Spaces of continuous functions over Dugundji compacta

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Abstract

We show that for every Dugundji compact K of weight $\aleph_1$ the Banach space $C(K)$ is 1-Plichko and the space $P(K)$ of probability measures on K is Valdivia compact. Combining this result with the existence of a non-Valdivia compact group, we answer a question of Kalenda.

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1 Introduction

Given an infinite-dimensional Banach space E, it is an important question how many nontrivial projections it does have. A *projection* is, by definition, a bounded linear operator $P\colon E \to E$ such that $PP = P$. A projection P is *nontrivial* if both $\operatorname{im} P$ and $\ker P$ are infinite-dimensional. There exist *indecomposable* Banach spaces, i.e. spaces on which every projection is trivial; by [9] such spaces can be even of the form $C(K)$ for a suitable (non-metrizable) compact K.

We are interested in non-separable Banach spaces which have "many" nontrivial norm one projections onto separable subspaces. Note that every retraction of a compact space K induces a norm one projection on $C(K)$. Thus, it is natural to ask whether there is a compact space K having few retractions onto metrizable subsets, while its Banach space $C(K)$ has "many" norm one projections onto separable subspaces. Of course such a question is not precise. A reasonable yet large enough class of Banach spaces with "many" norm one projections seems to be the class of 1-*Plichko spaces* (see the definitions below). On the other hand, one should say that a compact K has *few retractions* if for every retraction $r\colon K \to K$ with $X = r[K]$ being second countable, there are some restrictive conditions on the topological type of X. For example: the first cohomology group $H^1(X)$ be trivial. This is motivated by [14], where a compact (connected Abelian) group G with this property is described. Trying to investigate this particular $C(G)$ space, one can look for a topological property of G imposing the existence of many projections on $C(G)$. It turns out that the property of being *Dugundji compact* is good enough. Namely, we show that $C(K)$ is a 1-Plichko space for every Dugundji compact K of weight $\leqslant \aleph_1$. In particular, it follows that the space of probability measures $P(K)$ may be Valdivia compact (the property dual to being 1-Plichko), while at the same time K can be relatively far from Valdivia compacta (again witnessed by the compact group G from [14]). This answers a question of Ondřej Kalenda [7, Question 5.1.10] in the negative.

We do not know whether $C(K)$ is 1-Plichko, for a Dugundji compact K of weight $> \aleph_1$. One can prove that in this case $P(K)$ is a retract of a Valdivia compact and $C(K)$ is isomorphic to a 1-complemented subspace of a Plichko space. However, it is an open question whether Valdivia compacta are stable under retracts and whether Plichko spaces are stable under complemented subspaces. One should mention that every Plichko space admits an equivalent locally uniformly convex norm, see [3]. Such a norm on the space $C(G)$, where G is a compact group, has already been constructed by Aleksandrov in [1].

2 Preliminaries

We use standard notation concerning topology, set theory and Banach spaces. By a *map* we mean a continuous map. A *projection* in a Banach space E is a bounded linear operator $P\colon E \to E$ such that $PP = P$. One says that F is *complemented* in E if $F = \operatorname{im} P := \{Px\colon x \in E\}$ for some projection $P\colon E \to E$. More precisely, F is *k-complemented* if $\|P\| \leqslant k$. Let T be a linear operator between subspaces of Banach spaces of the form $C(K)$. Then T is called *regular* if T is *positive*, i.e. $Tf \geqslant 0$ whenever $f \geqslant 0$, and $T1 = 1$, where 1 denotes the constant function with value 1 (so it is assumed that this function belongs to the domain of T).

Let X, Y be two compact spaces and assume $f\colon X \to Y$ is a continuous surjection. We denote by f^* the operator $S\colon C(Y) \to C(X)$ defined by $S(\psi) = \psi f$, for $\psi \in C(Y)$. Clearly, f^* is linear and provides an isometric embedding of $C(Y)$ into $C(X)$. One usually identifies $C(Y)$ with the subspace of $C(X)$, via the quotient map f. A *regular averaging operator* associated with f, is a regular linear operator $T\colon C(X) \to C(Y)$ satisfying $T(\psi f) = \psi$ for every $\psi \in C(Y)$. Observe that given a regular averaging operator $T\colon C(X) \to C(Y)$, the map $P = f^*T$ is a regular (in particular: norm one) projection of $C(X)$ onto the subspace $\operatorname{im} f^* = \{\psi f\colon \psi \in C(Y)\}$, isomorphic to $C(Y)$. Regular averaging operators were introduced and studied by Pełczyński [16], motivated by Milyutin's Lemma [15], which says that there exists a continuous surjection of the Cantor set onto the unit interval admitting such an operator.

