DISCONTINUOUS SEPARATELY CONTINUOUS FUNCTIONS AND NEAR COHERENCE OF P-FILTERS

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Abstract. We prove that the problem of the existence of a discontinuous separately continuous function $f : X \times Y \to \mathbb{R}$ for any non-discrete Tychonov spaces X, Y of countable pseudocharacter is equivalent to NCPF (Near Coherence of P-filters) which is independent of ZFC. Also for every non-discrete Tychonov space X we find an abelian topological group G of countable cellularity and a discontinuous separately continuous function $f : X \times G \to \mathbb{R}$.

Introduction

All calculus students learn that a function of two real variables (x, y) can be continuous for each fixed x and for each fixed y without being continuous as a function of two real variables. The standard example illustrating this phenomenon is the function sp given by:

$$sp(x,y) = \frac{2xy}{x^2+y^2} \quad \text{if} \quad (x,y) \neq (0,0) \quad \text{while} \quad sp(0,0) = 0.$$

It is clear that sp is continuous as a function from the plane $\mathbb{R}^2$ to the real line $\mathbb{R}$ everywhere except at the origin.

Looking at this standard example, one could suggest that any non-discrete Tychonov spaces X, Y admit a discontinuous separately continuous function $f : X \times Y \to \mathbb{R}$. However that is not true. The following theorem proven in [HW, 6.14] supplies us with many counterexamples. We recall that a topological space X is called a *P-space* if each G_δ-subset of X is open.

Theorem HW. *Every separately continuous function $f : X \times Y \to \mathbb{R}$ defined on the product of a P-space X and a locally separable space Y is continuous.*

On the other hand, we have the following positive result, see [MMMS, Theorem 2.6].

Theorem MMMS. *Suppose X, Y are Tychonov spaces with non-isolated G_δ-points $a \in X$, $b \in Y$. If Y is first countable or locally connected at the point b, then there is a separately continuous function $f : X \times Y \to \mathbb{R}$ continuous everywhere except at the point (a, b). Moreover, this function f can be chosen of the form $f = sp \circ (g \times h)$, where $g : X \to \mathbb{R}$, $h : Y \to \mathbb{R}$ are suitable continuous functions.*

Therefore one should impose some restrictions on the nature of non-isolated points $a \in X$, $b \in Y$ to guarantee the existence of a separately continuous function

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$f : X \times Y \to \mathbb{R}$ discontinuous at (a, b). In light of Theorem MMMS it is natural to ask

Question A. *Suppose X, Y are Tychonov spaces with non-isolated G_δ-points $a \in X$, $b \in Y$. Is there a separately continuous function $f : X \times Y \to \mathbb{R}$ discontinuous at (a, b)? Can such a function f be chosen so that $f = sp \circ (g \times h)$ for suitable continuous functions $g : X \to \mathbb{R}$, $h : Y \to \mathbb{R}$?*

This question is tightly connected with the problem of relationship between various topologies on products of topological spaces, which appear naturally in the theory of separately continuous functions, see [HW], [My], [HK]. One of such natural topologies on the product $X \times Y$ of two topological spaces is the *cross topology* γ, i.e., the maximal topology coinciding with the product topology on each "line" $\{x\}\times Y$, $X \times \{y\}$ where $x \in X$ and $y \in Y$. Another one is the topology σ of *separate continuity*, i.e., the minimal topology on $X \times Y$ for which all separately continuous functions $X \times Y \to \mathbb{R}$ are continuous. It is known that $\gamma \subset \sigma \subset \tau$ for every Tychonov spaces X, Y, where τ stands for the usual product topology on $X \times Y$. Now the question appears: *for which Tychonov spaces are there inclusions strict?* According to Corollary 6.15 of [HW] the equality $(X \times Y, \sigma) = (X \times Y, \tau)$ holds for any P-space and any locally separable space Y. Thus the following question appears naturally.

Question B. *Is $(X \times Y, \sigma) \neq (X \times Y, \tau)$ for any Tychonov spaces X, Y which are not P-spaces?*

Let us remark that a topological space X is not a P-space if and only if X contains an *F_σ-accessible point*, that is a point $x \in X$ for which there is an F_σ-set $F \subset X$ with $x \in \bar{F} \setminus F$ where $\bar{F}$ stands for the closure of F in X. It is clear that each F_σ-accessible point is not isolated. Conversely, a non-isolated point x of a topological space X is F_σ-accessible if X has countable pseudocharacter, countable tightness or countable π-character at x.

Trying to answer Questions A and B we discovered (to our big surprise) that this can not be done in ZFC. Under Martin Axiom these questions have negative answer. On the other hand, there are models of ZFC in which answers to these questions are in positive. To describe these models we need to reminds the notion of *near coherence* of filters introduced and studied in [Bl], [BS$_1$], [BS$_2$].

