COMPATIBLE ALGEBRAIC STRUCTURES ON SCATTERED COMPACTA

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Abstract. It is proved that each hereditarily collectionwise Hausdorff compact scattered space with finite scattered height admits a continuous semilattice operation turning it into a topological semilattice with open principal filters. On the other hand a compactification $\gamma\mathbb{N}$ of a countable discrete space $\mathbb{N}$ whose remainder is homeomorphic to $[0,\omega_1]$ admits no (separately) continuous binary operation turning $\gamma\mathbb{N}$ into an inverse semigroup (semilattice). Also we construct a compactification $\psi\mathbb{N}$ of $\mathbb{N}$ admitting no separately continuous semilattice operation and such that the remainder $\psi\mathbb{N}\setminus\mathbb{N}$ is homeomorphic to the one-point compactification of an uncountable discrete space. To show that $\psi\mathbb{N}$ admits no continuous semilattice operation we prove that the set of isolated points of a compact scattered topological semilattice X of scattered height 2 is sequentially dense in X. Also we prove that each separable scattered compactum with scattered height 2 is a subspace of a separable compact scattered topological semilattice with open principal filters and scattered height 2. This allows us to construct an example of a separable compact scattered topological semilattice with open principal filters and scattered height 2, which fails to be Fréchet-Urysohn. Also we construct an example of a Fréchet-Urysohn separable non-metrizable compact scattered topological semilattice with open principal filters and scattered height 2.

In this note we pursue the investigations started in [6], [1] and consider a classical problem of topological algebra [19]: which compatible algebraic structures can live on a given topological space, or more precisely, on a given scattered compact space? Let us remind that a topological space X is *scattered* if every nonempty subspace of X contains an isolated point. In the sequel all topological spaces considered in the paper are Hausdorff. Observe that non-discrete scattered spaces fail to be topologically homogeneous and thus admit no compatible group structure. On the other hand, there are compatible algebraic structures, more general than the structure of a topological group, which can live on non-discrete scattered spaces. We have in mind the structure of a regular topological semigroup.

Let us recall that a *topological semigroup* is, by definition, a topological space S endowed with a continuous associative operation $*: S\times S\to S$. A semigroup S is *regular* if each element $x\in S$ has an *inverse*, that is an element $x^{-1}\in S$ such that $x*x^{-1}*x=x$ and $x^{-1}*x*x^{-1}=x^{-1}$. If for each $x\in S$ such an inverse element x^{-1} is unique, then S is called an *inverse semigroup*. By a *topological inverse semigroup* we understand an inverse semigroup X endowed with a topology such that the semigroup operation $*: X\times X\to X$ and the operation $(\cdot)^{-1}: X\to X$ of taking the inverse are continuous. The class of (topological) inverse semigroups is quite wide: it contains all (topological) groups and all (topological) semilattices. By a *semilattice* we understand a commutative semigroup $(S,\wedge)$ so that every element $x\in S$ is an *idempotent*, that is, $x\wedge x=x$. We say that a topological semilattice $(S,\wedge)$ has *open principal filters* if for any $x \in S$ the upper set $\uparrow x = \{y \in S : x \wedge y = x\}$ is open in S, see [6].

Observe that each topological space X admits a continuous binary operation $* : X \times X \to X$ turning X into a regular semigroup: just take $x * y = x$ for every $x, y \in X$. The situation with the structure of an inverse topological semigroup or a topological semilattice is much more subtle.

Considering scattered compacta, we see that in the metrizable case, each metrizable scattered compactum, being homeomorphic to a subset of the real line, admits a continuous semilattice operation max (or min). The simplest examples of non-metrizable scattered compacta such as the one-point compactification αD of an uncountable discrete space D or closed segments of ordinals also admit a continuous semilattice operation.

This observation led us to the following natural question (see [6]): *does every scattered compact space admit a continuous semilattice operation?* Surprisingly, the answer to this question is negative. We shall give two different counterexamples but first prove a positive result: we shall show that each hereditarily collectionwise Hausdorff scattered compact space with finite scattered height admits a continuous semilattice operation turning it into a topological semilattice with open principal filters.

We remind that a topological space X is *collectionwise Hausdorff* if for each closed discrete subspace D of X there is a discrete collection $\{U_x\}_{x\in D}$ of open subsets of X such that $x \in U_x$ for each $x \in D$ (a collection $\mathcal{F}$ of subsets of a topological space X is *discrete* if each point $x \in X$ has a neighborhood meeting at most one member of $\mathcal{F}$). A topological space X is *hereditarily collectionwise Hausdorff* if each subspace of X is collectionwise Hausdorff. Observe that the hereditary cellularity of a hereditarily collectionwise Hausdorff space X equals its cellularity.

Finally, we recall the definition of the scattered height of a scattered compactum, which is defined by transfinite induction. For a topological space X let $X_{(0)} = X$ and $X_{(1)}$ be the set of all nonisolated points of X. By transfinite induction for every ordinal $\alpha > 1$ define the α-th derived set $X_{(\alpha)}$ of X letting $X_{(\alpha)} = \bigcap_{\beta<\alpha}(X_{(\beta)})_{(1)}$. By the *scattered height* $\kappa(X)$ of a scattered topological space X we understand the smallest ordinal α such that $X_{(\alpha)}$ is finite. Thus the one-point compactification of an infinite discrete space has scattered height 1. Conversely, each scattered compact space X with $\kappa(X) = 1$ is a finite disjoint union of one-point compactifications of infinite discrete spaces and thus is hereditarily collectionwise Hausdorff. As we shall see later this is not true for scattered compacta of scattered height 2.

Theorem 1. *Each hereditarily collectionwise Hausdorff compact scattered space X with finite scattered height admits a continuous semilattice operation $\wedge$ turning X into a topological semilattice with open principal filters and also making any given element x_0 into the zero of X (that is $x_0 \wedge x = x_0$ for all $x \in X$).*

Proof. The theorem will be proved by induction on the scattered height of X. If $\kappa(X) = 0$ (which means that X is finite), then the theorem is trivial.

Suppose that $\kappa(X) = n$ for some $n \in \mathbb{N}$ and the theorem has been proved for all scattered compacta with scattered height $< n$. First we consider the case when the set $X_{(n)}$ consists of a unique point a_∞. Let x_0 be a fixed point of X.