Given a compact space K, we denote by $P(K)$ the space of all regular probability measures on K. In other words,

$$P(K) = \{\mu \in C(K)^*\colon \|\mu\| = 1 \text{ and } \mu(\varphi) \geqslant 0 \text{ for every } \varphi \geqslant 0\}.$$

We shall always consider $P(K)$ with the weak-star topology inherited from $C(K)^*$. Every continuous map of compact spaces $f\colon X \to Y$ induces a map $P(f)\colon P(X) \to P(Y)$ defined by $P(f)(\mu)(\varphi) = \mu(\varphi f)$, $\varphi \in C(Y)$. By this way P becomes a functor, usually called the *probability measures functor*. Note that $P(K)$ is a convex compact subset of $C(K)^*$ and it is second countable whenver K is so. Given a second countable compact K, a special case of Michael's Selection Theorem says that every lower semi-continuous map Φ, defined on a paracompact space X, whose values are closed convex subsets of $P(K)$, has a continuous selection, i.e. a map $h\colon X \to P(K)$ such that $h(x) \in \Phi(x)$ for every $x \in X$.

A Banach space E is *k-Plichko* if there are a linearly dense set $X \subseteq E$ and a k-norming set $Y \subseteq E^*$ such that for every $y \in Y$ the set $\{x \in X\colon y(x) \neq 0\}$ is countable. Recall that Y is *k-norming* if $\|v\| \leqslant k \sup\{|y(v)|/\|y\|\colon y \in Y\}$ for every $v \in E$. We shall be interested in 1-Plichko spaces. In the case of density $\aleph_1$, 1-Plichko spaces are characterized as Banach spaces with a *projectional resolution of the identity*, i.e. with a sequence $\{P_\alpha\}_{\alpha<\omega_1}$ of norm one projections onto separable subspaces satisfying the following conditions:

1. $\alpha < \beta \implies P_\alpha = P_\alpha P_\beta = P_\beta P_\alpha$;
2. $\bigcup_{\alpha<\omega_1} \operatorname{im} P_\alpha$ is the whole space and $\operatorname{im} P_\delta = \operatorname{cl}(\bigcup_{\alpha<\delta} \operatorname{im} P_{\alpha+1})$ for every limit ordinal δ,

where ω_1 denotes the first uncountable ordinal. For details we refer to Kalenda's survey [7]. The well known notion of a projectional resolution of the identity is defined for an arbitrary Banach space E, where it is required that the density of $P_\alpha E$ does not exceed the cardinality of $\alpha + \omega$, see e.g. [5, 3].

Plichko spaces are closely related to Valdivia compacta. Recall that a compact space K is called *Valdivia compact* if $K \subseteq [0,1]^\kappa$ so that $K \cap \Sigma(\kappa)$ is dense in K, where $\Sigma(\kappa)$ is the *Σ-product* of κ copies of $[0,1]$, i.e. $\Sigma(\kappa) = \{x \in [0,1]^\kappa\colon |\{\alpha\colon x(\alpha) \neq 0\}| \leqslant \aleph_0\}$. Let us recall that the Banach space $C(K)$ is 1-Plichko whenever K is Valdivia compact. On the other hand, straight from the definition it follows that for a 1-Plichko Banach space E, the dual unit ball of E endowed with the weak-star topology is Valdivia compact. For details we refer to [7].

We are going to use inverse sequences of compact spaces, so we briefly recall the necessary definitions. Let δ be an infinite limit ordinal. An *inverse sequence* of length δ is a triple of the form $\mathbb{S} = \langle X_\alpha, p_\alpha^\beta, \delta\rangle$, where for each $\alpha < \delta$, X_α is a topological (typically: compact) space and for each $\alpha < \beta < \delta$, $p_\alpha^\beta\colon X_\beta \to X_\alpha$ is a continuous (typically: quotient) map, called a *bonding map*. Moreover, the following compatibility is required: $p_\alpha^\gamma = p_\alpha^\beta p_\beta^\gamma$ for every $\alpha < \beta < \gamma < \delta$. The *limit* of $\mathbb{S}$ is a space $X = \varprojlim \mathbb{S}$ together with maps $p_\alpha\colon X \to X_\alpha$ $(\alpha < \delta)$ satisfying the following condition: given a topological space Y and a collection of maps $\{f_\alpha\}_{\alpha<\delta}$ such that $f_\alpha\colon Y \to X_\alpha$ and $f_\alpha = p_\alpha^\beta f_\beta$ for every $\alpha < \beta < \delta$, there exists a unique map $f\colon Y \to X$