Two filters $\mathcal{F}_1$, $\mathcal{F}_2$ on a set S are said to be *near coherent* if there is a finite-to-one function $h : S \to S$ such that $h(F_1) \cap h(F_2) \neq \emptyset$ for any $F_1 \in \mathcal{F}_1$, $F_2 \in \mathcal{F}_2$. A function $h : S \to S$ is said to be *finite-to-one* if $h^{-1}(s)$ is finite for every $s \in S$. We remind that a *filter* on a set S is a collection $\mathcal{F} \not\ni \emptyset$ of subsets of S, closed with respect to supersets and intersections. If $\bigcap_{F\in\mathcal{F}} F = \emptyset$, then the filter $\mathcal{F}$ is called *free*. Next, $\mathcal{F}$ is called a *P-filter* if every countable collection $\mathcal{C} \subset \mathcal{F}$ has a pseudointersection in $\mathcal{F}$, i.e., a set $A \in \mathcal{F}$ such that $A \setminus C$ is finite for every $C \in \mathcal{C}$.

The following theorem reduces the problem of the existence of discontinuous separately continuous functions to the near coherence of P-filters.

Theorem 1. *The following statements are equivalent.*

(1) *For any Tychonov spaces X, Y with non-isolated G_δ-points $a \in X$, $b \in Y$ there are continuous maps $g : X \to \mathbb{R}$, $h : Y \to \mathbb{R}$ and a function ψ :*

$\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *such that the function* $f=\psi\circ(g\times h):X\times Y\to\mathbb{R}$ *is separately continuous, discontinuous at* (a,b) *and continuous at other points of* $X\times Y$.

(2) *For any countable spaces* X,Y *with a unique non-isolated point there is a discontinuous separately continuous function* $f:X\times Y\to\mathbb{R}$.

(3) $(X\times Y,\gamma)\neq(X\times Y,\tau)$ *for any countable spaces* X,Y *with a unique non-isolated point.*

(4) *Any two free* P*-filters on a countable set are near coherent.*

The last statement of Theorem 1 will be abbreviated as NCPF (Near Coherence of P-Filters) by analogy with NCF (Near Coherence of Filters) introduced and studied in [Bl]. We remind that NCF means that any two free filters on a countable set are near coherent. NCF contradicts to Martin Axiom but holds in some models of ZFC, see [BS$_1$], [BS$_2$] or [HM, p.100]. The following theorem describes the relationship between NCPF, NCF, Martin Axiom, Continuum Hypothesis and the small cardinals $\mathfrak{d}$ (the dominating number) and $\mathfrak{t}$ (the tower number), see [vD] or [Va].

Theorem 2. *(NCF)⇒(NCPF)⇒* $(\mathfrak{d}\neq\mathfrak{t})\Rightarrow(\neg MA)\Rightarrow(\neg CH)$.

Since both NCF and CH are independent of ZFC, we get that NCPF as well as all the equivalent statements of Theorem 1 are independent of ZFC.

We do not known if NCPF is strictly stronger than NCF.

Problem 1. *Is there a model of ZFC in which NCPF holds but NCF fails?*

In this respect it is interesting to notice that assuming NCF in place of NCPF allows us to construct discontinuous separately continuous functions of some special type and take the function ψ from Theorem 1(1) equal to the standard separately continuous function *sp*, defined at the beginning of the paper.

Theorem 3. *Under NCF for any Tychonov spaces* X,Y *with* F_σ*-accessible* (G_δ*-*)*points* $a\in X$, $b\in Y$ *there exist continuous functions* $g:X\to\mathbb{R}$ *and* $h:Y\to\mathbb{R}$ *such that the separately continuous function* $f=sp\circ(g\times h):X\times Y\to\mathbb{R}$ *is discontinuous at* (a,b) *(and continuous at other points of* $X\times Y$*).*

This theorem implies

Corollary 1. *Assume NCF. If each separately continuous function* $f:X\times Y\to\mathbb{R}$ *defined on the product of two Tychonov spaces* X,Y *is continuous, then one of these spaces is a* P*-space.*

Problem 2. *Are Theorem 3 and Corollary 1 true under NCPF?*

According to Corollary 1 it is consistent to assume that any Tychonov non-P-spaces X,Y admit a discontinuous separately continuous function $f:X\times Y\to\mathbb{R}$. As Theorem HW shows the situation changes if one of the spaces X or Y is a P-space and the other is locally separable. It is natural to ask if the local separability in this theorem can be replaced by the countable cellularity. We shall show that this cannot be done.