First we consider the subcase $a_\infty = x_0$. Note that $D = X_{(n-1)} \setminus \{a_\infty\}$ is a closed discrete subspace of $X \setminus \{a_\infty\}$. Taking into account that X is zero-dimensional and hereditarily collectionwise Hausdorff, we can assign to each $a \in D$ a closed-and-open neighborhood $U_a \subset X$ such that $\{U_a\}_{a\in D}$ is a discrete collection in $X \setminus \{a_\infty\}$. Let $U_{a_\infty} = X \setminus \bigcup_{a\in D} U_a$ and observe that $\max\{\kappa(U_a) : a \in X_{(n-1)}\} < n$. By the induction assumption, for each $a \in X_{(n-1)}$ the space U_a admits a continuous semilattice operations $\wedge_a$ turning it into a topological semilattice with open principal filters. Moreover, we can assume that a_∞ is a zero of U_{a_∞}. Now define a semilattice operation $\wedge$ on X letting

$$x \wedge y = \begin{cases} x \wedge_a y & \text{if } x, y \in U_a \text{ for some } a \in X_{(n-1)}, \\ a_\infty & \text{otherwise.} \end{cases}$$

Let us show that X endowed with the operation $\wedge$ is a topological semilattice with open principal filters and zero a_∞.

We shall verify the continuity of $\wedge$ at the point (a_∞, a_∞). Take any neighborhood $W \subset X$ of $a_\infty = a_\infty \wedge a_\infty$. Since the collection $\{U_a\}_{a\in D}$ is discrete in $X \setminus \{a_\infty\}$ and $X \setminus W$ is compact, the set $E = \{a \in D : U_a \cap (X \setminus W) \neq \emptyset\}$ is finite and hence the set $U = X \setminus \bigcup_{a\in E} U_a$ is an open neighborhood of a_∞ with $U \wedge U \subset W$. This means that $\wedge$ is continuous at (a_∞, a_∞). The continuous of $\wedge$ at other points of $X \times X$ easily follows from the definition of $\wedge$. This finishes the proof in the particular case $|X_{(n)}| = 1$ and $x_0 = a_\infty$.

Now suppose that $x_0 \neq a_\infty$. Find a closed-and-open neighborhood U_0 of x_0 such that $a_\infty \notin U_0$. The preceding argument supplies us with a continuous semilattice operation $\wedge_1$ on $X \setminus U_0$ turning the scattered space $X \setminus U_0$ into a topological semilattice with open principal filters and a_∞ for a zero. Since $\kappa(U_0) < n$ we can apply the induction hypothesis to find a continuous semilattice operation $\wedge_0$ turning the space U_0 into a topological semilattice with open filters and x_0 as a zero. Now define a semilattice operation $\wedge$ on X letting

$$x \wedge y = \begin{cases} x \wedge_0 y & \text{if } x, y \in U_0, \\ x \wedge_1 y & \text{if } x, y \in X \setminus U_0, \\ x_0 & \text{otherwise.} \end{cases}$$

It is easy to see that X endowed with the operation $\wedge$ is a topological semilattice with open principal filters and zero x_0. This finishes the proof in the particular case $|X_{(n)}| = 1$.

Now let us pass to the general case. Write $X_{(n)} \cup \{x_0\} = \{x_0, \dots, x_m\}$. For every $i \le m$ find a closed-and-open neighborhood U_i of x_i in X so that $X = \bigcup_{i=0}^m U_i$ and $U_i \cap U_j = \emptyset$ for distinct $i, j \le m$. Since the n-th derived set of every U_i consists of at most one point, we can apply the previous argument to construct a continuous semilattice operation $\wedge_i$ on U_i turning U_i into a topological semilattice with open principal filters and zero at x_i. Now define a continuous semilattice operation $\wedge$ on X letting

$$x \wedge y = \begin{cases} x \wedge_i y & \text{if } x, y \in U_i \text{ for some } i \le m, \\ x_0 & \text{otherwise.} \end{cases}$$

It is easy to verify that $\wedge$ turns X into a topological semilattice with open principal filters and x_0 for zero. □ □

Now we construct examples of scattered compacta admitting no continuous semilattice operation. In fact, one of these examples has so wild topological structure that it admits no partial order turning it into a pospace, locally compatible at each point.

By a *pospace* we understand a topological space X endowed with a partial order $\le$ such that for every point $x \in X$ its lower set $\downarrow x = \{y \in X : y \le x\}$ and its upper set $\uparrow x = \{y \in X : y \ge x\}$ are closed in X. Two elements x, y of a partially ordered set $(X, \le)$ are *comparable* if $x \ge y$ or $x \le y$. We define a pospace $(X, \le)$ to be *locally compatible* at a point $x_0 \in X$ if for every neighborhood U of x_0 there is a neighborhood V of x_0 such that for every $x \in V$ there is $z \in U$ such that z is comparable with x_0 and x. We say that a pospace $(X, \le)$ is *locally compatible at a set* $W \subset X$ if it is locally compatible at each point of W.

Each semilattice operation $* : X \times X \to X$ on a set X induces a partial order $\le$ defined by $x \le y$ iff $x * y = x$. This order is locally compatible at a point $x_0 \in X$ if the map $s_{x_0} : X \to X$, $s_{x_0} : x \mapsto x * x_0$, is continuous at x_0.

Our first counterexample is based on

Theorem 2. *Suppose a topological space X contains a topological copy W of $[0, \omega_1)$ whose complement $N = X \setminus W$ is countable and dense in X. Then*

(a) *X admits no partial order turning X into a pospace locally compatible at W;*
(b) *X admits no separately continuous semilattice operation;*
(c) *X is homeomorphic to no topological inverse semigroup.*

Proof. We shall identify $[0, \omega_1)$ with $W = X \setminus N$.

(a). Suppose $\le$ is a partial order on X such that $(X, \le)$ is a pospace locally compatible at W. Let

$$N_1 = \{x \in N : [0, \omega_1) \cap \uparrow x \text{ is uncountable}\} \tag{1}$$

$$N_2 = \{x \in N : [0, \omega_1) \cap \downarrow x \text{ is uncountable}\}. \tag{2}$$

We can find a countable ordinal α such that

$$[\alpha, \omega_1) \cap \uparrow x = \varnothing \text{ for every } x \in N \setminus N_1 \tag{3}$$

$$[\alpha, \omega_1) \cap \downarrow x = \varnothing \text{ for every } x \in N \setminus N_2. \tag{4}$$

Since the sets $[0, \omega_1) \cap \uparrow x$, $x \in N_1$ and $[0, \omega_1) \cap \downarrow x$, $x \in N_2$, are closed and unbounded in $[0, \omega_1)$, the intersection

$$C = (\alpha, \omega_1) \cap \Big(\bigcap_{x \in N_1} [0, \omega_1) \cap \uparrow x \Big) \cap \Big(\bigcap_{x \in N_2} [0, \omega_1) \cap \downarrow x \Big) \tag{5}$$

is unbounded with respect to the well order of $[0, \omega_1)$ (here we assume that the intersection of the empty collection of subsets of a set S is equal to S).

