

A HOMOLOGICAL SELECTION THEOREM IMPLYING A DIVISION THEOREM FOR Q -MANIFOLDS

TARAS BANAKH AND ROBERT CAUTY

ABSTRACT. We prove that a space M with Disjoint Disk Property is a Q -manifold if and only if $M \times X$ is a Q -manifold for some C -space X . This implies that the product $M \times I^2$ of a space M with the disk is a Q -manifold if and only if $M \times X$ is a Q -manifold for some C -space X . The proof of these theorems exploits the homological characterization of Q -manifolds due to Daverman and Walsh, combined with the existence of G -stable points in C -spaces. To establish the existence of such points we prove (and afterward apply) homological versions of the Brouwer Fixed Point Theorem and of Uspenskij's Selection Theorem.

It was noticed in [BR] that the famous Ščepin's Characterization Theorem for Tychonov cube [Šč] implies a surprising

Theorem 1 (Division Theorem for Tychonov cubes). *If the product $X \times Y$ of two spaces is homeomorphic to a Tychonov cube $[0, 1]^\tau$ of uncountable weight τ , then X or Y is homeomorphic to $[0, 1]^\tau$.*

This theorem has essentially nonmetric nature and is not true for the Hilbert cube $Q = [0, 1]^\omega$. A suitable counterexample was constructed by Singh [Sin].

Example 1 (Singh). *There is a compact absolute retract S such that $S \times [0, 1]$ is homeomorphic to Q but S contains no closed ANR-subspace of dimension ≥ 2 .*

Singh's space S contains no topological copy of the 2-disk I^2 and hence does not possess the Disjoint Disks Property. We recall that a space X has the Disjoint Disks Property (briefly, DDP) if any two maps $f, g : I^2 \rightarrow X$ from a 2-dimensional cube can be uniformly approximated by maps with disjoint images. This property was introduced by Cannon [Can] and is crucial in the topological characterization of finite-dimensional manifolds (see [Dav]).

It turns out that the pathology appearing in Singh's example cannot occur among spaces with DDP. This fact was first noticed by Daverman and Walsh in [DW].

Theorem 2 (Daverman-Walsh). *A space M with DDP is a Q -manifold if and only if $M \times X$ is a Q -manifold for some finite-dimensional space X .*

We recall that a topological space X is a Q -manifold if it is paracompact and has an open cover by subsets homeomorphic to open subsets of the Hilbert cube $Q = [0, 1]^\omega$.

In this paper we show that the finite-dimensionality of X in the Daverman-Walsh Theorem can be weakened to the C -space property. We recall that a space X is a C -space if for any countable sequence $(\mathcal{U}_n)_{n \in \omega}$ of open covers of X there is a sequence $(\mathcal{V}_n)_{n \in \omega}$ such that $\bigcup_{n \in \omega} \mathcal{V}_n$ is a cover of X and each \mathcal{V}_n is a discrete family of open sets refining the cover \mathcal{U}_n .

The class of C -spaces includes all metrizable *countable-dimensional* spaces, i.e., spaces that can be written as countable unions of zero-dimensional subspaces, see [En, 6.3.8]. The famous Pol's example [En, 6.1.21] of a compact metrizable C -space that contains a strongly infinite-dimensional subspace shows that the class of C -spaces is not hereditary and is wider than the class of countable-dimensional spaces.

The following Division Theorem generalizing Daverman-Walsh Theorem, is one of the principal results of this paper.

Theorem 3 (Division Theorem for Q -manifolds). *A space M with DDP is a Q -manifold if and only if the product $M \times X$ is a Q -manifold for some C -space X .*

There is a version of this theorem not involving DDP. It generalizes Corollary 6.3 of [DW].

Theorem 4. *If the product $M \times X$ of a space M with a C -space X is a Q -manifold, then the product $M \times I^2$ is a Q -manifold, too.*

The proof of these theorems consists of two basic ingredients: the homological characterization of Q -manifolds due to Daverman and Walsh [DW] and Theorem 9 on existence of G -stable points in C -spaces. The proof of the latter theorem is not trivial and relies on homological versions of the Brouwer Fixed Point Theorem and Uspenskij's Selection Theorem [Us], see Theorems 7 and 8. The mentioned homological theorems seem to have independent value and may be helpful for some other applications.

1. HOMOLOGICAL PRELIMINARIES

All topological spaces considered in the paper are Tychonov. We use singular relative homology $H_*(X, A; G)$ with coefficients in a non-trivial abelian group G . By $\tilde{H}_*(X; G)$ we denote the singular homology of X , reduced in dimension zero. If $G = \mathbb{Z}$, then we omit the symbol of the group and will write $H_*(X, A)$ in place of $H_*(X, A; \mathbb{Z})$.

Among coefficient groups G , the most important for us are: the group $\mathbb{F}_0 = \mathbb{Q}$ of rational numbers, and the cyclic groups $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ of prime order p . Note that these groups carry the structure of a field, so we shall call them *basic fields*. Basic fields \mathbb{F}_p are parametrized by numbers $p \in \Pi_0 = \{0\} \cup \Pi$ where Π stands for the set of all prime numbers.

In the following lemma we collect four homological results that will be applied later. The first three of them follow from the Universal Coefficients Formula [Hat, 3.A.3] and the last one from the Künneth formula (see Theorem 10 [Spa, §5.3]).