We remind that a topological space X has *countable cellularity* if any collection of pairwise disjoint non-empty open subsets of X is at most countable. Among the most important examples of topological spaces with countable cellularity there are

σ-compact topological groups and their subgroups (which admit an inner characterization as σ-bounded groups), see [Tk$_1$]. We recall that a topological group G is called *σ-bounded* if it is a countable union of totally bounded subsets (a subset $A \subset G$ is *totally bounded* if for every non-empty open set $U \subset G$ there is a finite subset $F \subset G$ with $A \subset (F \cdot U) \cap (U \cdot F)$). According to the famous Tkachenko Theorem [Tk$_2$], each σ-bounded topological group has countable cellularity.

Our last theorem shows that the local separability of the space Y in Theorem HW is essential and cannot be replaced by the countable cellularity of Y.

Theorem 4. *For any non-discrete Tychonov space X there is a σ-bounded abelian topological group G and a discontinuous separately continuous function $h : X \times G \to \mathbb{R}$.*

Proof of Theorem 4

The proof of Theorem 4 relies on the following

Lemma 1. *For every non-discrete Tychonov (P-)space X there exist a Tychonov (P-)space Y with a unique non-isolated point and a bounded discontinuous separately continuous function $f : X \times Y \to \mathbb{R}$.*

Proof. The space X, being non-discrete, contains a non-isolated point $a \in X$. Let $\tau(a)$ be the set of all neighborhoods of the point a in X. For any $U \in \tau(a)$ let $\downarrow U = \{V \in \tau(a) : V \subset U\}$. Let $b \notin \tau(a)$ be any point. On the union $Y = \{b\} \cup \tau(a)$ consider the topology in which all points $y \in \tau(a)$ are isolated while the sets $\{b\} \cup (\downarrow U)$, $U \in \tau(a)$, are neighborhoods of the point b. Clearly, if X is a P-space, then so is the space Y.

Next, we define a discontinuous separately continuous function $f : X \times Y \to [0, 1]$. For every neighborhood $U \in \tau(a)$ of a, fix a continuous function $f_U : X \to [0, 1]$ such that $\max_{x \in X} f(x) = 1$ and $f_U(\{a\} \cup (X \setminus U)) \subset \{0\}$. Define a function $f : X \times Y \to [0, 1]$ by the formula

$$f(x,y) = \begin{cases} 0, & \text{if } y = b, \\ f_U(x), & \text{if } y = U \in \tau(a). \end{cases}$$

It is easy to see that the function f is separately continuous but discontinuous at the point (a, b). $\square$

Proof of Theorem 4. Given a non-discrete Tychonov space X let $f : X \times Y \to \mathbb{R}$ be a bounded discontinuous separately continuous function provided by Lemma 1. For every $x \in X$ let $\beta f_x : \beta Y \to \mathbb{R}$ be the continuous extension of the bounded continuous function $f_x : Y \to \mathbb{R}$, $f_x : y \mapsto f(x, y)$, onto the Stone-Čech compactification βY of Y. Next, let $A(\beta Y) \supset \beta Y$ be a free abelian topological group of βY (see [Ma]) and $h_x : A(\beta Y) \to \mathbb{R}$ be a unique continuous group homomorphism extending the function βf_x. Let G be the group hull of the set Y in $A(\beta Y)$. It is clear that G, being a subgroup of the σ-compact group $A(\beta Y)$, is a σ-bounded group and each element $g \in G$ can be written as $g = k_1 y_1 + \dots + k_n y_n$ for some $n \in \mathbb{N}$, $k_1, \dots, k_n \in \mathbb{Z}$, and $y_1, \dots, y_n \in Y$. Finally, consider the function $h : X \times G \to \mathbb{R}$ defined by $h(x, g) = h_x(g)$.

It is clear that for every fixed $x \in X$ the function $h(x,\cdot) : G \to \mathbb{R}$ is a continuous group homomorphism. To see that $h(\cdot, g) : X \to \mathbb{R}$ is continuous for every fixed $g \in G$, observe that

$$h(x,g) = h_x(g) = k_1 f_x(y_1) + \cdots + k_n f_x(y_n) = k_1 f(x,y_1) + \cdots + k_n f(x,y_n),$$

where $g = k_1 y_1 + \cdots + k_n y_n$.

Therefore, $h : X \times G \to \mathbb{R}$ is a separately continuous function. Since $h|X \times Y = f$, the function h is discontinuous. □

Proof of Theorem 3

Let X_1, X_2 be Tychonov spaces with F_σ-accessible (G_δ-)points $a_i \in X_i$ for $i = 1, 2$. Assuming NCF we have to find continuous maps $f_i : X_i \to \mathbb{R}$ for $i = 1, 2$ such that the separately continuous function $f = sp \circ (f_1 \times f_2) : X_1 \times X_2 \to \mathbb{R}$ is discontinuous at (a_1, a_2) (and continuous at other points of $X_1 \times X_2$).