We claim that there are three points $a, b, c \in C$ such that $b \notin (\downarrow a) \cup (\uparrow c)$. Indeed, if C contains two incomparable elements x, y put $a = x$, $b = y$, $c = x$. If any two elements of C are comparable, take any three distinct points $a, b, c \in C$ ordered so that $a \le b \le c$.

Note that $U = X \setminus \big((\downarrow a) \cup (\uparrow c) \cup [0, \alpha] \big)$ is a neighborhood of b. Using the density of N and the local compatibility of $(X, \le)$ at the point b find points $x \in N \setminus (\downarrow a \cup \uparrow c)$ and $z \in U$ such that z is comparable with x and b.

Let us show that $z \in W$. Assuming the converse, we would get $z \in N_1 \cup N_2$ according to (3) and (4) and the fact that z is compatible with $b \in (\alpha, \omega_1)$. On the other hand the equality (5) implies that $z \notin N_1 \cup N_2$ since $C \ni a \notin\uparrow z$ and $C \ni c \notin\downarrow z$. Therefore $z \in W \cap U$ and thus $z \in (\alpha, \omega_1)$. Since z is comparable with x, by (3) or (4) we would get $x \in N_1 \cup N_2$. Again (5) implies that $x \notin N_1 \cup N_2$ since $a \notin\uparrow x$ and $c \notin\downarrow x$. This contradiction completes the proof of the statement in (a).

(b). Suppose $* : X \times X \to X$ is a separately continuous semilattice operation. Define a partial order $\le$ on X letting $x \le y$ iff $x * y = x$. It can be easily seen that X endowed with this order is a pospace locally compatible at each point. Now the preceding statement completes the proof.

(c). Suppose $* : X \times X \to X$ is an operation turning X into a topological inverse semigroup. Denote by $E = \{x \in X : x*x = x\}$ the set of idempotents of X. Clearly, E is a closed subset of X. Moreover, according to [2, Theorem 1.1.7], E is a semilattice with respect to the operation $*$. Let us show that E is uncountable. Assuming the converse we shall prove that X is countable. Since X is a topological inverse semigroup, the operation $(\cdot)^{-1} : X \to X$ of taking the inverse is continuous. Then the functions $r_1, r_2 : X \to E$ defined by $r_1(x) = x * x^{-1}$ and $r_2(x) = x^{-1} * x$ for $x \in X$ are continuous retractions of X onto E.

It is clear that $X = \bigcup_{e,f \in E} G_{e,f}$, where $G_{e,f} = r_1^{-1}(e) \cap r_2^{-1}(f)$. To show that X is countable, it suffices to verify that for every $e, f \in E$ the set $G_{e,f}$ is at most countable. A routine verification shows that for every idempotent $e \in E$ the set $G_{e,e}$ is a closed subgroup of the inverse semigroup X.

We claim that this subgroup $G_{e,e}$ is countable. Observe that the set $\overline{W} \setminus W$ consists of at most one point and the space X is locally countable at each point $x \notin \overline{W} \setminus W$. This implies that the topological group $G_{e,e}$, being topologically homogeneous, is locally countable. The subspace $G_{e,e} \cap W$ of $G_{e,e}$, being countably compact, is totally bounded in $G_{e,e}$ (the latter means that for any neighborhood U of the unit of the group $G_{e,e}$ there is a finite subset $F \subset G_{e,e}$ with $G_{e,e} \cap W \subset F \cdot U$). Now the local countability of $G_{e,e}$ and the total boundedness of $G_{e,e} \cap W$ in $G_{e,e}$ imply that the set $G_{e,e} \cap W$ is countable. Since the set $X \setminus W$ is countable, we conclude that the group $G_{e,e} \subset (G_{e,e} \cap W) \cup (X \setminus W)$ is countable too.

To show that for any $e, f \in E$ the set $G_{e,f}$ is countable, fix any element $a \in G_{e,f}$ and consider the map $h : G_{e,f} \to X$ defined by $h(x) = x * a^{-1}$ for $x \in G_{e,f}$. Since $x * a^{-1} * a = x * f = x * (x^{-1} * x) = x$, the map h is injective. Given any point $x \in G_{e,f}$, we get

$$r_1(h(x)) = x * a^{-1} * (x * a^{-1})^{-1} = x * a^{-1} * a * x^{-1} = xx^{-1} = r_1(x) = e$$

and

$$\begin{aligned} r_2(h(x)) =& (x * a^{-1})^{-1} * (x * a^{-1}) = a * x^{-1} * x * a^{-1} = \\ =& a * r_2(x) * a^{-1} = a * f * a^{-1} = a * r_2(a) * a^{-1} = \\ =& a * (a^{-1} * a) * a^{-1} = a * a^{-1} = r_1(a) = e, \end{aligned}$$

which means that $h(x) \in G_{e,e}$.

It follows from the injectivity of h that $|G_{e,f}| \le |G_{e,e}|$. Consequently, the sets $G_{e,f}$, $e, f \in E$, are countable and so is the semigroup X, a contradiction which shows that the set E is uncountable.

