ω-space X is strongly universal if and only if it has the compact unknotting property.*

Another fundamental feature of strongly universal k_ω-spaces is described by

Theorem 2.12 (Uniqueness Theorem)**.** *Two k_ω-spaces X, Y are homeomorphic provided they are strongly universal for some class $\mathcal{K}$ of compact Hausdorff spaces. In particular, two strongly universal k_ω-spaces X, Y are homeomorphic if and only if $\mathcal{K}(X) = \mathcal{K}(Y)$.*

Both the theorems can be proved by the standard back-and-forth argument. In light of the above results it would be helpful to detect ordinals for which the space $I^{<\alpha}$ is strongly universal or has the compact unknotting property.

Theorem 2.13. *For an ordinal α the following conditions are equivalent:*

(1) *$I^{<\alpha}$ is a strongly universal k_ω-space;*
(2) *$I^{<\alpha}$ is a k_ω-space with the compact unknotting property;*
(3) *$\mathrm{cf}(\alpha) = \omega$ and $\beta + |\beta| < \alpha$ for any uncountable ordinal $\beta < \alpha$;*
(4) *$I^{<\alpha}$ is homeomorphic to a topological group;*
(5) *$I^{<\alpha}$ is homeomorphic to a locally convex linear topological lattice.*

Observe that this theorem characterizes ordinals α with countable cofinality for which the space $I^{<\alpha}$ is strongly universal. For ordinals with uncountable cofinality we get another theorem characterizing strongly universal spaces $I^{<\alpha}$.

Theorem 2.14. *For an ordinal α with uncountable cofinality the following conditions are equivalent:*

(1) *$I^{<\alpha}$ is a strongly universal space;*
(2) *$I^{<\alpha}$ has the compact unknotting property;*
(3) *α is a regular cardinal.*

These two theorems imply that for spaces $I^{<\alpha}$ the strong universality is equivalent to the compact unknotting property.

Let us note that for the smallest uncountable ordinal ω_1 the class $\mathcal{K}(I^{<\omega_1})$ of compact subspaces of $I^{<\omega_1}$ is well-understood: it consists of all Corson compacta of weight $\le \omega_1$. We recall that a topological space X is called *Corson compact* if it is homeomorphic to a compact subset of a Σ-product of lines $\Sigma(\mathbb{R}) = \{f \in \mathbb{R}^\tau : |\{i \in \tau : f(i) \neq 0\}| \le \omega\} \subset \mathbb{R}^\tau$ for some cardinal τ.

For an infinite ordinal α with countable cofinality the class $\mathcal{K}(I^{<\alpha})$ also admits a simple description: if $\alpha > \omega$, then $\mathcal{K}(I^{<\alpha})$ consists of all compact Hausdorff spaces with weight $< \alpha$. For the ordinal $\alpha = \omega$ the class $\mathcal{K}(I^{<\omega})$ consists of all finite-dimensional metrizable compact spaces. Using this description and the Uniqueness Theorem we get the following characterization theorems. The first two of them belong to K.Sakai [Sa].

Characterization 2.15 (Sakai)**.** *A topological space X is homeomorphic to the space $I^\infty = I^{<\omega}$ if and only if X is a strongly universal k_ω-space for the class of finite-dimensional compact metrizable spaces.*

Characterization 2.16 (Sakai)**.** *A topological space X is homeomorphic to the space $(I^\omega)^\infty = (I^\omega)^{<\omega}$ if and only if X is homeomorphic to $I^{<\alpha}$ for some countable limit ordinal $\alpha > \omega$ if and only if X is a strongly universal k_ω-space for the class of compact metrizable spaces.*

The latter characterization theorem of Sakai was generalized to spaces $(I^\tau)^\infty = (I^\tau)^{<\omega}$ by T. Banakh [Ba1].

Characterization 2.17 (Banakh)**.** *A topological space X is homeomorphic to the space $(I^\tau)^{<\omega}$ for some infinite cardinal τ if and only if X is a strongly universal k_ω-space for the class of compact spaces of weight $\le \tau$.*

Finally, the topology of the spaces $I^{<\tau}$ for cardinals τ of countable cofinality was characterized by O. Shabat and M. Zarichnyi in [SZ].

Characterization 2.18 (Shabat, Zarichnyi)**.** *A topological space X is homeomorphic to $I^{<\tau}$ for some cardinal with $\mathrm{cf}(\tau) = \omega$ if and only if X is a strongly universal k_ω-space for the class of compact spaces of weight $< \tau$.*

These theorems give us topological characterizations of strongly universal spaces of the form $I^{<\alpha}$ for ordinals α with countable cofinality. Next, we turn to the problem of topological characterization of the spaces $I^\tau \times (I^\kappa)^{<\omega}$ with $\tau > \kappa$. For $\kappa = 1$ this problem was posed in the paper [SZ]. It should be mentioned that unlike the spaces considered in Theorems 2.15–2.18 the spaces $I^\tau \times (I^\kappa)^\infty$ for $\tau > \kappa$ are not strongly universal.

First we recall two notion. Let κ be a cardinal. A closed subset A of a topological space X is called

- a G_κ*-set* in X if $A = \cap\mathcal{U}$ for some family $\mathcal{U}$ of open subsets of X with $|\mathcal{U}| = \kappa$;
- a $Z_{<\kappa}$*-set* in A is for every map $f : I^\kappa \to X$ and a family $\mathfrak{U}$ of open covers of X with $|\mathfrak{U}| < \kappa$ there is a map $g : X \to X \setminus A$ which is $\mathcal{U}$-near to f for every cover $\mathcal{U} \in \mathfrak{U}$.

Observe that for the cardinal $\kappa = \omega$, the notion of a $Z_{<\omega}$-set coincides with the classical notion of a Z-set introduced by Anderson, see [Ch].

Our final theorem gives a characterization of the spaces $I^\tau \times (I^\kappa)^{<\omega}$ and hence answers the mentioned problem from [SZ].

Characterization 2.19. *For a topological space X and infinite cardinals $\tau \ge \kappa$ the following conditions are equivalent:*

1. *X is homeomorphic to $I^\tau \times (I^\kappa)^{<\omega}$;*
2. *X is homeomorphic to $I^{<\alpha}$ for some ordinal with $|\alpha| = \tau$, $\mathrm{cf}(\alpha) = \omega$, and $|\mathrm{tl}(\alpha)| = \kappa$;*
3. *X is a k_ω-space such that each compact subset $K \subset X$ lies as a $Z_{<\kappa}$-set in some compact G_κ-subset $K \subset X$, homeomorphic to the Tychonov cube I^τ.*

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