Lemma 1. *Let $A \subset X$, $B \subset Y$ be closed subsets of topological spaces X, Y .*

- (1) *If $H_*(X, A) = 0$, then $H_*(X, A; G) = 0$ for any coefficient group G .*
- (2) *If $H_n(X, A)$ contains an element of infinite order, then $H_n(X, A; \mathbb{Q}) \neq 0$.*
- (3) *If $H_n(X, A)$ contains an element of prime order p , then $H_{n+1}(X, A; \mathbb{Z}_p) \neq 0$.*
- (4) *If for some field G the groups $H_*(X, X \setminus A; G)$ and $H_*(Y, Y \setminus B; G)$ are not trivial, then so is the group $H_*(X \times Y, X \times Y \setminus A \times B; G)$.*

Homologies appeared to be very efficient in the study of (homological) Z_∞ -sets, see [BCK]. We recall that a closed subset A of a space X is called

- a Z_∞ -set if every map $f : I^n \rightarrow X$ of a finite-dimensional cube can be approximated by maps into $X \setminus A$;
- a *homotopical* Z_∞ -set if for every open set $U \subset X$ the relative homotopy groups $\pi_k(U, U \setminus A)$ are trivial for all k ;
- a *G -homological* Z_∞ -set if for every open set $U \subset X$ the relative homology groups $H_k(U, U \setminus A; G)$ are trivial for all k ;
- a *homological* Z_∞ -set if it is a \mathbb{Z} -homological Z_∞ -set in X .

In [DW] homological Z_∞ -sets are referred to as closed sets with infinite codimension. Lemma 1(1) implies that each homological Z_∞ -set is a G -homological Z_∞ -set for any coefficient group G .

A point $x \in X$ is called a (*homotopical, homological*) Z_∞ -point if the singleton $\{x\}$ is a (homotopical, homological) Z_∞ -set in X . The Excision Axiom for singular homology [Hat, 2.20] implies that a point $x \in X$ is a G -homological Z_∞ -point if and only if $H_k(X, X \setminus \{x\}; G) = 0$ for all k .

In the following lemma (whose proof can be found in [DW] or [BCK]) we collect some basic properties of homological Z_∞ -sets.

Lemma 2. *Let X be a Tychonov space and G be a non-trivial abelian group.*

- (1) *If A is a G -homological Z_∞ -set in X , then every closed subset B of A is a G -homological Z_∞ -set in X .*
- (2) *For any G -homological Z_∞ -set in X and any Tychonov space Y the product $A \times Y$ is a G -homological Z_∞ -set in $X \times Y$.*
- (3) *A closed Čech-complete σ -hereditarily disconnected subspace A of X is a G -homological Z_∞ -set in X if and only if each point $a \in A$ is a G -homological Z_∞ -point in X .*

A topological space X is *σ -hereditarily disconnected* if X can be written as the countable union $X = \bigcup_{n \in \omega} X_n$ of hereditarily disconnected subspaces X_n . We recall that a space X is *hereditarily disconnected* if X contains no connected subspace containing more than one point. By [Kra], each compact metrizable σ -hereditarily disconnected space is weakly infinite-dimensional (and in fact, a C -space by a recent result of R.Pol [Pol]).

We define a point x of a space X to be G -stable if x fails to be a G -homological Z_∞ -point in X (equivalently, if $H_*(X, X \setminus \{x\}; G) \neq 0$).

The last item of Lemma 2 implies

Corollary 1. *Each Čech-complete σ -hereditarily disconnected space X contains a G -stable point $x_G \in X$ for every coefficient group G .*

A C -space version of this corollary will be proved in Section 5 by more sophisticated methods. Now let us prove a result, useful for detecting homological Z_∞ -sets. We recall that by a basic field we understand either the field $\mathbb{F}_0 = \mathbb{Q}$ or rational numbers or the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ of a prime order p .

Lemma 3. *Let X be a locally path-connected space containing an \mathbb{F}_p -stable point $x_p \in X$ for every basic field \mathbb{F}_p , $p \in \Pi_0$. A subset A of a topological space M is a homological Z_∞ -set in M if and only if for every $p \in \Pi_0$ the product $A \times \{x_p\}$ is a homological Z_∞ -set in $M \times X$.*

Proof. The “only if” part of the lemma follows from Lemma 2. To prove the “if” part, assume that a closed subset A of M fails to be a homological Z_∞ -set in M , which means that $H_*(U, U \setminus A) \neq 0$ for some open set $U \subset M$. By Lemma 1(2,3), $H_*(U, U \setminus A; \mathbb{F}_p) \neq 0$ for some basic field \mathbb{F}_p . Since $H_*(X, X \setminus \{x_p\}; \mathbb{F}_p) \neq 0$, we may apply Lemma 1(4) to conclude that $H_*(U \times X, U \times X \setminus A \times \{x_p\}; \mathbb{F}_p) \neq 0$, which means that $A \times \{x_p\}$ is not an \mathbb{F}_p -homological Z_∞ -set in $M \times X$. \square

2. HOMOLOGICAL CHARACTERIZATION OF Q-MANIFOLDS

In this section we survey some homological and product characterizations of Q -manifolds essentially due to Daverman and Walsh [DW].

Theorem 5 ([DW, 6.1]). *A space X is a Q -manifold if and only if X is a locally compact ANR with DDP, containing a countable family \mathcal{Z} of homological Z_∞ -sets such that each closed subset A of X missing $\cup \mathcal{Z}$ is a homological Z_∞ -set in X .*

Theorem 6. *A space M with DDP is a Q -manifold if and only if $M \times X$ is a Q -manifold for some space X containing an \mathbb{F}_p -stable point $x_p \in X$ for every basic field \mathbb{F}_p .*

Proof. This theorem was implicitly proved in Appendix of [DW] but not stated there as a separate result. So we give a proof for the convenience of the reader.