Since the set N is dense in X, we get that its image $r_1(N)$ is dense in $E = r_1(X)$. Let $\alpha = \sup r_1(N) \cap [0,\omega_1)$ (the supremum taken with respect to the natural well order of $[0,\omega_1)$). Since $r_1(N)$ is countable, we get $\alpha < \omega_1$. Next, since the set E is uncountable and closed in X, $\Omega = E \cap [\alpha+1,\omega_1)$ is a closed unbounded subset of $[0,\omega_1)$ which is homeomorphic to $[0,\omega_1)$ while $M = E \setminus \Omega \supset r_1(N)$ is a countable dense subset of E. Thus E is a topological semilattice containing a topological copy Ω of $[0,\omega_1)$ whose complement $E \setminus \Omega$ is countable and dense in E, which is impossible according to the previous statement. □ □

Remark 1. In case of compact X the second statement of Theorem 2 can be derived from the third one with help of a classical result of Lawson [11, Theorem II.1.5] according to which any separately continuous semilattice operation on a zero-dimensional compact space K is continuous (moreover, K has a base of the topology consisting of subsemilattices). Unfortunately, this Lawson result is not valid in the class of commutative inverse semigroups. To construct a suitable counterexample, consider the one-point compactification $\bar{\mathbb{Z}} = \mathbb{Z} \cup \{\infty\}$ of the space of integers endowed with the discontinuous separately continuous binary operation $* : \bar{\mathbb{Z}} \times \bar{\mathbb{Z}} \to \bar{\mathbb{Z}}$ defined by

$$x * y = \begin{cases} x + y, & \text{if } x, y \in \mathbb{Z}; \\ \infty, & \text{otherwise.} \end{cases}$$

□

It follows from Parovichenko Theorem [17] (see also [7, 3.5.H] or [8]) that the ordinal space $[0,\omega_1]$ is the image of the remainder $\beta\mathbb{N} \setminus \mathbb{N}$ of the Stone-Čech compactification of $\mathbb{N}$ under a continuous map $f : \beta\mathbb{N}\setminus\mathbb{N} \to [0,\omega_1]$. The map f induces a closed equivalence relation $\sim$ on $\beta\mathbb{N}$: $x \sim y$ iff either $x, y \in \beta\mathbb{N} \setminus \mathbb{N}$ and $f(x) = f(y)$ or $x, y \in \mathbb{N}$ and $x = y$. The quotient space $\gamma\mathbb{N} = \beta\mathbb{N}/\sim$ of $\beta\mathbb{N}$ with respect to the equivalence relation $\sim$ is a compactification of $\mathbb{N}$ whose remainder $\gamma\mathbb{N} \setminus \mathbb{N}$ is homeomorphic to $[0,\omega_1]$. For this compactification Theorem 2 implies

Corollary 1. *Let $\gamma\mathbb{N}$ be a compactification of $\mathbb{N}$ whose remainder $\gamma\mathbb{N} \setminus \mathbb{N}$ is homeomorphic to the ordinal segment $[0,\omega_1]$. Then the compact scattered space $\gamma\mathbb{N}$ does not admit a separately continuous semilattice operation and is homeomorphic to no topological inverse semigroup.* □

The compactification $\gamma\mathbb{N}$ has a rather complex structure. In particular, $\gamma\mathbb{N}$ has uncountable scattered height. Next, we construct a scattered compact space $\psi\mathbb{N}$ with scattered height 2 admitting no continuous semilattice operation.

The construction of the compactification $\psi\mathbb{N}$ is based on the notion of MAD family. We remind that a family $\mathcal{F}$ of infinite subsets of $\mathbb{N}$ is called *almost disjoint* if $A \cap B$ is finite for any distinct $A, B \in \mathcal{F}$. Under a *maximal almost disjoint* (briefly MAD) family we understand any maximal element in the set of infinite almost disjoint families of infinite subsets of $\mathbb{N}$.

Given a MAD family $\mathcal{F}$ endow the set $\mathcal{F} \cup \mathbb{N}$ with the topology generated by the subbase $\{\{n\}, \{A\} \cup A \setminus K : n \in \mathbb{N},\ A \in \mathcal{F},\ K \text{ is a finite subset of } \mathbb{N}\}$. It is easy to see that this topology on $\mathcal{F} \cup \mathbb{N}$ is Hausdorff and locally compact. Finally, let $\psi_{\mathcal{F}}(\mathbb{N}) = \{\infty\} \cup \mathcal{F} \cup \mathbb{N}$ be the one-point compactification of the space $\mathcal{F} \cup \mathbb{N}$.

Theorem 3. *For any infinite MAD family $\mathcal{F}$ on $\mathbb{N}$ the space $\psi_{\mathcal{F}}(\mathbb{N})$ is a scattered compact space with scattered height 2 admitting no separately continuous semilattice operation.*

Proof. It is clear that $\psi_{\mathcal{F}}(\mathbb{N})$ is compact, scattered and has scattered height 2. Observe also that no sequence $(x_n)_{n\in\omega} \subset \mathbb{N}$ converges to the point at infinity ∞ of $\psi_{\mathcal{F}}(\mathbb{N})$. Indeed, for any infinite set $X = \{x_n : n \in \omega\} \subset \mathbb{N}$ there is a set $A \in \mathcal{F}$ such that $A \cap X$ is infinite. Then the subsequence $\{x_n : x_n \in A\}$ of (x_n) converges to the point $A \neq \infty$ and thus (x_n) cannot converge to ∞. Now we see that Theorem 3 is a particular case of the following more general theorem. □ □

We call a subset A of a topological space X *sequentially dense* in X if each point $x \in X$ is the limit of a sequence $(a_n) \subset A$. In particular, $\mathbb{N}$ is not sequentially dense in $\psi_{\mathcal{F}}(\mathbb{N})$.

Theorem 4. *If a separable compact scattered space X of scattered height 2 admits a separately continuous semilattice operation, then the set* $\mathrm{Iso}(X)$ *of isolated points of X is sequentially dense in X.*

To prove Theorem 4 we need the following fact, see [1, Lemma 1].

Lemma 1. *If K is a compact subset with a unique non-isolated point e in a scattered topological semilattice X, then the set $K\setminus \uparrow e$ is at most countable.*

PROOF of Theorem 4. Let $\wedge : X\times X \to X$ be a separately continuous semilattice operation on a scattered compact space X with $\kappa(X) \le 2$. Applying [11, Theorem II.1.5] we conclude that the operation $\wedge$ is continuous. Fix any point $e \in X$. We have to find a sequence $(x_n)_{n\in\omega}$ of isolated points of X, tending to e. If $e \in X_{(1)} \setminus X_{(2)}$, then e, being an isolated point of $X_{(1)}$, possesses a clopen neighborhood $U \subset X$ such that e is a unique non-isolated point of U. Take any infinite subset $\{x_n : n \in \omega\} \subset U \setminus \{e\}$ and observe that the sequence (x_n) converges to e.