For every $i = 1, 2$ fix an F_σ-subset $F_i \subset X_i$ with $a_i \in \bar{F}_i \setminus F_i$ and a continuous function $\alpha_i : X_i \to [0,1]$ such that $\alpha_i(a_i) = 0$ and $\alpha_i(F_i) \subset (0,1]$. Let $B_i = \alpha_i^{-1}(0)$. Since $a \in \bar{F}_i$ and $F_i \cap B_i = \emptyset$, we conclude that B_i is not a neighborhood of a_i in X_i. If a_i is a G_δ-point of X_i, then we may take $F_i = X_i \setminus \{a_i\}$ and get that $B_i = \{a_i\}$.

Now consider the separately continuous function $sp \circ (\alpha_1 \times \alpha_2) : X_1 \times X_2 \to \mathbb{R}$. It is easy to see that it is continuous on the set $X_1 \times X_2 \setminus B_1 \times B_2$.

If this function is discontinuous at the point (a_1, a_2), then we finish the proof. This is so if for some $i = 1, 2$ the image $\alpha_i(W)$ of any neighborhood $W \subset X_i$ of a_i is a neighborhood of zero in $[0,1]$.

Next, consider the (non-trivial) case when $sp \circ (\alpha_1 \times \alpha_2)$ is continuous at (a_1, a_2) and for every $i \in \{1, 2\}$ there is a neighborhood $W_i \subset X_i$ of the point a_i whose image $\alpha_i(W_i)$ is not a neighborhood of zero in $[0,1]$. This means that $\alpha_i(W_i)$ contains many "holes" tending to zero. Using this fact, construct a continuous map $r_i : \alpha_i(W_i) \to S_0$ of $\alpha_i(W_i)$ onto the convergent sequence $S_0 = \{0\} \cup S$ where $S = \{\frac{1}{n} : n \in \mathbb{N}\}$. This map may be chosen so that $r_i^{-1}(0) = \{0\}$. Observe that the family

$$\mathcal{F}_i = \{S \cap r_i \circ \alpha_i(U) : U \subset W_i \text{ is a neighborhood of } a_i\}$$

is a free filter on the countable set S.

NCF implies that the filters $\mathcal{F}_1$ and $\mathcal{F}_2$ are near coherent. Hence there is a finite-to-one function $\beta : S \to S$ such that $\beta(A_1) \cap \beta(A_2) \neq \emptyset$ for any $A_i \in \mathcal{F}_i$, $i = 1, 2$. The function β admits a unique continuous extension $\bar{\beta} : S_0 \to S_0$ with $\bar{\beta}(0) = 0$. For every $i = 1, 2$, pick a continuous function $\lambda_i : X_i \to [0,1]$ such that $\lambda_i(X_i \setminus W_i) = \{0\}$ while $\lambda_i^{-1}(1)$ is a neighborhood of a_i in X_i. Define a continuous function $f_i : X_i \to [0,1]$ letting

$$f_i(x) = \begin{cases} \lambda_i(x) \cdot \bar{\beta} \circ r_i \circ \alpha_i(x) + (1 - \lambda_i(x))\alpha_i(x), & \text{if } x \in W_i; \\ \alpha_i(x), & \text{if } x \notin W_i. \end{cases}$$

We claim that the separately continuous function $f = sp \circ (f_1 \times f_2) : X_1 \times X_2 \to \mathbb{R}$ is discontinuous at (a_1, a_2) and continuous on the set $X_1 \times X_2 \setminus B_1 \times B_2$.

The continuity of f on the set $X_1 \times X_2 \setminus B_1 \times B_2$ follows from the continuity of sp on $\mathbb{R}^2 \setminus \{(0,0)\}$ and the inclusions $f_i(X_i \setminus B_i) \subset (0,1]$ for $i = 1,2$.

Assuming that f is continuous at (a_1, a_2), we may find neighborhoods $U_i \subset \lambda_i^{-1}(1) \subset W_i$ of the points a_i, $i = 1,2$ such that $f(U_1 \times U_2) \subset (-1,1)$. Let $A_i = S \cap r_i \circ \alpha_i(U_i) \in \mathcal{F}_i$ for $i = 1,2$. Since $\beta(A_1) \cap \beta(A_2) \ni y$ for some $y \in S$, there exist points $x_i \in U_i$, $i = 1,2$, such that $y = \beta \circ r \circ \alpha_i(x_i)$ for $i = 1,2$ which yields that $f(x_1, x_2) = sp(y,y) = 1$, a contradiction with $f(x_1,x_2) \in f(U_1 \times U_2) \subset (-1,1)$. $\square$

Proof of Theorem 1

To prove Theorem 1 we shall verify the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1). In fact, the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are trivial.