The “if” part of the theorem is trivial. To prove the “only if” part, assume that the product $M \times X$ is a Q -manifold for some space X containing an \mathbb{F}_p -stable point x_p for every basic field \mathbb{F}_p . It follows that both spaces M and X are locally compact ANR’s.

Since all points of the Q -manifold $M \times X$ are homological Z_∞ -points, we may apply Lemma 3 to conclude that all points of the space M are homological Z_∞ -points.

The product $M \times X$, being a Q -manifold, contains a countable family $\{A_n : n \in \omega\}$ of finite-dimensional Z_∞ -sets such that each closed subset of $M \times X$ missing the union $\bigcup_n A_n$ is a homological Z_∞ -set in $M \times X$.

For every $n \in \omega$ and $p \in \Pi_0$ consider the closed finite-dimensional set $B_{n,p} = \{z \in M : (z, x_p) \in A_n\}$ in M . Since each point of M is a homological Z_∞ -point, the sets $B_{n,p}$, being finite-dimensional, are homological Z_∞ -sets in M , see Lemma 2(3) or Corollary 2.5 of [DW]. It remains to prove that each closed subset $F \subset X$ that misses the union $\bigcup_{n,p} B_{n,p}$ is a homological Z_∞ -set in X . Observe that for every $p \in \Pi_0$ the set $F \times \{x_p\}$ misses the union $\bigcup_{n \in \omega} A_n$ and thus is a homological Z_∞ -set in $M \times X$ by the choice of the family $\{A_n\}$. By Lemma 3, F is a homological Z_∞ -set in M .

Now it is legal to apply the Daverman-Walsh Theorem 5 to conclude that the space M is a Q -manifold. \square

The following corollary of Theorem 6 generalizes Corollary 6.3 of [DW].

Corollary 2. *For a space M the following conditions are equivalent:*

- (1) $M \times I^2$ is a Q -manifold;
- (2) $M \times X$ is a Q -manifold for some space X containing an \mathbb{F}_p -stable point x_p for every basic field \mathbb{F}_p .

Proof. The implication (1) \Rightarrow (2) is trivial since the point $x_0 = (\frac{1}{2}, \frac{1}{2})$ of the square I^2 is G -stable for every group G (because $H_2(I^2, I^2 \setminus \{x_0\}; G) = H_1(I^2 \setminus \{x_0\}; G) = G$).

(2) \Rightarrow (1) Assume that $M \times X$ is a Q -manifold for some space X containing an \mathbb{F}_p -stable point x_p for every basic field \mathbb{F}_p . Applying Lemma 3, conclude that each point of the space M is a homological Z_∞ -point. By Proposition 5.2 of [DW], the space $X \times I^2$ has DDP. Since $M \times I^2 \times X$ is a Q -manifold, we may apply Theorem 6, to conclude that $M \times I^2$ is a Q -manifold. \square

Combining the preceding results with Corollary 1 we obtain “ σ -hereditarily disconnected” versions of Theorems 3 and 4:

Proposition 1. *A space M with DDP is a Q -manifold if and only if the product $M \times X$ is a Q -manifold for some σ -hereditarily disconnected space X .*

Proposition 2. *If the product $M \times X$ of a space M with a σ -hereditarily disconnected space X is a Q -manifold, then the product $M \times I^2$ is a Q -manifold, too.*

Theorems 3 and 4 will follow in the same manner as soon as we prove that each locally compact locally contractible C -space X contains an \mathbb{F}_p -stable point x_p for every basic field \mathbb{F}_p . For this we need to elaborate some tools like homological versions of the Brouwer Fixed Point Theorem and of the Uspenskij Selection Theorem [Us].

3. A HOMOLOGICAL FIXED POINT THEOREM

In this section we prove a homological version of the classical Brouwer Fixed Point Theorem asserting that each continuous map $f : X \rightarrow X$ on a compact absolute retract X has a fixed point. The homological version of this result guarantees the existence of a fixed point for chain morphisms between complexes of singular chains on X .

First we recall some definitions. Throughout this section, G is a field. For a subset A of a topological space X and a cover \mathcal{U} of X let $\mathcal{St}(A, \mathcal{U}) = \cup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$ be the star of A with respect to the cover \mathcal{U} . Also we shall write $\text{diam}A < \mathcal{U}$ if $A \subset U$ for some $U \in \mathcal{U}$.

For a space X by $S(X; G)$ we denote the complex of singular chains of the space X . So $S(X; G) = \oplus_{k \geq 0} S_k(X; G)$ is the direct sum of free modules $S_k(X; G)$ over G generated by singular k -simplexes in X . The modules $S_k(X; G)$ are linked via the boundary homomorphisms $\partial : S_k(X; G) \rightarrow S_{k-1}(X; G)$ where $S_{-1}(X; G) = G$.

By $\Delta^k = \{(t_0, \dots, t_k) \in [0, 1]^{k+1} : t_0 + \dots + t_k = 1\}$ we denote the standard k -simplex. For a singular k -simplex $\sigma : \Delta^k \rightarrow X$ by $\|\sigma\| = \sigma(\Delta^k)$ we denote the carrier of σ and for a k -chain $c \in S_k(X; G)$ we put $\|c\| = \cup_i \|g_i \sigma_i\|$, where $c = \sum_i g_i \sigma_i$ is the irreducible representation of c as a linear combination of singular k -simplexes.