So it rests to consider the case $e \in X_{(2)}$. If e has a countable closed neighborhood W in X, then by the compactness of X, W is metrizable and by the density of the set $\mathrm{Iso}(X)$ in X there is a sequence $(x_n) \subset W \cap \mathrm{Iso}(X)$ convergent to e. So we assume that no neighborhood of e is countable and no sequence of isolated points of X tends to e.

The scattered topological semilattice $(X, \wedge)$, being zero-dimensional, is Lawson [11, Theorem II.1.5] which means that X has a base consisting of subsemilattices. Since $X_{(2)}$ is finite, we can find a clopen neighborhood $Y \subset X$ of e such that $Y \cap X_{(2)} = \{e\}$ and Y is a subsemilattice of X. Without loss of generality we can assume that $Y = X$, i.e., the set $X_{(2)}$ consists of a unique point e. Then $X_{(1)}$ is an uncountable compact set with a unique non-isolated point e. Applying Lemma 1 we get that the set $X_{(1)}\setminus \uparrow e$ is at most countable. Then the set $X\setminus \uparrow e$ also is at most countable and thus can be written as $X\setminus \uparrow e = \bigcup_{n\in\omega} F_n$ where $F_n \subset F_{n+1}$ are finite subsets. By the compactness of the set $\uparrow e$, for every $n \in \omega$ find a closed neighborhood $\bar{W}_n \subset X$ of $\uparrow e$ such that for each $z \in \bar{W}_n$ both z and $z \wedge e$ are not in F_n . Without loss of generality we can assume that $\bar{W}_{n+1} \subset \bar{W}_n$ for all $n \in \omega$. Thus $\bigcap_{n\in\omega} \bar{W}_n = \uparrow e$.

Let $\langle \mathrm{Iso}(X)\rangle$ be the smallest subsemilattice of X containing the set $\mathrm{Iso}(X)$. Since X is separable, the set $\mathrm{Iso}(X)$ is countable as well as the semilattice $\langle \mathrm{Iso}(X)\rangle$. We claim that the partial order induced by the semilattice operation is well-founded on the set $C = (X_{(1)} \cap \uparrow e) \setminus (\langle \mathrm{Iso}(X)\rangle \cup \{e\})$. The latter means that each subset of C has a minimal element, or equivalently, there is no infinite strictly decreasing sequences $(e_n)_{n\in\omega}$ in C. To prove this fact, assume that $(e_n)_{n\in\omega} \subset C$ is such a decreasing sequence (i.e. $e_n \wedge e_{n+1} = e_{n+1}$ and $e_n \neq e_{n+1}$ for all n). For every

$n \in \omega$ fix a neighborhood $U_n \subset \bar{W}_n$ of e_n such that $U_n \cap X_{(1)} = \{e_n\}$. We can take the neighborhoods U_n to be pair-wise disjoint. Next, for every $n \in \omega$, fix a sequence $(x_{n,m})_{m\in\omega} \subset U_n \cap \mathrm{Iso}(X)$ convergent to e_n (such a sequence exists since $e_n \in X_{(1)} \setminus X_{(2)}$).

Using the continuity of the semilattice operation, for every $n \in \omega$ fix a function $f_n : \omega \to \omega$ such that $y_n = x_{0,f_n(0)} \wedge x_{1,f_n(1)} \wedge \cdots \wedge x_{n,f_n(n)} \in U_n$. We can assume that $f_n(i) < f_m(i)$ for all $i \in \omega$ and $n < m$. Taking into account that $U_n \cap X_{(1)} = \{e_n\} \not\subset \langle \mathrm{Iso}(X)\rangle$ and $y_n \in \langle \mathrm{Iso}(X)\rangle \cap U_n$, we conclude that $y_n \in U_n \setminus X_{(1)} \subset \mathrm{Iso}(X)$ for all $n \in \omega$. The space X, being compact of scattered height 2, is sequentially compact [16]. Consequently, some subsequence of (y_n) converges to a point $y_\infty \in X_{(1)}$. Without loss of generality, we can assume that the sequence (y_n) tends to y_∞. The continuity of the semilattice operation and the equality $y_n = x_{k,f_n(k)} \wedge y_n$ holding for all $k \le n$ imply that $y_\infty = \lim_n y_n = \lim_n(x_{k,f_n(k)} \wedge y_n) = \lim_n x_{k,f_n(k)} \wedge \lim_n y_n = e_k \wedge y_\infty$. Taking into account that e is the limit point of the sequence $(e_k)_{k\in\omega}$, we conclude that $y_\infty = y_\infty \wedge e$ and thus $y_\infty \le e$. We claim that $y_\infty = e$.

Assuming the converse we would find $m \in \omega$ such that $y_\infty \in F_m$. Now recall that for any $k \ge m$ we have $y_k \in U_k \subset \bar{W}_k \subset \bar{W}_m$. Since the set $\bar{W}_m$ is closed in X, we get $y_\infty \in \bar{W}_m$. Now the inclusion $y_\infty = y_\infty \wedge e \in F_m$ contradicts the choice of the set $\bar{W}_m$. Therefore $y_\infty = e$ and $(y_n) \subset \mathrm{Iso}(X)$ is a sequence of isolated points of X tending to e which contradicts our assumption. This contradiction shows that the partial order on the set C is well-founded.

Consider the subset $L = \{e\} \cup \bigcup\{\uparrow x : x \in \langle \mathrm{Iso}(X)\rangle \cap \uparrow e \setminus \{e\}\}$ and observe that it is at most countable. This follows from the finiteness of the set $X_{(1)} \cap \uparrow x$ for any $x \not\le e$. Observe also that $(X_{(1)} \cap \uparrow e) \setminus L \subset C$.

By induction for every $n \in \omega$ we shall construct an open subsemilattice $V_n \subset V_{n-1} \cap \bar{W}_n$ of X containing the element e, a minimal element e_n of $(V_{n-1} \cap X_{(1)} \cap \uparrow e) \setminus L$, and an element $x_n \in V_{n-1} \cap \mathrm{Iso}(X)$ such that

a) $x_n \wedge z \in X_{(1)}$ for each $z \in V_n \cap \mathrm{Iso}(X)$ and

b) $x_n \wedge x_{n-1} \ne x_{i-1} \wedge x_i$ for every $i \in \{1, \dots, n-1\}$ with $x_{i-1} \wedge x_i \ne e$.

