WEAKLY P-SMALL NOT P-SMALL SUBSETS IN GROUPS

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Abstract. Answering a question of D.Dikranjan and I.Protasov we prove that each infinite group G contains a weakly P-small subset $A \subset G$ which is not P-small. This means that for every n there are n pairwise disjoint translation copies of A in G but there is no infinitely many such disjoint copies.

In this paper we answer a question of D.Dikranjan and I.Protasov concerning the relation between P-small and weakly P-small sets in groups. We recall that a subset A of a group G is called

- *P-small* (or else *small in the sense of Prodanov*) if there is an infinite set $B \subset G$ such that $xA \cap yA = \emptyset$ for all distinct elements $x, y \in B$;
- *weakly P-small* if for every $n \in \mathbb{N}$ the re is a subset $B_n \subset G$ of size $|B_n| \geq n$ such that $xA \cap yA = \emptyset$ for all distinct elements $x, y \in B_n$.

It is clear that each P-small set is weakly P-small. The converse is not true: each infinite Abelian group G contains a weakly P-small subset A that is not P-small, see [1]. In this paper we prove a non-commutative version of this result.

Main Theorem. *Each infinite group G contains a weakly P-small subset $A \subset G$ that is not P-small.*

The proof of this theorem relies on the existence of a subset $\mathbb{B} = \mathbb{B}^{-1}$ of G with the following properties:

(1) for every $n \in \mathbb{N}$ there is a subset $B_n \subset G$ of size $|B_n| \geq n$ such that $B_n^{-1} B_n \subset \mathbb{B}$;
(2) $c_1\mathbb{B} \cap c_2\mathbb{B} \cap c_3\mathbb{B}$ is finite for any pairwise distinct points $c_1, c_2, c_3 \in G$;
(3) $F\mathbb{B}F \neq G$ for any subset $F \subset G$ of size $|F| < |G|$.

Here by $|A|$ we denote the cardinality of a set A.

Assuming for a moment that such a set $\mathbb{B}$ exists we shall construct a weakly P-small subset $A \subset G$ which is not P-small.

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Let $\mathbb{B}^\circ = \mathbb{B} \setminus \{e\}$, where e stands for the neutral element of G. We shall construct a subset A so that $A \cap \mathbb{B}^\circ A = \emptyset$ and $G \setminus \mathbb{B}^\circ \subset AA^{-1}$.

Let $\kappa = |G\setminus\mathbb{B}^\circ|$ and $G\setminus\mathbb{B}^\circ = \{g_\alpha : \alpha < \kappa\}$ be an enumeration of $G\setminus\mathbb{B}^\circ$ by ordinals $\alpha < \kappa$.

The set A will be of the form $A = \bigcup_{\alpha<\kappa}\{a_\alpha, g_\alpha a_\alpha\}$ for a suitable sequence $(a_\alpha)_{\alpha<\kappa}$ in G. This clearly yields $G\setminus\mathbb{B}^\circ \subset AA^{-1}$. So it remains to define the sequence $(a_\alpha)_{\alpha<\kappa}$ so that $A \cap \mathbb{B}^\circ A = \emptyset$. This will be done by induction.

We start with $a_0 = e$.

Assuming that for some α the points $a_\beta, \beta < \alpha$, have been constructed, consider the set $F_\alpha = \{a_\beta, g_\beta a_\beta : \beta < \alpha\} \cup \{e, g_\alpha^{-1}\}$ containing $< \kappa$ elements. By the property (3) of the set $\mathbb{B}$ we have $F_\alpha \mathbb{B} F_\alpha \neq G$.

Therefore we can pick any point $a_\alpha \in G$ with

$$a_\alpha \notin F_\alpha \mathbb{B} F_\alpha.$$

This gives $A \cap (\mathbb{B}^\circ A) = \emptyset$.

It remains to show that A satisfies the conclusion of the theorem.

It follows from $A \cap \mathbb{B}^\circ A = \emptyset$ and $B_n^{-1}B_n \subset \mathbb{B}$ that for any distinct points $x, y \in B_n$ we get $x^{-1}yA \cap A = \emptyset$, which is equivalent to $xA \cap yA = \emptyset$. This clearly yields that A is weakly P-small.