For a cover \mathcal{U} of X let $S(X, \mathcal{U}; G)$ stand for the submodule of $S(X; G)$, generated by singular simplexes whose supports have diameter less than \mathcal{U} (i.e., lie in elements of \mathcal{U}).

A homomorphism $\varphi : S(X, \mathcal{U}; G) \rightarrow S(Y; G)$ is called a *chain morphism* if $\partial \circ \varphi = \varphi \circ \partial$ and $\varphi(S_k(X, \mathcal{U}; G)) \subset S_k(Y; G)$ for all $k \in \omega$. Each continuous map $f : X \rightarrow Y$ induces a chain morphism $f_\# : S(X, \mathcal{U}; G) \rightarrow S(Y; G)$ assigning to a singular simplex $\sigma : \Delta^k \rightarrow X$ the simplex $f \circ \sigma : \Delta^k \rightarrow Y$.

A point $x \in X$ is called a *fixed point* of a chain morphism $\varphi : S(X, \mathcal{U}; G) \rightarrow S(X; G)$ if for any neighborhood $V \subset X$ of x there is a chain $c \in S(V; G) \cap S(X, \mathcal{U}; G)$ such that $\|\varphi(c)\| \cap V \neq \emptyset$.

Observe that the set of fixed points of a continuous map $f : X \rightarrow X$ coincide with the set of fixed points of the induced chain morphism $f_\# : S(X, \mathcal{U}; G) \rightarrow S(X; G)$. So, detecting fixed points of chain morphisms can help in detecting fixed points of continuous maps.

The principal result of this section is the following homological version of the Brouwer Fixed Point Theorem. For the field $G = \mathbb{Q}$ of rational numbers a much more general result (treating compact maps of so-called algebraic ANR's) was proved in Theorem 5 of [Ca]. However in the framework of usual AR's it has a simpler proof, which we include here for the convenience of the reader.

Theorem 7 (A Homological Fixed Point Theorem). *Let G be a field and \mathcal{U} be an open cover of a compact absolute retract X . Any chain morphism $\varphi : S(X, \mathcal{U}; G) \rightarrow S(X; G)$ has a fixed point $x \in X$.*

Proof. First, we reduce the problem to the case $X = I^n$ for a suitable $n \in \mathbb{N}$. Suppose that some chain morphism $\varphi : S(X, \mathcal{U}; G) \rightarrow S(X; G)$ has no fixed point.

Embed X into the Hilbert cube Q and take any retraction $r : Q \rightarrow X$. Since the chain morphism φ fails to have a fixed point, then so does the morphism $\varphi \circ r_\# : S(Q, r^{-1}(\mathcal{U}); G) \rightarrow S(X; G) \subset S(Q; G)$. Therefore, we lose no generality assuming that $X = Q$.

The absence of a fixed point for φ implies the existence of an open cover \mathcal{V} of X inscribed into \mathcal{U} and such that $\|\varphi(c)\| \cap \mathcal{St}(V, \mathcal{V}) = \emptyset$ for every $V \in \mathcal{V}$ and every singular chain $c \in S(V; G)$. We can assume that all sets $V \in \mathcal{V}$ are of cylindrical form, that is $V = p^{-1}(p(V))$, where $p : Q \rightarrow I^n$ is the natural projection of $Q = [0, 1]^\omega$ onto the n -dimensional cube $I^n = \{(t_i) \in Q : t_i = 0 \text{ for all } i \geq n\}$. Denoting by $i : I^n \rightarrow Q$

the identity embedding, we see that the morphism $p_{\#} \circ \varphi \circ i_{\#} : S(I^n, p(\mathcal{V}); G) \rightarrow S(I^n; G)$ also has no fixed point. So, we lose no generality assuming that $X = I^n$ for some n .

Let \mathcal{T} be a triangulation of I^n so fine that the closed star $\overline{St}(\sigma)$ of every simplex $\sigma \in \mathcal{T}$ lies in some set $V \in \mathcal{V}$. Denote by \mathcal{W} the cover of I^n by open stars of vertices of the triangulation \mathcal{T} . Let \mathcal{T}' be the barycentric subdivision of \mathcal{T} and let $C(\mathcal{T}'; G)$ denote the complex of oriented simplicial chains of \mathcal{T}' with coefficients in the field G . For every simplex σ of \mathcal{T}' fix a corresponding generator of $C(\mathcal{T}'; G)$ and denote it by the same symbol σ . By $\|\sigma\| \subset I^n$ we denote the geometric realization of a simplex σ and for a simplicial chain $c \in C(\mathcal{T}'; G)$ we put $\|c\| = \bigcup_i \|\sigma_i\|$ where $c = \sum_i g_i \sigma_i$ is an irreducible representation of c as a linear combination of oriented simplexes σ_i with nonzero coefficients.

Using the acyclicity of geometric simplexes, by induction on dimension one can construct a chain morphism $\eta : C(\mathcal{T}'; G) \rightarrow S(I^n; G)$ such that $\|\eta(\sigma)\| \subset \|\sigma\|$ for all simplexes σ of \mathcal{T}' . According to [Spa, 4.4.14], there is an operator of subdivision $Sd : S(I^n; G) \rightarrow S(I^n, \mathcal{W}; G)$ such that $\|Sd(c)\| \subset \|c\|$ for any singular chain $c \in S(I^n; G)$.

Next, we are going to construct a chain morphism $\psi : S(I^n, \mathcal{W}; G) \rightarrow C(\mathcal{T}'; G)$ such that $\|\psi(c)\| \subset St(\|c\|, \mathcal{W})$ for every singular chain $c \in S(I^n, \mathcal{W}; G)$. The construction of ψ is inductive and relies on the notion of the traverse $Tr(\sigma)$ of a simplex σ of \mathcal{T} .