The proof of implication (4) $\Rightarrow$ (1) repeats the proof of Theorem 3, except for one additional lemma (acting in the case when one of the filters $\mathcal{F}_1$ or $\mathcal{F}_2$ fails to be a P-filter). All the denotations in the following lemma are taken from the proof of Theorem 3.

Lemma 3. *If $B_1 = \{a_1\}$ and $\mathcal{F}_1$ is not a P-filter, then there is a function $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and continuous maps $f_i : X_i \to [0,1]$ for $i = 1,2$ such that the function $f = \psi \circ (f_1 \times f_2) : X_1 \times X_2 \to \mathbb{R}$ is separately continuous, discontinuous at (a_1, a_2) and continuous on the set $X_1 \times X_2 \setminus B_1 \times B_2$.*

Proof. Since $\mathcal{F}_1$ is not a P-filter, there is a sequence $(A_n)_{n=1}^\infty \subset \mathcal{F}_1$ having no pseudointersection in $\mathcal{F}_1$. Replacing each set A_n by $A_1 \cap \dots \cap A_n$, we may assume that the sequence $(A_n)_{n=1}^\infty$ is decreasing. Let $\mathbb{R}_0$ stand for the real line endowed with the topology τ coinciding with the usual topology of the real line at all points except for 0. A subset $U \subset \mathbb{R}$ is declared a neighborhood of 0 in the topology τ if $U \setminus \{0\}$ is open in $\mathbb{R}$ and $U \supset A_n \cup \{0\}$ for some $n \in \mathbb{N}$.

It is easy to construct a separately continuous function $\psi : \mathbb{R}_0 \times \mathbb{R}$ which is continuous on the set $\mathbb{R}_0 \times \mathbb{R} \setminus \{(0,0)\}$ and satisfies the condition:

$$\psi(x,y) = \begin{cases} 1, & \text{if } 0 < x \le y = \frac{1}{n} \text{ and } x \notin A_n \text{ for some } n \in \mathbb{N}; \\ 0, & \text{otherwise} \end{cases}$$

for any $(x,y) \in S_0 \times S_0$.

For every $i = 1,2$, pick a continuous function $\lambda_i : X_i \to [0,1]$ such that $\lambda_i(X_i \setminus W_i) = \{0\}$ and $\lambda_i^{-1}(1)$ is a neighborhood of a_i in X_i. Next, for every $i = 1,2$ consider the continuous function $f_i : X_i \to [0,1]$ defined by

$$f_i(x) = \begin{cases} \lambda_i(x) \cdot r_i \circ \alpha_i(x) + (1 - \lambda_i(x))\alpha_i(x), & \text{if } x \in W_i; \\ \alpha_i(x), & \text{if } x \notin W_i. \end{cases}$$

Observe that $f_2^{-1}(0) = B_2$, $f_1^{-1}(0) = B_1 = \{a_1\}$, and f_1 is continuous as a function from X_1 into $\mathbb{R}_0$.

Then the function $f = \psi \circ (f_1 \times f_2) : X_1 \times X_2 \to \mathbb{R}$ is separately continuous and continuous outside the set $B_1 \times B_2 = (f_1 \times f_2)^{-1}(0,0)$.

It rests to verify that f is discontinuous at the point (a_1, a_2). Assuming the converse we would find a neighborhood $U_1 \times U_2 \subset \lambda_1^{-1}(1) \times \lambda_2^{-1}(1)$ of (a_1, a_2) in

$X_1 \times X_2$ such that $f(U_1 \times U_2) \subset (-1,1)$. Let $P = S \cap r_1 \circ \alpha_1(U_1)$. Since $P \in \mathcal{F}_1$, P is not a pseudointersection of the collection $\{A_n\}_{n=1}^{\infty}$ and thus $P \setminus A_n$ is infinite for some $n \in \mathbb{N}$. Since the sequence $(A_n)_{n=1}^{\infty}$ is decreasing, $P \setminus A_m$ is infinite for all $m \geq n$. Pick any point $\frac{1}{m} \in r_2 \circ \alpha_2(U_2)$ with $m \geq n$. Since $P \setminus A_m$ is infinite, we can find a point $\frac{1}{k} \in P \setminus A_m$ with $k \geq m$. By the definition of ψ, $\psi(\frac{1}{k}, \frac{1}{m}) = 1$. Pick any points $x_1 \in U_1$, $x_2 \in U_2$ with $r_1 \circ \alpha_1(x_1) = \frac{1}{k}$ and $r_2 \circ \alpha_2(x_2) = \frac{1}{m}$. Then $f(x_1, x_2) = \psi(\frac{1}{k}, \frac{1}{m}) = 1$, a contradiction which proves the discontinuity of f at (a_1, a_2). $\square$

This proves the implication (4) $\Rightarrow$ (1) of Theorem 1. To prove the implication (3)$\Rightarrow$(4) we need several auxiliary results.