such that $p_\alpha f = f_\alpha$ holds for every $\alpha < \delta$. The maps p_α are called *projections*. Typically, $\varprojlim \mathbb{S}$ is represented as $X = \{x \in \prod_{\alpha<\delta} \colon p_\alpha^\beta(x(\beta)) = x(\alpha)$ for every $\alpha < \beta < \delta\}$, where p_α is the projection onto α-th coordinate. The inverse sequence $\mathbb{S} = \langle X_\alpha, p_\alpha^\beta, \delta\rangle$ is *continuous* if for every limit ordinal $\gamma < \delta$ the space X_γ together with the collection $\{p_\alpha^\gamma\}_{\alpha<\gamma}$ is the limit of the sequence $\langle X_\alpha, p_\alpha^\beta, \gamma\rangle$.

We recall the definition of the class $\mathcal{R}$, introduced in [2]. It is the smallest class of (compact) spaces that contains all metric compacta and which is stable under limits of continuous inverse sequences whose bonding maps are retractions. Every Valdivia compact has a decomposition into a continuous inverse sequence of retractions onto smaller Valdivia compacta (see e.g. [7]), therefore it belongs to $\mathcal{R}$. For more information concerning class $\mathcal{R}$ and its properties, we refer to [11].

A *Dugundji compact* is a compact space K which is an *absolute extensor* for the class of all 0-dimensional compact spaces; that is: given a 0-dimensional compact X and a continuous map $f\colon A \to K$ defined on a closed subset of X, there exists a continuous map $F\colon X \to K$ such that $F \restriction A = f$. We shall use the following useful characterization of Dugundji compacta, due to Haydon [6]:

Haydon's Theorem. *Let K be a compact space. Then K is Dugundji compact if and only if $K = \varprojlim \mathbb{S}$, where $\mathbb{S} = \langle K_\xi, p_\xi^\eta, \kappa\rangle$ is a continuous inverse sequence such that K_0 is metrizable and each $p_\xi^{\xi+1}$ is an open surjection with a metrizable kernel.*

Recall that a quotient map of compact spaces $f\colon X \to Y$ has a *metrizable kernel* if there is a map $h\colon X \to Z$ such that Z is second countable and the diagonal map $f \triangle h$ is one-to-one. Equivalently: there exists a second countable space Z such that X embeds into $Y \times Z$ so that f is homeomorphic to the projection onto the first coordinate.

Interesting and important examples of Dugundji spaces are compact groups, see Uspenskij's article [17]. Let us note that 0-dimensional Dugundji compacta are Valdivia [13], although by [14] there exist compact Abelian groups which are not in the class $\mathcal{R}$.

The following lemma is well known. We give the proof for the sake of completeness.

Lemma 2.1. *Assume X, Y are compact spaces and $f\colon X \to Y$ is an open surjection with a metrizable kernel. Then f admits a regular averaging operator.*

Proof. Let Q be a metric compact such that $X \subseteq Y \times Q$ and f is the projection onto the first coordinate. Let $\pi\colon X \to Q$ denote the projection onto the second coordinate. Define a multifunction Φ, from Y to the power set of Q, by setting

$$\Phi(y) = \pi[f^{-1}(y)].$$

Then Φ has nonempty compact values. Since f is open, Φ is lower semi-continuous. We identify Q with a suitable subset of the metrizable locally convex space $C(Q)^*$, endowed with the weak-star topology. By Michael's Selection Theorem, there exists a continuous map $h_0\colon Y \to C(Q)^*$

such that $h_0(y) \in \mathrm{cl}_*(\mathrm{conv}\,\Phi(y))$ for every $y \in Y$, where cl_* denotes the weak-star closure. It follows that $h_0(y)$ is a probability measure whose support is contained in $\pi[f^{-1}(y)]$. Now define $h(y) \in P(X)$ by setting