First some notation. We write $\tau \leq \sigma$ if a simplex τ is a face of a simplex σ and $\tau < \sigma$ if τ is a proper face of σ . By b_σ we denote the barycenter of a simplex σ .

The *traverse* $Tr(\sigma)$ of a simplex σ of the triangulation \mathcal{T} is the subcomplex of the barycentric subdivision \mathcal{T}' formed by geometric simplexes with vertices $b_{\sigma_0}, b_{\sigma_1}, \dots, b_{\sigma_n}$ where $\sigma \leq \sigma_0 < \dots < \sigma_n$ are simplexes of \mathcal{T} . It is well known (and easy to see) that the traverse $Tr(\sigma)$, being a cone with vertex b_σ , is a contractible subcomplex of \mathcal{T}' . It is also clear that $Tr(\sigma) \subset St(\|\sigma\|, \mathcal{W})$ and $Tr(\tau) \supset Tr(\sigma)$ if τ is a face of σ .

For every subset $A \subset I^n$ of diameter $\text{diam} A < \mathcal{W}$ denote by $s(A)$ the simplex of \mathcal{T} with the vertices v such that $A \subset St(v, \mathcal{W})$. Observe that for subsets $A \subset B$ of I^n with $\text{diam} B < \mathcal{W}$ we get $s(A) \supset s(B)$ and $Tr(s(A)) \subset Tr(s(B))$.

By induction we shall construct a chain morphism $\psi : S(I^n, \mathcal{W}; G) \rightarrow C(\mathcal{T}'; G)$ such that $\|\psi(\sigma)\| \subset Tr(s(\|\sigma\|))$ for every simplex $\sigma \in S(I^n, \mathcal{W}; G)$. For a 0-dimensional singular simplex σ we let $\psi(\sigma)$ be the barycenter $b_{s(\|\sigma\|)}$ of the simplex $s(\|\sigma\|)$. Assuming that for some $k \geq 0$ the chain morphism ψ is defined on the space $S_k(I^n, \mathcal{W}; G)$, take any singular $(k+1)$ -simplex $\sigma \in S(I^n, \mathcal{W}; G)$ and consider the chain $\partial\sigma$. By the inductive assumption, for every face τ of σ we get $\|\psi(\tau)\| \subset Tr(s(\|\tau\|)) \subset Tr(s(\|\sigma\|))$. Then $\|\psi(\partial\sigma)\| \subset Tr(s(\|\sigma\|))$ and the contractibility of the traverse $Tr(s(\|\sigma\|))$ will help us to find a chain $c \in C_{k+1}(\mathcal{T}'; G)$ with $\|c\| \subset Tr(s(\|\sigma\|))$ and $\partial c = \psi(\partial\sigma)$. Letting $\psi(\sigma) = c$ we define the morphism ψ on each $(k+1)$ -simplex $\sigma \in S(I^n, \mathcal{W}; G)$, and this morphism can be extended onto $S_{k+1}(I^n, \mathcal{W}; G)$ by linearity.

It follows from the construction that for any singular simplex $\sigma \in C(I^n, \mathcal{W}; G)$ we get $\|\psi(\sigma)\| \subset Tr(s(\|\sigma\|)) \subset St(s(\|\sigma\|), \mathcal{W}) \subset St(\|\sigma\|, \mathcal{W})$. The linearity of ψ implies that $\|\psi(c)\| \subset St(\|c\|, \mathcal{W})$ for any chain $c \in S(I^n, \mathcal{W}; G)$.

Having constructed the morphism ψ , consider the chain morphism

$$\xi = \psi \circ Sd \circ \varphi \circ \eta : C(\mathcal{T}'; G) \rightarrow C(\mathcal{T}'; G).$$

We claim that for every $k \geq 0$ the k th component $\xi_k : C_k(\mathcal{T}'; G) \rightarrow C_k(\mathcal{T}'; G)$ of this morphism has zero trace; more precisely, the matrix of ξ_k in the canonic base of $C(\mathcal{T}'; G)$ has zero trace. The canonic base of $C(\mathcal{T}'; G)$ consists of oriented simplexes σ of the triangulation \mathcal{T}' . For any such a simplex σ take any $V \in \mathcal{V}$ with $\|\sigma\| \subset V$ and observe that $\|\eta(\sigma)\| \subset \|\sigma\| \subset V$ and hence $\|\varphi(\eta(\sigma))\| \cap St(V, \mathcal{V}) = \emptyset$. Then

$$\|Sd \circ \varphi \circ \eta(\sigma)\| \subset \|\varphi \circ \eta(\sigma)\| \subset I^n \setminus St(V, \mathcal{V})$$

and

$$\begin{aligned} \|\xi(\sigma)\| &= \|\psi \circ Sd \circ \varphi \circ \eta(\sigma)\| \subset St(\|Sd \circ \varphi \circ \eta(\sigma)\|, \mathcal{W}) \subset \\ &\subset St(I^n \setminus St(V, \mathcal{V}), \mathcal{W}) \subset I^n \setminus V \subset I^n \setminus \|\sigma\|, \end{aligned}$$

which means that for the representation $\xi(\sigma) = \sum_{\tau} g_{\tau} \cdot \tau$ of $\xi(\sigma)$ as a linear combination of k -simplexes, the coefficient $g_{\sigma} = 0$. Therefore the matrix of the transformation ξ_k has zero diagonal and zero trace $\text{tr}(\xi_k) = 0$.