Let $V_{-1} = \bar{W}_0$ and assume that for some $n \in \omega$ the neighborhoods V_i, and points e_i, x_i are constructed for all $i < n$. Since the point e has no countable neighborhood, the set $V_{n-1} \cap X_{(1)}$ is uncountable. Applying Lemma 1 we conclude that the set $V_{n-1} \cap X_{(1)} \cap \uparrow e$ is uncountable and that the set $(V_{n-1} \cap X_{(1)} \cap \uparrow e) \setminus L$, being non-empty and well-founded, contains a minimal element e_n. Since $e_n \in X_{(1)} \setminus X_{(2)}$, there is a sequence $(y_k)_{k\in\omega} \subset \mathrm{Iso}(X) \cap V_{n-1} \cap \bar{W}_n$ tending to e_n. Find a closed neighborhood $O_1 \subset V_{n-1}$ of e such that $e_n \wedge z \ne e_n$ for any $z \in O_1$. We claim that there is a number $k_1 \in \omega$ and a neighborhood $O_2 \subset O_1$ of e such that $z \wedge y_k \in X_{(1)}$ for all $z \in O_2 \cap \mathrm{Iso}(X)$ and $k \ge k_1$. Otherwise we would find an increasing number sequence $(k_i)_{i\in\omega}$ and a sequence $(z_i)_{i\in\omega}$ of pairwise distinct isolated points of O_1 such that $z_i \in \bar{W}_i$ for every $i \in \omega$ and $z_i \wedge y_{k_i} \in \mathrm{Iso}(X)$. By the sequential compactness of X, we can assume that the sequence $(z_i)_{i\in\omega}$ converges to some point $z_\infty \in O_1$. It follows that $z_\infty \in \bigcap_{i\in\omega} \bar{W}_i = \uparrow e$ and $z_\infty \in X_{(1)}$. Note that the sequence $(z_i \wedge y_{k_i})_{i\in\omega} \subset \mathrm{Iso}(X)$ converges to the point $z_\infty \wedge e_n \ne e_n$. Taking into account that no sequence of isolated points tends to e, we conclude that $z_\infty \wedge e_n \ne e$. Recall that $V_{n-1} \cap X_{(1)} \cap \uparrow e \ni z_\infty \wedge e_n < e_n$ and e_n is a minimal element of the set $(V_{n-1} \cap X_{(1)} \cap \uparrow e) \setminus L$. It follows that $z_\infty \wedge e_n \in L$. By the definition of L this yields that $z_\infty \wedge e_n \in \uparrow x$ for some $x \in \langle \mathrm{Iso}(X)\rangle \cap \uparrow e \setminus \{e\}$. Since $e_n \ge z_\infty \wedge e_n$, we get $e_n \in \uparrow x \subset L$ which contradicts the choice of the point e_n.

This contradiction shows that there is a number $k_1 \in \omega$ and a neighborhood $O_2 \subset O_1$ of e such that $z \wedge y_k \in X_{(1)}$ for all $k \ge k_1$ and $z \in O_2 \cap \mathrm{Iso}(X)$. Taking into account that X is a zero-dimensional Lawson semilattice, find a clopen subsemilattice $V_n \subset O_2$ containing e. We can choose V_n so small that $z \wedge y_{k_1} \notin \{x_{i-1} \wedge x_i : i \in \{1, \dots, n-1\}$ and $x_{i-1} \wedge x_i \neq e\}$ for any $z \in V_n$. Finally, put $x_n = y_{k_1}$. This completes the inductive step.

Now consider the sequence $(x_n)_{n\in\omega} \in \mathrm{Iso}(X)$. By the sequential compactness of X a subsequence of (x_n) tends to some element $x_\infty \in X$. Without loss of generality we can assume that all the sequence (x_n) tends to x_∞. Since $x_k \in V_{k-1} \subset \bar{W}_m$ for any $k > m$, we get $x_\infty \in \bigcap_{m\in\omega} \bar{W}_m = \uparrow e$. Our hypothesis yields that $x_\infty \neq e$. Then the sequence $(x_n \wedge x_{n+1})_{n\in\omega}$ tends to $x_\infty \wedge x_\infty = x_\infty$. By the choice of the sequence (x_n) we get $x_n \wedge x_{n+1} \in X_{(1)}$. Since e is a unique non-isolated point of $X_{(1)}$ we conclude that there are numbers $n < m$ such that $x_m \wedge x_{m+1} = x_n \wedge x_{n+1} = x_\infty$, which contradicts the choice of the sequence (x_i). This contradiction finishes the proof of theorem. □

Unlike the compactification $\gamma\mathbb{N}$ from Corollary 1 which has size $\aleph_1$, the size of the compactification $\psi_{\mathcal{F}}(\mathbb{N})$ cannot be too small. More precisely, $|\psi_{\mathcal{F}}(\mathbb{N})| = |\mathcal{F}| \ge \mathfrak{a}$ where $\mathfrak{a}$ is the smallest size of an infinite MAD family on $\mathbb{N}$. It is known that $\aleph_1 \le \mathfrak{a} \le \mathfrak{c}$. Martin's Axiom implies $\mathfrak{a} = \mathfrak{c}$ but there are models of ZFC with $\mathfrak{a} < \mathfrak{c}$, see [4], [18].

We define a topological space X to be a *ψ-space* if X is homeomorphic to the compactification $\psi_{\mathcal{F}}(\mathbb{N})$ for some MAD family $\mathcal{F}$ on $\mathbb{N}$.

The following proposition shows that the cardinality restriction on ψ-spaces is essential. We remind that a topological space X is *Fréchet-Urysohn* if for every point a from the closure $\bar{A}$ of a subset $A \subset X$ in X there is a sequence $(a_n) \subset A$ convergent to a.

Proposition 1. *Suppose that X is a compact scattered space of scattered height 2. If X is not Fréchet-Urysohn, then X contains a copy of a ψ-space and consequently, $|X| \ge \mathfrak{a}$.*

Proof. By [16], the space X, being scattered compact, is sequentially compact. Assuming that X is not Fréchet-Urysohn, we could find a countable subset $A \subset X$ and a point $a \in \bar{A}$ such that no sequence $(a_n) \subset A$ converges to a. It is easy to see that $a \in X_{(2)}$. Let $W \subset X$ be a closed-and-open neighborhood of a such that $W \cap X_{(2)} = \{a\}$. It follows that the intersection $X_{(1)} \cap W \cap A$ is finite and thus we can assume that $A \subset W \setminus X_{(1)}$. Let K be the closure of the set A in X. It follows that $K_{(2)} = \{a\}$, $A = \mathrm{Iso}(K)$ and no sequence $(a_n) \subset A$ converges to a.