We define a subset $A \subset \mathbb{N}\times\mathbb{N}$ to be *cross-finite* if for every $n \in \mathbb{N}$ the intersection $A\cap(\mathbb{N}\times\{n\}\cup\{n\}\times\mathbb{N})$ is finite. Under a *standard cross-finite* subset we understand a subset $A \subset \mathbb{N} \times \mathbb{N}$ of the form $A = \bigcup_{n\in\mathbb{N}} h^{-1}(n) \times h^{-1}(n)$ for some finite-to-one function $h : \mathbb{N} \to \mathbb{N}$.

Lemma 4. *Every cross-finite subset $A \subset \mathbb{N} \times \mathbb{N}$ lies in the union of two standard cross-finite sets.*

Proof. Inductively, we can construct a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) \geq \max\{k \in \mathbb{N} : \exists i \leq n \text{ with } (k,i) \in A \text{ or } (i,k) \in A\}$. Next, define recursively the function $g : \mathbb{N} \to \mathbb{N}$ letting $g(1) = f(1)$ and $g(n) = f(g(n-1))$ for $n > 1$. Let also $g(-1) = g(0) = 1$. Finally, define two finite-to-one functions $h_1, h_2 : \mathbb{N} \to \mathbb{N}$ letting $h_1^{-1}(k) = [g(2k-3), g(2k-1))$ and $h_2^{-1}(k) = [g(2k-2), g(2k))$ for $k \in \mathbb{N}$. We claim that $A \subset A_1 \cup A_2$, where $A_i = \bigcup_{k\in\mathbb{N}} h_i^{-1}(k) \times h_i^{-1}(k)$ for $i = 1, 2$. Indeed, fix any $(i,j) \in A$. Without loss of generality, $i \leq j$. Find $n \in \mathbb{N}$ with $g(n-1) \leq i < g(n)$. Then $g(n-1) \leq i \leq j \leq f(i) < f(g(n)) = g(n+1)$ and $(i,j) \in [g(n-1), g(n+1))^2 \subset A_1 \cup A_2$. $\square$

Lemma 5. *Two filters $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\mathbb{N}$ are near coherent if and only if there is a cross-finite subset $A \subset \mathbb{N} \times \mathbb{N}$ such that $A \cap (F_1 \times F_2) \neq \emptyset$ for every $F_i \in \mathcal{F}_i$, $i = 1, 2$.*

Proof. To prove the "only if" part, assume that the filters $\mathcal{F}_1$ and $\mathcal{F}_2$ are near coherent. Then there is a finite-to-one function $h : \mathbb{N} \to \mathbb{N}$ such that $h(F_1) \cap h(F_2) \neq \emptyset$ for every sets $F_i \in \mathcal{F}_i$, $i = 1, 2$. Consider the standard cross-finite subset $A = \bigcup_{n\in\mathbb{N}} h^{-1}(n) \times h^{-1}(n)$ of $\mathbb{N} \times \mathbb{N}$. It follows that $(F_1 \times F_2) \cap A \neq \emptyset$ for every $F_i \in \mathcal{F}_i$, $i = 1, 2$.

To prove the "if" part, assume that A is a cross-finite subset of $\mathbb{N} \times \mathbb{N}$ such that $(F_1 \times F_2) \cap A \neq \emptyset$ for every $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$. By Lemma 4, there are two standard cross-finite subsets A_1, A_2 of $\mathbb{N}$ such that $A \subset A_1 \cup A_2$. Find finite-to-one functions $h_1, h_2 : \mathbb{N} \to \mathbb{N}$ with $A_i = \bigcup_{n\in\mathbb{N}} h_i^{-1}(n) \times h_i^{-1}(n)$ for $i = 1, 2$. Assuming that the filters $\mathcal{F}_1$ and $\mathcal{F}_2$ are not near coherent we would find sets $F_1 \in \mathcal{F}_1$ and $F_2 \in \mathcal{F}_2$ such that $h_i(F_1) \cap h_i(F_2) = \emptyset$ for every $i = 1, 2$. This yields $(F_1\times F_2)\times(A_1\cup A_2) = \emptyset$, a contradiction with $A \subset A_1\cup A_2$ and $(F_1\times F_2)\cap A \neq \emptyset$. $\square$

Now we are able to prove the implication (3)$\Rightarrow$(4). Given a free filter $\mathcal{F}$ on $\mathbb{N}$ let $\mathbb{N}_{\mathcal{F}} = \mathbb{N} \cup \{\infty\}$ be the one-point extension of the discrete space $\mathbb{N}$ with the sets $\{\infty\} \cup F$, $F \in \mathcal{F}$, being neighborhoods of the unique non-isolated point ∞ of $\mathbb{N}_{\mathcal{F}}$.