$$h(y)(\psi) = h_0(\psi_y),$$

where $\psi_y \in C(Q)$ is defined by $\psi_y(t) = \psi(y,t)$. The map $y \mapsto \psi_y$ is continuous with respect to the norm topology on $C(Q)$, therefore $h\colon Y \to P(X)$ is continuous with respect to the weak-star topology on $P(X)$. Now define $T\colon C(X) \to C(Y)$ by

$$(T\psi)(y) = \int_X \psi \, dh(y).$$

By the continuity of h, $T\psi$ is indeed a continuous function. Thus T is a regular linear operator. Now assume $\psi = \varphi f$. Then ψ has constant value $\varphi(y)$ on the set $f^{-1}(y)$. Recalling that the support of $h(y)$ is contained in $f^{-1}(y)$, we deduce that $(T\psi)(y) = \varphi(y)$. Thus T is a regular averaging operator. □

A collection of sets $\{S_\alpha\}_{\alpha<\lambda}$ is a *chain* if $S_\alpha \subseteq S_\beta$ whenever $\alpha < \beta$. A chain $\{E_\alpha\}_{\alpha<\lambda}$ of closed subspaces of a Banach space is *continuous* if $E_\delta = \mathrm{cl}(\bigcup_{\alpha<\delta} E_\alpha)$ for every limit ordinal $\delta < \lambda$.

Lemma 2.2 ([10])**.** *Let E be a Banach space and assume $\{E_\alpha\}_{\alpha<\lambda}$ is a continuous increasing chain of closed subspaces of E with $E = \mathrm{cl}(\bigcup_{\alpha<\lambda} E_\alpha)$ (λ is a limit ordinal). Assume that for each $\alpha < \lambda$, $R_\alpha\colon E_{\alpha+1} \to E_\alpha$ is a norm one projection. Then there exists a sequence $\{P_\alpha\}_{\alpha<\lambda}$ of projections of E such that*

(1) $\|P_\alpha\| = 1$ and $P_\alpha E = E_\alpha$,

(2) $\alpha \leqslant \beta < \lambda \implies P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$.

(3) $P_\alpha \restriction E_{\alpha+1} = R_\alpha$.

If, additionally, E is of the form $C(K)$ and each R_α is regular, then we may assume that each P_α is regular.

Proof. We construct inductively norm one linear projections $P_\alpha^\beta\colon E_\beta \to E_\alpha$, where $\alpha < \beta \leqslant \lambda$ and $E_\lambda = E$, satisfying the following condition

$$(*) \qquad \alpha < \beta < \gamma \implies P_\alpha^\beta P_\beta^\gamma = P_\alpha^\gamma.$$

Suppose $\delta > 0$ and P_α^β have been defined for all $\alpha \leqslant \beta < \delta$. If $\delta = \varrho + 1$ then define $P_\alpha^{\varrho+1} = P_\alpha^\varrho Q_\alpha$ for $\alpha < \delta$. Assume now that δ is a limit ordinal. Let $D = \bigcup_{\xi<\delta} E_\xi$. Then D is a dense linear subspace of E_δ. Fix $\alpha < \delta$ and define $P_\alpha^\delta(x) = P_\alpha^\xi(x)$, where ξ is any ordinal satisfying $\alpha < \xi < \delta$ and $x \in E_\xi$. Then P_α^δ is a well defined norm one projection of D onto E_α, therefore it extends uniquely onto E_δ. Given $\alpha < \beta < \delta$, the formula $P_\alpha^\beta P_\beta^\delta(x) = P_\alpha^\delta(x)$ is valid for every $x \in D$, therefore by continuity it holds for every $x \in E_\delta$.

Now define $P_\alpha = P_\alpha^\lambda$. Clearly (1) holds. Fix $\alpha < \beta$. Condition (*) for $\gamma = \lambda$ says that $P_\alpha = P_\alpha^\beta P_\beta$, therefore $P_\alpha \restriction E_\beta = P_\alpha^\beta$. Thus $P_\alpha P_\beta = P_\alpha$ and also (3) holds, because $P_\alpha^{\alpha+1} = R_\alpha$. On the other hand, $P_\beta P_\alpha = P_\alpha$, because $E_\alpha \subseteq E_\beta$. This shows (2).