Then the Lefschetz number

$$\Lambda(\xi) = \sum_{k \geq 0} (-1)^k \text{tr}(\xi_k)$$

of the chain morphism ξ equals zero. On the other hand, the Hopf Trace Formula [Bre, 2.3.2] implies that $\Lambda(\xi) = \Lambda(\xi_*)$, where $\xi_* : H_*(I^n; G) \rightarrow H_*(I^n; G)$ is the induced homomorphism in homologies. All

homological groups of I^n , except for $H_0(I^n; G) = G$ are trivial, which implies that $1 = \Lambda(\xi_*) = \Lambda(\xi) = 0$ and this is a contradiction, completing the proof of the theorem. \square

4. A HOMOLOGICAL VERSION OF USPENSKIJ'S SELECTION THEOREM

In this section we prove another important ingredient for the proof of Theorem 9 – a homological version of Uspenskij's Selection Theorem [Us]. We recall that Uspenskij's Theorem guarantees the existence of a continuous selection for each strongly lower semicontinuous multivalued function $\Phi : X \rightrightarrows Y$ defined on a paracompact C -space X and having aspherical values.

A multivalued function $\Phi : X \rightrightarrows Y$ is called *strongly lower semicontinuous* if for each compact subset $K \subset Y$ the set $\{x \in X : K \subset \Phi(x)\}$ is open in X .

A space X is *aspherical* if its homotopy groups all are trivial. A homological version of Uspenskij's Selection Theorem treats multivalued functions having G -acyclic values, where a space X is defined to be G -acyclic if its singular homology $\tilde{H}_*(X; G)$ (reduced in dimension zero) with the coefficient group G is trivial.

Theorem 8. *Let G be a ring with unit, X be a paracompact C -space, Y be a topological space, and $\Phi : X \rightrightarrows Y$ be a strongly lower semicontinuous multivalued function having G -acyclic values $\Phi(x)$ for all $x \in X$. Then there are an open cover \mathcal{U} of X and a chain morphism $\varphi : S(X, \mathcal{U}; G) \rightarrow S(Y; G)$ such that $\varphi(S(U; G)) \subset S(\Phi(x); G)$ for every $U \in \mathcal{U}$ and every point $x \in U$.*

Proof. We shall construct two sequences of locally finite open covers of X , $\mathcal{V}_n = \{V_\alpha : \alpha \in A_n\}$ and $\mathcal{W}_n = \{W_\alpha : \alpha \in A_n\}$ of X so that

$$(1) \overline{W}_\alpha \subset V_\alpha \text{ for all } \alpha.$$

The index sets A_n will be taken pairwise disjoint and we shall construct an increasing sequence $K_1 \subset K_2 \subset \dots$ of simplicial complexes and chain morphisms $\mu_n : C(K_n; G) \rightarrow S(Y; G)$ satisfying

$$(2) \mu_n|_{C(K_{n-1}; G)} = \mu_{n-1}.$$

To every index $\alpha \in A_n$ we shall assign a finite contractible subcomplex L_α of K_n in such a way that the compact set

$$C_\alpha = \bigcup_{\sigma \subset L_\alpha} \|\mu_n(\sigma)\|.$$

will satisfy the inclusion

$$(3) C_\alpha \subset \Phi(x) \text{ for all } x \in V_\alpha.$$

For every $x \in X$, fix a point $y_x \in \Phi(x)$ and consider the set $O_x = \{x' \in X : y_x \in \Phi(x')\}$ which is an open neighborhood of x because Φ is strongly lower semicontinuous. Let $\mathcal{V}_1 = \{V_\alpha : \alpha \in A_1\}$ be an open locally finite cover of X inscribed into $\{O_x : x \in X\}$, and let $\mathcal{W} = \{W_\alpha : \alpha \in A_1\}$ be a locally finite open cover satisfying (1). Let K_1 be the zero-dimensional complex having A_1 for the set of vertices. For every $\alpha \in A_1$ let $L_\alpha = \{\alpha\}$ and choose $x_\alpha \in X$ so that O_{x_α} contains V_α . Let $\mu_1 : C(K_1; G) \rightarrow S(Y; G)$ be the chain morphism assigning to the generator corresponding to α the 0-simplex y_{x_α} . Then the set $C_\alpha = \{y_{x_\alpha}\}$ satisfies (3) by the definition of O_{x_α} .

Assume that for some $n \geq 1$, the covers \mathcal{V}_n and \mathcal{W}_n have been constructed. For every $x \in X$ let P_x be an open neighborhood of x meeting only finitely many sets of the locally finite cover $\bigcup_{p \leq n} \mathcal{V}_p$ and such that $P_x \subset V_\alpha$ for every $\alpha \in A_p$ with $P_x \cap \overline{W}_\alpha \neq \emptyset$. Let $J(x)$ be the set of all $\alpha \in A_1 \cup \dots \cup A_n$ such that $P_x \subset V_\alpha$ and let $D_x = \bigcup\{C_\alpha \mid \alpha \in J(x)\}$. The set $J(x)$ is finite, so D_x is compact, and the condition (3) guarantees that $D_x \subset \Phi(x)$. Let $M_x = \bigcup\{L_\alpha \mid \alpha \in J(x)\}$; this is a finite subcomplex of K_n , and the definition of C_α ensures that $\mu_n(C(M_x; G)) \subset S(D_x; G) \subset S(\Phi(x); G)$. The G -acyclicity of $\Phi(x)$ guarantees that, for the cone L_x with base M_x and vertex $v_x \notin M_x$, there is a chain morphism $\mu_x : C(L_x; G) \rightarrow S(\Phi(x); G)$ extending $\mu_n|_{C(M_x; G)}$. Then $C_x = \bigcup_{\sigma \subset L_x} \|\mu_x(\sigma)\|$ is a compact set containing D_x and lying in $\Phi(x)$. Let $O_x^n = \{x' \in P_x : C_x \subset \Phi(x')\}$. Since Φ is strongly lower semicontinuous, O_x^n is an open neighborhood of x .