We claim that K is a ψ-space. Since $D = K_{(1)} \setminus K_{(2)}$ is a discrete subspace of K, to each point $x \in D$ we can assign a clopen subset U_x of K such that $U_x \cap K_{(1)} = \{x\}$. We claim that $\mathcal{F} = \{U_x \cap A : x \in D\}$ is a MAD family on A.

First we verify that for any distinct points $x, y \in D$ the intersection $U_x \cap U_y$ is finite. Otherwise by the compactness of K, we would find a non-trivial sequence $(a_n) \subset U_x \cap U_y$ having a cluster point $a_\infty \in U_x \cap U_y \cap K_{(1)}$ which is not possible since $U_x \cap U_y \cap K_{(1)} = \emptyset$. Consequently, $|\mathcal{F}| = |D|$ and hence the family $\mathcal{F}$ is infinite.

Next, we show that $\mathcal{F}$ is a *maximal* almost disjoint family. Assuming the converse we would find an infinite set $U \subset A$ such that $U \cap U_x$ is finite for all $x \in D$. By the sequential compactness of K we could find a non-trivial sequence $(a_n)_{n\in\omega} \subset U$ convergent to a point $a_\infty \in K_{(1)}$. Taking into account that $U_x \cap \{a_n : n \in \omega\}$ is finite for all $x \in D$, we conclude that $a_\infty \neq x$ for any $x \in D$ and thus $a_\infty = a \in K_{(2)}$. Therefore, the sequence $(a_n) \subset A$ converges to a, which is a contradiction.

Thus $\mathcal{F}$ is a MAD family on A. Now consider the bijective map $h : K \to \psi_{\mathcal{F}}(A)$ defined by $h(x) = x$ if $x \in A = K \setminus K_{(1)}$, $h(x) = U_x$ if $x \in K_{(1)} \setminus K_{(2)}$, and $h(a) = \infty$. Observe that h is continuous and thus is a homeomorphism by the compactness of K. Therefore X contains the topological copy K of the ψ-space $\psi_{\mathcal{F}}(A)$ and hence $|X| \geq |K| = |\mathcal{F}| \geq \mathfrak{a}$. □ □

Besides the size and the scattered height, there is another essential difference between the scattered compacta $\gamma\mathbb{N}$ and $\psi\mathbb{N}$ from Theorems 2 and 3: unlike the space $\gamma\mathbb{N}$, the space $\psi\mathbb{N}$ embeds into a compact semilattice S so that the remainder $S \setminus \psi\mathbb{N}$ is countable. This follows from the subsequent general

Theorem 5. *Each scattered compactum K of scattered height $\kappa(K) \leq 2$ is a subspace of a scattered compact semilattice $S \supset K$ with open principal filters such that $\kappa(S) = \kappa(K)$, $S_{(1)} = K_{(1)}$ and $|\operatorname{Iso}(S)| = |\operatorname{Iso}(K)|$.*

To prove this theorem we need to make some preliminary work. For an infinite discrete space D let $\alpha D = D \cup \{\infty\}$ be the one-point compactification of D. The compactification αD can be identified with the subspace $\{\chi_\emptyset, \chi_{\{x\}} : x \in D\}$ of the Cantor cube $\{0,1\}^D$ (here for a subset $A \subset D$ by $\chi_A : D \to \{0,1\}$ we denote the characteristic function of the set A, i.e., $\chi_A(x) = 1$ iff $x \in A$). We can think of $\{0,1\}^D$ as a compact semilattice endowed with the usual min-operation. In this case $\alpha D \subset \{0,1\}^D$ is an ideal in $\{0,1\}^D$ in the sense that $\min(x,y) \in \alpha D$ for each $x \in \alpha D$ and $y \in \{0,1\}^D$.

For a subset $A \subset \{0,1\}^D$ let $\langle A\rangle$ be the smallest subsemilattice of $\{0,1\}^D$ containing the set A.

Lemma 2. *If K is a subspace of $\{0,1\}^D$ such that $K_{(1)} = \alpha D$, then $\langle K\rangle$ is a compact scattered semilattice with $\langle K\rangle_{(1)} = \alpha D$ and $|\operatorname{Iso}(\langle K\rangle)| = |\operatorname{Iso}(K)|$.*

Proof. First we verify that the semilattice $\langle K\rangle$ is compact. Given an open cover $\mathcal{U}$ of $\langle K\rangle$ find a finite subcover $\mathcal{V}$ of the compact subset $\alpha D = K_{(1)} \subset \langle K\rangle$. Taking into account that αD is an ideal of $\{0,1\}^D$, find an open ideal V of $\{0,1\}^D$ such that $\alpha D \subset V \subset \cup\mathcal{V}$. Since $K_{(1)} \subset V$, the set $F = K \setminus V$ is finite. Consequently, the set $\langle K\rangle \setminus V = \langle F\rangle$ is finite as well and hence can be covered by a finite subcover $\mathcal{W}$ of $\mathcal{U}$. Then $\langle K\rangle$ is covered by the finite subcover $\mathcal{V} \cup \mathcal{W}$ of $\mathcal{U}$ which yields the compactness of $\langle K\rangle$.

The above argument implies that the set $\langle K\rangle \setminus V$ is finite for any neighborhood V of $K_{(1)}$. This observation implies that $\langle K\rangle_{(1)} = K_{(1)} = \alpha D$. Then $\operatorname{Iso}(K) \subset \operatorname{Iso}(\langle K\rangle)$ and hence $|\operatorname{Iso}(K)| \leq |\operatorname{Iso}(\langle K\rangle)|$.

Since $K_{(1)} = \alpha D$ is an ideal of $\{0,1\}^D$, $\langle K\rangle = K_{(1)} \cup \langle \operatorname{Iso}(K)\rangle$ and thus $\operatorname{Iso}(\langle K\rangle) \subset \langle \operatorname{Iso}(K)\rangle$ which yields $|\operatorname{Iso}(K)| = |\operatorname{Iso}(\langle K\rangle)|$. □ □

Now we are able to prove Theorem 5.

PROOF of Theorem 5. Suppose that K is a compact scattered space of scattered height $\kappa(K) \leq 2$. Repeating the argument in the proof of Theorem 1 we can reduce the proof to the special case when the second derived set $K_{(2)}$ of K consists of a unique point ∞.