Finally, in case where $E = C(K)$, it suffices to recall that regular operators are closed under compositions and pointwise limits. □

3 Main result

Theorem 3.1. *Let K be a Dugundji compact of weight $\aleph_1$. Then $C(K)$ has a projectional resolution of the identity $\{P_\alpha\}_{\alpha<\omega_1}$ such that each P_α is regular. In particular, $C(K)$ is 1-Plichko and the space of probability measures $P(K)$ is Valdivia compact.*

Proof. By Haydon's Theorem, $K = \varprojlim \langle K_\alpha, p_\alpha^\beta, \omega_1 \rangle$, where each K_α is a compact metric space, each p_α^β is an open surjection and the sequence is continuous. It suffices to show that for every $\alpha < \omega_1$, $C(K_\alpha)$ is complemented by a regular projection of $C(K_{\alpha+1})$, where we identify each $C(K_\alpha)$ with $p_\alpha^*[C(K_\alpha)]$ ($p_\alpha \colon K \to K_\alpha$ is the projection). Indeed, Lemma 2.2 will give us a projectional resolution of the identity $\{P_\alpha\}_{\alpha<\omega_1}$ on $C(K)$ which consists of regular operators. Thus $C(K)$ is 1-Plichko (see [7]). Since P_α's are regular, the dual operators P_α^* provide a continuous inverse sequence of retractions of $P(K)$ onto metrizable subspaces which, by [13, Corollary 4.3], shows that $P(K)$ is Valdivia compact.

Fix $\alpha < \omega_1$. Since $q := q_\alpha^{\alpha+1}$ is open, by Lemma 2.1, it admits a regular averaging operator $T \colon C(K_{\alpha+1}) \to C(K_\alpha)$. Then $R_\alpha := q^* T$ is a norm one regular projection of $C(K_{\alpha+1})$ onto $q^*[C(K_\alpha)]$, identified with $C(K_\alpha)$. □

The following statement gives a negative answer to a question of O. Kalenda [7, Question 5.1.10(i)].

Theorem 3.2. *There exists a compact space K of weight $\aleph_1$, such that $K \notin \mathcal{R}$, while $P(K)$ is Valdivia compact and $C(K)$ is 1-Plichko.*

Proof. Let K be the Abelian group described in [14]. Then $\mathrm{w}(K) = \aleph_1$, $K \notin \mathcal{R}$ and K is a Dugundji space (being a compact group [17]). Thus, by Theorem 3.1, $P(K)$ is Valdivia and $C(K)$ is 1-Plichko. □

Remark 3.3. One should point out two things concerning Theorem 3.1. First, we did not have to show that $P(K)$ is Valdivia compact, because it follows from [7, Theorem 5.1.2]. Second, Theorem 3.1 is valid (with the same proof) for every compact space K which can be represented as $K = \varprojlim \mathbb{S}$, where $\mathbb{S} = \langle K_\alpha, p_\alpha^\beta, \omega_1 \rangle$ is a continuous inverse sequence of metric compacta such that for every $\alpha < \omega_1$ the map $p_\alpha^{\alpha+1} \colon K_{\alpha+1} \to K_\alpha$ admits a (not necessarily regular!) norm one averaging operator, i.e. such that $C(K_\alpha)$ is 1-complemented in $C(K_{\alpha+1})$, under the suitable identification. This statement fails when ω_1 is replaced by ω_2 (the first ordinal of cardinality $\aleph_2$). For example, consider $K = \omega_2 + 1$ as the linearly ordered space. It has been proved by Kalenda [8] that $C(K)$ is not a Plichko space. On the other hand, K is the limit of a continuous inverse sequence of spaces of weight $\aleph_1$ in which all bonding maps are retractions (define $K_\alpha = \alpha + 1$, $p_\alpha^{\alpha+1} \restriction K_\alpha = \mathrm{id}_{K_\alpha}$ and $p_\alpha^{\alpha+1}(\alpha+1) = \alpha$).

Remark 3.4. It is well known that for every Dugundji compact K there exists a continuous surjection $f\colon 2^\kappa \to K$ which admits a regular averaging operator. This gives a one-complemented isometric embedding of $C(K)$ into $C(2^\kappa)$ and a retraction of $P(2^\kappa)$ onto $P(K)$. The space $P(2^\kappa)$ is Valdivia compact and the space $C(2^\kappa)$ is 1-Plichko.

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