Let $\mathcal{V}_{n+1} = \{V_\alpha : \alpha \in A_{n+1}\}$ be a locally finite open cover of X inscribed into $\{O_x^n : x \in X\}$, and let $\mathcal{W}_{n+1} = \{W_\alpha : \alpha \in A_{n+1}\}$ be a locally finite open cover of X , satisfying (1). For every $\alpha \in A_{n+1}$, choose a point $x_\alpha \in X$ such that $O_{x_\alpha}^n$ contains V_α , and let L_α be the cone with base M_{x_α} and vertex α . Put $K_{n+1} = K_n \cup \bigcup_{\alpha \in A_{n+1}} L_\alpha$. Let ψ_α be the simplicial isomorphism of L_α onto L_{x_α} which is identity on M_{x_α} and maps α onto v_{x_α} . Define the chain morphism $\mu_{n+1} : C(K_{n+1}; G) \rightarrow S(Y; G)$ by $\mu_{n+1}|_{C(K_n; G)} = \mu_n$

and $\mu_{n+1}|C(L_\alpha; G) = \mu_{x_\alpha} \circ (\psi_\alpha)_\#$ for all $\alpha \in A_{n+1}$. Then $C_\alpha = C_{x_\alpha}$, and the definition of $O_{x_\alpha}^n$ guarantees that (3) holds. Moreover, since $O_{x_\alpha}^n \subset P_{x_\alpha}$, the definition of P_x guarantees that

(4) If $\alpha' \in A_1 \cup \dots \cup A_n$ and $\alpha \in A_{n+1}$ satisfy $W_\alpha \cap W_{\alpha'} \neq \emptyset$, then $W_\alpha \subset V_{\alpha'}$, and thus $L_{\alpha'} \subset L_\alpha$.

Let $K = \bigcup_{n=1}^\infty K_n$. Since X is a C -space, there are disjoint families $\mathcal{U}_n = \{U_\lambda : \lambda \in \Lambda_n\}$, $n \geq 1$, of open sets in X such that every \mathcal{U}_n refines \mathcal{W}_n and $\mathcal{U} = \bigcup \mathcal{U}_n$ covers X . We suppose that the index sets Λ_n are pairwise disjoint and put $\Lambda = \bigcup_{n=1}^\infty \Lambda_n$, $\Lambda_{\leq n} = \Lambda_1 \cup \dots \cup \Lambda_n$, and

$$S_n = \sum_{\lambda \in \Lambda_{\leq n}} S(U_\lambda; G).$$

The sets S_n form an increasing sequence of subcomplexes of $S(X; G)$ with union equal to $S(X, \mathcal{U}; G)$. For every $\lambda \in \Lambda_n$, select an $\alpha_\lambda \in A_n$ such that $U_\lambda \subset W_{\alpha_\lambda}$. We are going to construct a chain morphism $\nu : S(X, \mathcal{U}; G) \rightarrow C(K; G)$ such that

(5) $\nu(S(U_\lambda; G)) \subset C(L_{\alpha_\lambda}; G)$ for all $\lambda \in \Lambda$.

Since L_{α_λ} is contractible, for every $\lambda \in \Lambda_1$ there is a chain morphism $\nu_\lambda : S(U_\lambda; G) \rightarrow C(L_{\alpha_\lambda}; G)$. Since the family U_λ , $\lambda \in \Lambda_1$, is disjoint, S_1 is a direct sum of $S(U_\lambda; G)$ for $\lambda \in \Lambda_1$, so we may define the restriction ν_1 of ν to S_1 letting $\nu_1|S(U_\lambda; G) = \nu_\lambda$ for all $\lambda \in \Lambda_1$.

Suppose that for some $n \geq 1$ the restriction ν_n of ν to S_n has been constructed. If λ_1, λ_2 are two distinct elements of Λ_{n+1} , then $S(U_{\lambda_1}; G) \cap S(U_{\lambda_2}; G) = \{0\}$. So, to extend ν_n to S_{n+1} , it suffices to construct extensions ν_λ of $\nu_n|S(U_\lambda; G) \cap S_n$ onto $S(U_\lambda; G)$ for every $\lambda \in \Lambda_{n+1}$. If $\lambda' \in \Lambda_{\leq n}$ satisfies $U_\lambda \cap U_{\lambda'} \neq \emptyset$, then $W_{\alpha_\lambda} \cap W_{\alpha_{\lambda'}} \neq \emptyset$, so $L_{\alpha_{\lambda'}} \subset L_{\alpha_\lambda}$ according to (4). The condition (5) implies $\nu_n(S(U_\lambda; G) \cap S_n) \subset C(L_{\alpha_\lambda}; G)$, and the G -acyclicity of L_{α_λ} ensures the existence of the desired extension ν_λ .