Next we check that A is not P-small. According to the property (2) of the set $\mathbb{B}$, for any infinite subset B_∞ of G and distinct elements $c_1, c_2, c_3 \in B_\infty$ the set $c_1\mathbb{B} \cap c_2\mathbb{B} \cap c_3\mathbb{B}$ is finite and so there is $b \in B_\infty$ with $c_1^{-1}b \notin \mathbb{B}$ or $c_2^{-1}b \notin \mathbb{B}$ or $c_3^{-1}b \notin \mathbb{B}$. In any case B_∞ contains two distinct points x, y with $x^{-1}y \notin \mathbb{B}$. Then $x^{-1}y \in G\setminus\mathbb{B}^\circ \subset AA^{-1}$ and hence the intersections $xA \cap yA$ is not empty, which yields that A is not P-small.

Thus it remains to prove that each group G contains a subset $\mathbb{B}$ with properties (1)—(3). In fact, it suffices to consider the case of a countable group G. Otherwise we can take any countable infinite subgroup H of G and find a subset $\mathbb{B}$ of H with properties (1)–(3). It is clear that $\mathbb{B}$ has the properties (1),(2) in the whole group G. Property (3) follows from the fact that the set $\mathbb{B}$ is countable while G is not.

So we can assume that G is a countable group which can be injectively enumerated as $G = \{g_i : i \in \omega\}$ with $g_0 = e$ being the neutral element of G. For every $n \in \omega$ consider the finite subset $F_n = \{g_i, g_i^{-1} : 1 \le i \le n\}$ of G.

The set $\mathbb{B}$ will be constructed in the form

$$\mathbb{B} = \bigcup_{n=1}^{\infty} B_n^{-1} B_n$$

where $B_n = \{b_n^i : 1 \le i \le n\}$ are to be chosen later. Obviously $\mathbb{B}$ has property (1).

To simplify notation we write $\mathbb{B}_{\le k}$ instead of $\bigcup_{n=1}^{k} B_n^{-1}B_n$ and $\mathbb{B}_{>k}$ instead of $\bigcup_{n=k+1}^{\infty} B_n^{-1}B_n$. Also we put $B_n^{<i} = \{b_n^j : 1 \le j < i\}$.

Now we are in a position to define a sequence of sets $B_n = \{b_n^i : 1 \le i \le n\}$ such that the set $\mathbb{B}$ will have properties (2) and (3).

To ensure property (3) we also will construct a sequence $(h_n)_{n\in\omega}$ in G, such that $h_n \notin F_n\mathbb{B}F_n$.

We start with $B_1 = \{e\}$ and pick any point h_1 with $h_1 \notin F_1^2$. Next we proceed by induction. Suppose that for some n the sets B_k and the elements h_k, $k < n$, have been constructed.

Then pick any point h_n with

$$h_n \notin F_n\mathbb{B}_{<n}F_n.$$

Such a point exists because the set $F_n\mathbb{B}_{<n}F_n$ is finite and G is infinite. Let

$$H_n = \{h_i, h_i^{-1} : 1 \le i \le n\}.$$

Next we define inductively the elements of $B_n = \{b_n^i : 1 \le i \le n\}$.

We pick any b_n^1 with $b_n^1 \in G\backslash\mathbb{B}_{<n}$. Next, for $1 < i \le n$ we choose b_n^i, to satisfy four conditions:

(a) $b_n^i \notin B_n^{<i}F_n\mathbb{B}_{<n} \cup B_n^{<i}\mathbb{B}_{<n}F_n$;
(b) $b_n^i \notin B_n^{<i}(B_n^{<i})^{-1}B_n^{<i}F_n \cup B_n^{<i}F_n(B_n^{<i})^{-1}B_n^{<i}$;
(c) $b_n^i \notin B_n^{<i}F_nH_nF_n$;
(d) $b_n^i \in \{x \in G : x^{-1}B_n^{<i}(B_n^{<i})^{-1}x \cap F_n = \emptyset\}$.

The existence of such a point b_n^i will follow as soon as we check that the set $\{x \in G : x^{-1}B_n^{<i}(B_n^{<i})^{-1}x \cap F_n = \emptyset\}$ is infinite. It follows from the property (a) the set $B_n^{<i}(B_n^{<i})^{-1} \cap F_n$ is empty. Now the existence of a point b_n^i satisfying (a)–(d) follows from the following lemma that will be proved later with help of ultrafilters.

Lemma 1. *For any finite disjoint subsets $F, S \subset G$ the set $\{x \in G : x^{-1}Sx \cap F = \emptyset\}$ is infinite.*

This allows us to complete the inductive construction of the sets B_n and define the set $\mathbb{B} = \bigcup_{n=1}^{\infty} B_n^{-1}B_n$. Now let us prove that the constructed set $\mathbb{B}$ has properties (1)—(3). In fact, the property (1) is evident while (3) follows immediately from (c).