For $i = 0, 1$ let $D_i = K_{(i)} \setminus K_{(i+1)}$ where $K_{(0)} = K$. Let $f_0 : K \to \{0,1\}^{D_0}$ be the continuous function defined by $f_0(x) = \chi_\emptyset$ if $x \in K_{(1)}$ and $f_0(x) = \chi_{\{x\}}$ if $x \in D_0 = K \setminus K_{(1)}$. By analogy, define a function $f_1 : K_{(1)} \to \{0,1\}^{D_1}$ letting $f_1(x) = \chi_\emptyset$ if $x \in K_{(2)}$ and $f_1(x) = \chi_{\{x\}}$ if $x \in D_1 = K_{(1)} \setminus K_{(2)}$. Since K is a zero-dimensional compact space, the function f_1 extends to a continuous function $\bar{f}_1 : K \to \{0,1\}^{D_1}$. Now consider the function $f = (f_0, \bar{f}_1) : K \to \{0,1\}^{D_0} \times \{0,1\}^{D_1}$ and observe that it is injective and hence is a topological embedding.

We can think of the product $P = \{0,1\}^{D_0} \times \{0,1\}^{D_1}$ as a compact topological semilattice endowed with the usual min-operation. Let S be the smallest subsemilattice of P containing the compactum $f(K)$. The previous lemma implies that S is a compact scattered semilattice with $S_{(1)} = f(K_{(1)})$, $\kappa(S) = \kappa(K)$ and $|\operatorname{Iso}(S)| = |\operatorname{Iso}(K)|$. It is easy to see that S is a compact semilattice with open principal filters. This finishes the proof of the theorem. □

Theorem 5 implies that the space $\psi\mathbb{N}$ from Theorem 3 is a subspace of a separable scattered compact semilattice S with open principal filters such that $\kappa(S) = 2$. Since $\psi(S)$ is not Fréchet-Urysohn, S is not Fréchet-Urysohn either. Thus we get the following surprising result nicely complementing Theorem 4.

Corollary 2. *There is a separable scattered compact semilattice S with open principal filters and scattered height $\kappa(S) = 2$ which fails to be a Fréchet-Urysohn space.*

There are also simple examples of Fréchet-Urysohn separable non-metrizable topological semilattices with open principal filters and scattered height 2. Such an example can be constructed as follows. Let C be the Cantor cube $\{0,1\}^\omega$ and $\mathcal{B}$ be the standard base of the topology of C, i.e., $\mathcal{B} = \{\mathrm{pr}_n^{-1}(x) : n \in \omega,\ x \in \{0,1\}^n\}$ where $\mathrm{pr}_n : C \to \{0,1\}^n$, $\mathrm{pr}_n : (x_i)_{i=0}^\infty \mapsto (x_i)_{i=0}^{n-1}$, is the natural projection. For a subset $A \subset C$ denote by $\chi_A : C \to \{0,1\}$ the characteristic function of the set A (that is, $\chi_A(x) = 1$ iff $x \in A$). Such a characteristic function is an element of the non-metrizable Cantor cube $\{0,1\}^C$. In this cube consider the subspace

$$S = \{\chi_\emptyset, \chi_{\{x\}}, \chi_U : x \in C, U \in \mathcal{B}\} \subset \{0,1\}^C.$$

It is easy to see that S is a compact scattered subspace of $\{0,1\}^C$ of scattered height 2. It should be mentioned that the space S is well-known in topology as an example of a compact scattered space with is neither supercompact, nor hereditarily normal, see [15].

Observe that S is a subsemilattice of $\{0,1\}^C$ with respect to the coordinate-wise min-operation. Moreover, it can be easily shown that the topological semilattice $(S, \min)$ has the following properties.

Proposition 2. *The space S endowed with the min-operation is a Fréchet-Urysohn separable non-metrizable scattered compact topological semilattice with open principal filters and scattered height 2.*

Now let us pass to open problems, first of which is suggested by Theorem 4.

Problem 1. *Suppose X is a compact topological semilattice of finite scattered height. Is the set $Iso(X)$ of isolated points of X sequentially dense in X?*

Another question is suggested by Theorem 5.

Problem 2. *Is every (separable) scattered compact space [with finite scattered height] a subspace of a (separable) scattered compact topological semilattice?*

As we saw in Corollary 2 a separable scattered compact semilattice needs not be Fréchet-Urysohn. It should be mentioned that the class of compact Fréchet-Urysohn spaces includes all Corson compacta (and consequently, Eberlein, Gulko, Talagrand compacta) and all Rosenthal and Rosenthal-Banach compacta, see [10], [3], [9], [14]. The compactifications $\gamma\mathbb{N}$ and $\psi_{\mathcal{F}}(\mathbb{N})$ from Theorems 2 and 4 are not Fréchet-Urysohn.

Problem 3. *Does every scattered Fréchet-Urysohn (Eberlein) compact space X (with scattered height 2) admit a continuous semilattice operation?*

In light of Theorem 2 and Remark 1 it is reasonable to ask

Problem 4. *Does every scattered compact space admit a separately continuous operation turning it into an inverse semigroup?*

A semilattice operation is a particular case of a *mean*, i.e., a function $m : X \times X \to X$ such that $m(x, y) = m(y, x)$ and $m(x, x) = x$ for every $x, y \in X$, see [5].

Problem 5. *Does every scattered compact space X admit a (separately) continuous mean $m : X \times X \to X$?*

Following [5] we say that a compact space K is *hyadic* if X is a continuous image of the space of all closed subsets of some compact space in the Vietoris topology (equivalently, X is a continuous image of a zero-dimensional compact topological semilattice). By [5] each non-discrete G_δ-subspace of a hyadic compact space contains a non-trivial convergent sequence. Consequently, the space $\beta\mathbb{N}$ is not hyadic. In fact each Tychonov space X with hyadic βX is pseudocompact, see [5, p.42].

Problem 6. *Is every scattered compact space hyadic? In particular, are the spaces $\gamma\mathbb{N}$ and $\psi_{\mathcal{F}}(\mathbb{N})$ from Corollary 1 and Theorem 3 hyadic? Is $\gamma\mathbb{N}$ (resp. $\psi_{\mathcal{F}}(\mathbb{N})$) a continuous image of a compact topological semilattice?*

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