Let $\varphi = \mu \circ \nu : S(X, \mathcal{U}; G) \rightarrow S(Y; G)$. Using (5) and the definition of C_α , we get, for every $\lambda \in \Lambda$,

$$\varphi(S(U_\lambda; G)) \subset \mu(C(L_{\alpha_\lambda}; G)) \subset S(C_{\alpha_\lambda}; G).$$

Since $U_\lambda \subset W_{\alpha_\lambda} \subset V_{\alpha_\lambda}$, (3) implies that $C_{\alpha_\lambda} \subset \Phi(x)$ for every $x \in U_\lambda$, which completes the proof of the theorem. \square

5. G -STABLE POINTS IN C -SPACES

Now we are able to prove a promised theorem, which being combined with Theorem 6 and Corollary 2, implies Theorems 3 and 4 announced in the Introduction. For the ring $G = \mathbb{Z}$ this theorem was proved in [BCK].

Theorem 9. *Each metrizable locally compact locally contractible C -space X contains a G -stable point $x_G \in X$ for every field G .*

Proof. By Gresham's Theorem [Gre], X , being a locally contractible C -space, is an ANR. Then Edwards' ANR-Theorem [Chap, 44.1] guarantees that the product $M = X \times Q$ is a Q -manifold. Take an open subset $W \subset M$ such that the closure \overline{W} of W in M is homeomorphic to the Hilbert cube and the remainder $\overline{W} \setminus W$ is a homotopical Z_∞ -set in \overline{W} . Let $\text{pr} : \overline{W} \rightarrow X$ stand for the natural projection.

To derive a contradiction, assume that for some field G the space X contains no G -stable point; in other terms, each point x of X is a G -homological Z_∞ -point. Then Lemma 2 ensures that the product $\{x\} \times Q$ is a G -homological Z_∞ -set in $M \times X$, which guarantees that $H_*(W, W \setminus (\{x\} \times Q); G) = 0$. The exact sequence of the pair $(W, W \setminus (\{x\} \times Q)) = (W, W \setminus \text{pr}^{-1}(x))$ implies that $\tilde{H}_*(W \setminus \text{pr}^{-1}(x); G) = H_*(W, W \setminus \text{pr}^{-1}(x); G) = 0$,

Since the remainder $\overline{W} \setminus W$ is a homotopical Z_∞ -set in \overline{W} , the space $\overline{W} \setminus \text{pr}^{-1}(x)$ is homotopy equivalent to $W \setminus \text{pr}^{-1}(x)$ and hence $\tilde{H}_*(\overline{W} \setminus \text{pr}^{-1}(x); G) = 0$, which means that the multivalued function

$$\Phi : X \rightrightarrows \overline{W}, \quad \Phi : x \mapsto \overline{W} \setminus \text{pr}^{-1}(x)$$

has G -acyclic values.

Observe that for every compact subset $K \subset \overline{W}$ the set

$$\{x \in X : K \subset \Phi(x)\} = X \setminus \text{pr}(K)$$

is open in X , which means that the multivalued function $\Phi : X \rightrightarrows \overline{W}$ is strongly lower semicontinuous.

Now it is legal to apply Theorem 8 to find an open cover \mathcal{U} on X and a chain morphism $\varphi : S(X, \mathcal{U}; G) \rightarrow S(\overline{W}; G)$ such that $\varphi(S(U; G)) \subset S(\Phi(x); G)$ for every $U \in \mathcal{U}$ and every point $x \in X$.

Consider the open cover $\text{pr}^{-1}(\mathcal{U}) = \{\text{pr}^{-1}(U) : U \in \mathcal{U}\}$ of a Hilbert cube \overline{W} , and let $\text{pr}_* : S(\overline{W}, \text{pr}^{-1}(\mathcal{U}); G) \rightarrow S(X, \mathcal{U}; G)$ be the chain morphism induced by the projection $\text{pr} : \overline{W} \rightarrow X$.

By the Homological Fixed Point Theorem 7, for the chain morphism

$$\varphi \circ \text{pr}_* : S(\overline{W}, \text{pr}^{-1}(U); G) \rightarrow S(\overline{W}; G)$$

there are a set $U \in \mathcal{U}$ and a chain $c \in S(\text{pr}^{-1}(U); G)$ such that $\|\varphi \circ \text{pr}_*(c)\| \cap \text{pr}^{-1}(U)$ contains some point (x, q) . Since $\text{pr}_*(c) \in S(U; G)$, the choice of the morphism φ guarantees that $\varphi(\text{pr}_*(c)) \in S(\Phi(x); G)$ and hence $\|\varphi \circ \text{pr}_*(c)\| \subset \Phi(x) = \overline{W} \setminus \text{pr}^{-1}(x) \subset \overline{W} \setminus \{(x, q)\}$, which contradicts the choice of the point (x, q) . \square

6. OPEN PROBLEMS

Theorems 3 and 4 would follow from (more simple) Propositions 1 and 2 if the answer to the following question were affirmative.

Problem 1. *Is each compact C -space σ -hereditarily disconnected?*

The other our problem asks if Theorem 9 is true for weakly infinite-dimensional compacta.

Problem 2. *Let X be a compact weakly infinite-dimensional ANR-space. Does X contain a G -stable point x_G for every (basic) field G ?*

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INSTYTUT MATEMATYKI, AKADEMIA ŚWIĘTOKRZYSKA, KIELCE (POLAND),
 DEPARTMENT OF MATHEMATICS, IVAN FRANKO LVIV NATIONAL UNIVERSITY, LVIV (UKRAINE),
 NIPISSING UNIVERSITY, NORTH BAY (CANADA)
E-mail address: `tbanakh@franko.lviv.ua`

UNIVERSITÉ PARIS VI (FRANCE)
E-mail address: `cauty@math.jussieu.fr`