Lemma 2. *The set $\mathbb{B}$ has property (2).*

Proof. Given pairwise distinct points $c_1, c_2, c_3 \in G$, find $k \in \omega$ such that $c_i^{-1}c_j \in F_k$ for all distinct $i, j \in \{1,2,3\}$. Assuming that the intersection $c_1\mathbb{B}\cap c_2\mathbb{B}\cap c_3\mathbb{B}$ is infinite, we may find a point $b \in \bigcap_{i=1}^{3} c_i\mathbb{B}_{>k}\backslash\{c_i\}$. A contradiction will be reached in three steps.

Step 1. First we show that there is $n>k$ with $b\in\bigcap_{i=1}^3 c_iB_n^{-1}B_n$.

Otherwise, $c_p^{-1}b\in B_n^{-1}B_n$ and $c_q^{-1}b\in B_m^{-1}B_m$ for some $m>n>k$ and some $p\ne q$. Write $c_q^{-1}b$ as $c_q^{-1}b=b_m^{-i}b_m^j$. The inequality $b\ne c_q$ yields $i\ne j$.

If $i<j$, then

$$b_m^j=b_m^ic_q^{-1}b\in b_m^ic_q^{-1}c_pB_n^{-1}B_n\subset B_m^{<j}F_k\mathbb{B}_{<m},$$

which contradicts (a).

If $i>j$, then

$$b_m^i=b_m^jb^{-1}c_q\in b_m^j(c_pB_n^{-1}B_n)^{-1}c_q=b_m^iB_n^{-1}B_nc_p^{-1}c_q\subset B_m^{<i}\mathbb{B}_{<m}F_k,$$

again contradiction to (a).

Step 2. We claim that if $c_1^{-1}b=(b_n^i)^{-1}b_n^j$ and $c_2^{-1}b=(b_n^s)^{-1}b_n^t$, then $\max\{i,j\}=\max\{s,t\}$.

Conversely suppose $\max\{i,j\}>\max\{s,t\}$. If $i>j$, then $c_1^{-1}c_2=(b_n^i)^{-1}b_n^j(b_n^t)^{-1}b_n^s$ and hence

$$b_n^i=b_n^j(b_n^t)^{-1}b_n^sc_2^{-1}c_1\in B_n^{<i}(B_n^{<i})^{-1}B_n^{<i}F_k,$$

which contradicts (b).

If $i<j$, then

$$b_n^j=b_n^ic_1^{-1}c_2(b_n^s)^{-1}b_n^t\in B_n^{<j}F_k(B_n^{<j})^{-1}B_n^{<j},$$

again contradicting (b).

Step 3. According to the previous step, there exist $n>k$ and l such that

$c_1^{-1}b=(b_n^i)^{-1}b_n^j$ where $\max\{i,j\}$ is equal to l;
$c_2^{-1}b=(b_n^s)^{-1}b_n^t$ where $\max\{s,t\}$ is equal to l;
$c_3^{-1}b=(b_n^q)^{-1}b_n^r$ where $\max\{q,r\}$ is equal to l.

In this case we obtain a dichotomy: either two among the three numbers i,s,q are equal to l or two among j,t,r are equal to l.

In the first case we lose no generality assuming that $i=s=l$; in the second, that $j=t=l$.

In the first case, we get

$$F_k\ni c_1^{-1}c_2=(b_n^l)^{-1}b_n^j(b_n^t)^{-1}b_n^l\in(b_n^l)^{-1}B_n^{<l}(B_n^{<l})^{-1}b_n^l,$$

a contradiction with (d).

In the second case, $F_k\ni c^{-1}c_2=(b_n^i)^{-1}b_n^s$ with $i\ne s$. If $i<s$, then we get $b_n^s\in b_n^iF_k\subset B_n^{<s}F_n$, a contradiction with (a). If $i>s$, then $b_n^i\in b_n^sF_k^{-1}\subset B_n^{<i}F_k$ again contradicts (a). □

It remains to prove Lemma 1. This will be done by the technique of ultrafilters. Recall that a family $\mathcal{F}$ of non-empty subsets of a set X is called *a filter* if it is closed under intersections and taking supersets.

The simplest example of a filter is $\mathcal{F}_x = \{F \subset X : x \in F\}$ where x is a point of X. Such a filter is called *principal*.