Lemma 6. *Two free P-filters $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\mathbb{N}$ are near coherent if and only if the cross topology γ on $\mathbb{N}_{\mathcal{F}_1} \times \mathbb{N}_{\mathcal{F}_2}$ is strictly stronger than the product topology.*

Proof. If free filters $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\mathbb{N}$ are near coherent, then by Lemma 5, there is a cross-finite set $A \subset \mathbb{N} \times \mathbb{N}$ such that $(F_1 \times F_2) \cap A \neq \emptyset$ for every $F_i \in \mathcal{F}_i$, $i = 1, 2$. It is clear that the set $U = (\mathbb{N}_{\mathcal{F}_1} \times \mathbb{N}_{\mathcal{F}_2}) \setminus A$, being a neighborhood of (∞, ∞) in the cross topology, is not a neighborhood of (∞, ∞) in the product topology. Thus the cross topology in strictly stronger than the product topology on $\mathbb{N}_{\mathcal{F}_1} \times \mathbb{N}_{\mathcal{F}_2}$.

Next, we prove that the cross topology on $\mathbb{N}_{\mathcal{F}_1} \times \mathbb{N}_{\mathcal{F}_2}$ coincides with the product topology if the free P-filters $\mathcal{F}_1$ and $\mathcal{F}_2$ are not near coherent. In fact, it suffices to verify that these topologies coincide at (∞, ∞). Let $U \subset \mathbb{N}_{\mathcal{F}_1} \times \mathbb{N}_{\mathcal{F}_2}$ be a neighborhood of (∞, ∞) in the cross topology. Since $\mathcal{F}_1$, $\mathcal{F}_2$ are P-filters, we may find sets $P_1 \in \mathcal{F}_1$, $P_2 \in \mathcal{F}_2$ such that $(P_1 \times \{j\}) \setminus U$ and $(\{i\} \times P_2) \setminus U$ are finite for every $i, j \in \mathbb{N}$ with $(\infty, j), (i, \infty) \in U$. Moreover, we can assume that $P_1 \times \{\infty\} \cup \{\infty\} \times P_2 \subset U$. Then the set $A = (P_1 \times P_2) \setminus U$ is cross-finite. By Lemma 5, there are sets $F_1 \in \mathcal{F}_1$, $F_2 \in \mathcal{F}_2$ such that $(F_1 \times F_2) \cap A = \emptyset$. Replacing F_i by $F_i \cap P_i$, if necessary, we may assume that $F_i \subset P_i$ for $i = 1, 2$. Then $F_1 \times F_2 \subset U$ and $(\{\infty\} \cup F_1) \times (\{\infty\} \cup F_2) \subset U$ which shows that U is a neighborhood of (∞, ∞) in the product topology of $\mathbb{N}_{\mathcal{F}_1} \times \mathbb{N}_{\mathcal{F}_2}$. $\square$

Proof of Theorem 2

First we remind some standard notation, see [vD], [Va]. We identify cardinals with the smallest ordinals of the corresponding size. Let ω stand for the set of finite ordinals, $[\omega]^\omega$ denote the set of all infinite subsets of ω and ${}^\omega\omega$ be the set of all functions from ω into ω. On ${}^\omega\omega$ we consider the usual partial order: $f \leq g$ iff $f(n) \leq g(n)$ for all $n \in \omega$. A set $D \subset {}^\omega\omega$ is *dominating* if for every $f \in {}^\omega\omega$ there is $g \in D$ with $f \leq g$.

For A, B in $[\omega]^\omega$ we say that A is *almost included* in B (denoted $A \subset^* B$) if $A \setminus B$ is finite. A set $A \subset \omega$ is called a *pseudo-intersection* of a family $\mathcal{T} \subset [\omega]^\omega$ if $A \subset^* B$ for every $B \in \mathcal{T}$. A family $\mathcal{T} \subset [\omega]^\omega$ well-ordered by $\supset^*$ is called a (decreasing) *scale*. A family $\mathcal{T} \subset [\omega]^\omega$ is a *tower* if $\mathcal{T}$ is a scale having no infinite pseudo-intersection. Let

$$\begin{aligned} \mathfrak{d} &= \min\{|D| : \ D \text{ is a dominating set in } {}^\omega\omega\}, \\ \mathfrak{t} &= \min\{|\mathcal{T}| : \ \mathcal{T} \subset [\omega]^\omega \text{ is a tower}\}. \end{aligned}$$

It is known that $\aleph_1 \leq \mathfrak{t} \leq \mathfrak{d} \leq \mathfrak{c}$, where $\mathfrak{c}$ stands for the cardinality of continuum. It is consistent to assume that any of the above inequalities is strict, see [vD]. Martin Axiom (MA) implies the equality $\mathfrak{t} = \mathfrak{d} = \mathfrak{c}$, see [Ru, 8] or [Va].

Among the implications (NCF)$\Rightarrow$(NCPF)$\Rightarrow$ ($\mathfrak{d} \neq \mathfrak{t}$) $\Rightarrow$ ($\neg$MA) $\Rightarrow$ ($\neg$CH) of Theorem 2 all except for the second one are trivial or well-known. The second implication follows from

Lemma 7. *If $\mathfrak{t} = \mathfrak{d}$, then there are two P-filters $\mathcal{F}_1$, $\mathcal{F}_2$ on ω that are not near coherent.*

Proof. It is well known that each ordinal α can be uniquely written as $\alpha = \beta + n$, where β is a limit ordinal and $n \in \omega$ (such n will be called the *integer part* of α and will be denoted by $n(\alpha)$).

Under a *normal scale* we shall understand a family $\mathcal{T} \subset [\omega]^\omega$ for which there are an ordinal α and a map $f : \alpha \to \mathcal{T}$ such that $f(\beta) \subset^* f(\gamma)$, $f(\gamma+1) \subset f(\gamma)$, and $f(\beta) \not\ni n(\beta)$ for any ordinals $\gamma < \beta < \alpha$.

Assume that $\mathfrak{t} = \mathfrak{d}$ and let $D = \{g_\alpha : \alpha < \mathfrak{d}\}$ be a dominating set in ${}^\omega\omega$ with $g_0 \equiv 0$. By transfinite induction we shall construct normal scales $\mathcal{T}_1 = \{X_\alpha : \alpha < \mathfrak{d}\}$ and $\mathcal{T}_2 = \{Y_\alpha : \alpha < \mathfrak{d}\}$ of subsets of ω such that for every ordinal $\alpha < \mathfrak{d}$ the following condition is satisfied:

$$X_\alpha \cap Y_\alpha = \emptyset \text{ and } g_\alpha(x) < y \text{ for every } x, y \in X_\alpha \cup Y_\alpha \text{ with } x < y. \qquad (*_\alpha)$$

Let $X_0 = \{2n+1 : n \in \omega\}$ and $Y_0 = \{2n+2 : n \in \omega\}$. Assume that for some ordinal $\alpha < \mathfrak{d}$ the sets X_β, Y_β satisfying $(*_\beta)$ are constructed for all $\beta < \alpha$. Since $\alpha < \mathfrak{d} = \mathfrak{t}$, the scale $\{X_\beta : \beta < \alpha\}$ is not a tower and hence has an infinite pseudo-intersection $I_1 \not\ni n(\alpha)$. In case of a non-limit ordinal α, we may additionally assume that $I_1 \subset X_{\alpha-1}$. The same argument allows us to find an infinite pseudo-intersection I_2 for the family $\{Y_\beta : \beta < \alpha\}$ such that $I_2 \not\ni n(\alpha)$ and $I_2 \subset Y_{\alpha-1}$ if α is not limit.

By induction, construct infinite disjoint subsets $X_\alpha \subset I_1$ and $Y_\alpha \subset I_2$ such that $g_\alpha(x) < y$ for any $x, y \in X_\alpha \cup Y_\alpha$ with $x < y$. This competes the inductive step.

For every $i = 1, 2$ consider the filter $\mathcal{F}_i = \{F \subset \omega : F \supset T \text{ for some } T \in \mathcal{T}_i\}$ whose base is the scale $\mathcal{T}_i$. Using the facts that the cofinality of the cardinal $\mathfrak{d}$ is uncountable [vD, 3.1] and $\mathcal{T}_i$ is a scale, we can easily show that $\mathcal{F}_i$ is a P-filter. Assuming that the P-filters $\mathcal{F}_1$, $\mathcal{F}_2$ are near coherent, we would find a finite-to-one function $h : \omega \to \omega$ such that $h(X_\alpha) \cap h(Y_\alpha) \neq \emptyset$ for every $\alpha < \mathfrak{d}$. Since the set D is dominating, there is an ordinal $\alpha < \mathfrak{d}$ such that $g_\alpha(i) \geq \max\{k \in \omega : h(k) = h(i)\}$ for each $i \in \omega$. Since $h(X_\alpha) \cap h(Y_\alpha) \neq \emptyset$, there are $i \in X_\alpha$ and $j \in Y_\alpha$ with $h(i) = h(j)$. Since $X_\alpha \cap Y_\alpha = \emptyset$, we get $i \neq j$. If $i < j$, then by $(*_\alpha)$ we get $g_\alpha(i) < j$ and by the choice of g_α, $j > g_\alpha(i) \geq \max\{k \in \omega : h(k) = h(i)\} \geq j$, a contradiction. So, $j < i$. In this case, we use $(*_\alpha)$ to get a contradiction: $i > g_\alpha(j) \geq i$. $\square$

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