The family of filters on a fixed set X is partially ordered by the inclusion relation. A filter which is maximal with respect to this partial order is called *an ultrafilter*. Zorn Lemma implies that each filter can be enlarged to an ultrafilter. Any principal filter is ultrafilter. Non-principal ultrafilters are called *free*. The following useful result can be found in [2, §2] or in [5, §1].

Lemma 3. *If $F = F_1 \cup \dots \cup F_n$ is a finite partition of an element $F \in \mathcal{F}$ of an ultrafilter $\mathcal{F}$, then one of the sets F_i belongs to $\mathcal{F}$.*

The set βX of all ultrafilters carries a compact Hausdorff topology generated by the base $\bar{A} = \{\mathcal{F} \in \beta X : A \in \mathcal{F}\}$ where A runs over subsets of X. Identifying each point $x \in X$ with the principal ultrafilter $\mathcal{F}_x$ we embed X into βX as an open discrete subspace.

We shall use the fact that the group operation on G can be extended to a semigroup operation on βG as follows. For two ultrafilters $\mathcal{F}$,$\mathcal{U}$ on G the product $\mathcal{F}\mathcal{U}$ is the ultrafilter generated by the sets $\bigcup_{x\in F} U_x x$ where $F \in \mathcal{F}$ and $U_x \in \mathcal{U}$ for all $x \in F$. It is well-known that the so-defined operation turns βG into a left-topological semigroup (which means that left translations on βG are continuous), see [5, §4] or [4]. Since the product of two free ultrafilters is a free ultrafilter, $\beta G \setminus G$ is a closed subsemigroup of βG. It is well-known that each compact Hausdorff left-topological semigroup contains an idempotent (see [5, Th.4.1]). Therefore, there is a free ultrafilter $\mathcal{E}$ on G with $\mathcal{E}\mathcal{E} = \mathcal{E}$. This ultrafilter has the following nice property.

Lemma 4. *For any set $A \in \mathcal{E}$ we get $AA^{-1} \in \mathcal{E}$.*

Proof. Given a set $A \in \mathcal{E}$ find a an element $\bigcup_{x\in F} E_x x \subset A$ of the canonical base of $\mathcal{E}\mathcal{E}$, where $F \in \mathcal{E}$ and $E_x \in \mathcal{E}$ for all $x \in F$. Pick any point $a \in F \cap A$ and observe that $E_x a \subset A$. Then $\mathcal{E} \ni E_x \subset Aa^{-1} \subset AA^{-1}$ implies $AA^{-1} \in \mathcal{E}$. □

Now we are in position to prove our lemma.

Proof of Lemma 1. We have to show that the set $\{x \in G : x^{-1}Sx \cap F = \emptyset\}$ is infinite if $S \cap F = \emptyset$. It suffices to check that this set belongs to the idempotent $\mathcal{E}$ of the semigroup $\beta G \setminus G$. In its turn, this will follow as soon as we prove that the set $\{x \in G : x^{-1}sx \notin F\}$ belongs to $\mathcal{E}$ for every $s \in S$. Assuming the converse, we would get $\{x \in G : x^{-1}sx \in F\} \in \mathcal{E}$ and thus $A = \{x \in G : x^{-1}sx = f\} \in \mathcal{E}$ for some $f \in F$ by Lemma 3. By Lemma 4, $AA^{-1} \in \mathcal{E}$. On the other hand, for any $x, y \in A$, we get $x^{-1}sx = y^{-1}sy$. Then $yx^{-1}sxy^{-1} = s \neq f$ and hence $AA^{-1} \cap A = \emptyset$, a contradiction with $A, AA^{-1} \in \mathcal{E}$.

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References

[1] T.Banakh, N.Lyaskovska, Weakly P-small not P-small subsets in Abelian group, *Algebra and Discrete Mathematics.* N.3 (2006) 29-34.
[2] W.W.Comfort, S.Negrepontis, *The Theory of Ultrafilters* (Springer, Berlin, 1974).
[3] D. Dikranjan, I. Protasov, Every infinite group can be generated by a P-small subset, *General Applied Topology,* (to appear); available at http://unicyb.kiev.ua/Site-Eng/admin/download/P-small subset.pdf.
[4] N.Hidman, D.Strauss, *Algebra in Stone-Čech compactification. Theory and Applications* (de Gruyter Exposition in Mathematics, **27**, Walter de Gruyter& Co., Berlin, 1998).
[5] I. Protasov, *Combinatorics of Numbers* (Monograph Series, Vol.2, VNTL, Lviv, 1997).

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