ON LINEAR REALIZATIONS AND LOCAL SELF-SIMILARITY OF THE UNIVERSAL ZARICHNYI MAP

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Abstract. Answering a question M.Zarichnyi we show that the universal Zarichnyi map $\mu : \mathbb{R}^\infty \to Q^\infty$ is not locally self-similar. Also we characterize linear operators homeomorphic to μ and on this base give a simple construction of a universal Zarichnyi map μ.

In this paper we investigate the properties of the universal map $\mu : \mathbb{R}^\infty \to Q^\infty$ constructed by M.Zarichnyi in [Za$_3$] and subsequently studied in [Za$_4$] and [Za$_5$]. Answering a question posed in [Za$_5$] we prove that the map μ is not locally self-similar. Also we characterize linear operators homeomorphic to μ and on this base give a simple construction of a universal Zarichnyi map μ.

Strongly universal maps

All topological spaces considered in this paper are Tychonov, all compact spaces are metrizable, and all maps are continuous; $\omega = \{0, 1, 2, \dots\}$ stands for the set of all finite ordinals.

Given a class $\mathcal{C}$ of compacta, by $\mathcal{C}^\infty$ we denote the class of topological spaces X admitting a countable cover $\mathcal{U}$ by subsets of the class $\mathcal{C}$, generating the topology of X in the sense that a subset $F \subset X$ is closed in X if and only if $F \cap K$ is closed in K for every $K \in \mathcal{U}$. In our subsequent considerations $\mathcal{C} = \mathcal{K}$ or $\mathcal{K}_{fd}$, where $\mathcal{K}$ ($\mathcal{K}_{fd}$) is the class of all (finite-dimensional) metrizable compacta.

Given a space X with a fixed point $*$ let X^∞ denote the set

$$X^\omega_f = \{(x_i)_{i\in\omega} \in X^\omega : x_i = * \text{ for almost all } i\}$$

endowed with the strongest topology inducing the product topology on each space $X^n = \{(x_i)_{i\in\omega} \in X : x_i = * \text{ for all } i \ge n\}$, $n \in \omega$. If the space X is topologically homogeneous (like the real line $\mathbb{R}$ or the Hilbert cube $Q = [0, 1]^\omega$), then the topology of the space X^∞ does not depend on the particular choice of a fixed point $* \in X$.

Among the spaces X^∞ the spaces $\mathbb{R}^\infty$ and Q^∞ occupy the special place: they are universal for the classes $\mathcal{K}^\infty_{fd}$ and $\mathcal{K}^\infty$ in the sense that each space from the class $\mathcal{K}^\infty$ (resp. $\mathcal{K}^\infty_{fd}$) is homeomorphic to a closed subspace of Q^∞ (resp. $\mathbb{R}^\infty$). Topological copies of

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the spaces $\mathbb{R}^\infty$ and Q^∞ very often appear in topological algebra and functional analysis, see [Ba$_2$]—[BS$_2$], [Sa$_2$]—[Za$_2$]. In particular, every infinite-dimensional linear topological space $X \in \mathcal{K}^\infty_{fd}$ is homeomorphic to $\mathbb{R}^\infty$ [Ba$_2$] while each locally convex space $Y \in \mathcal{K}^\infty$ with uncountable Hamel basis is homeomorphic to Q^∞ [Ba$_3$]. A topological characterization of the spaces $\mathbb{R}^\infty$ and Q^∞ was given by K.Sakai [Sa$_1$]: *Up to a homeomorphism $\mathbb{R}^\infty$ (resp. Q^∞) is a unique strongly $\mathcal{K}_{fd}$-universal (resp. strongly $\mathcal{K}$-universal) space in the class $\mathcal{K}^\infty_{fd}$ (resp. $\mathcal{K}^\infty$).*

A topological space X is defined to be *strongly $\mathcal{C}$-universal* if every embedding $f : B \to X$ of a closed subset B of a space $A \in \mathcal{C}$ can be extended to an embedding $\bar{f} : A \to X$. Replacing the words "embedding" by "map" we obtain the definition of an *absolute extensor* for the class $\mathcal{C}$ (briefly, AE($\mathcal{C}$)).

In [Za$_3$] the notion of the strong universality was generalized to maps. A map $\pi : X \to Y$ between topological spaces is defined to be *strongly $\mathcal{C}$-universal* if for every embedding $f : B \to X$ of a closed subset B of a space $A \in \mathcal{C}$ and a map $g : A \to Y$ with $\pi \circ f = g|B$ there exists an embedding $\bar{f} : A \to X$ such that $\bar{f}|B = f$ and $\pi \circ \bar{f} = g$. Replacing the words "embedding" by "map" we obtain the definition of a *$\mathcal{C}$-soft* map. Observe that a space X is strongly $\mathcal{C}$-universal (resp. is an AE($\mathcal{C}$)) if and only if the constant map $X \to \{*\}$ is strongly $\mathcal{C}$-universal (resp. $\mathcal{C}$-soft).

The following theorem was proven in [Ba$_1$].

Uniqueness Theorem. *If $\pi : X \to Y$, $\pi' : X' \to Y$ are strongly $\mathcal{C}$-universal maps from spaces $X, X' \in \mathcal{C}^\infty$, then there is a homeomorphism $h : X \to X'$ such that $\pi' \circ h = \pi$.*

In case of a one-point space Y we obtain Uniqueness Theorem of strongly $\mathcal{C}$-universal spaces (see [Sa$_1$]): *Any two strongly $\mathcal{C}$-universal spaces $X, X' \in \mathcal{C}^\infty$ are homeomorphic.*

Thus (up to a homeomorphism) there is at most one strongly $\mathcal{K}_{fd}$-universal map from a space $X \in \mathcal{K}^\infty_{fd}$ onto a given space Y. For which spaces Y does such a map exist? If $Y \in \mathcal{K}^\infty_{fd}$, then the answer is easy: just consider the projection $\pi : Y \times \mathbb{R}^\infty \to Y$. If $Y \notin \mathcal{K}^\infty_{fd}$ (for example, if $Y = Q^\infty$) the situation is not so obvious. Nonetheless, applying certain non-trivial results of A.Dranishnikov [Dr], M.Zarichnyi has constructed in [Za$_3$] a strongly $\mathcal{K}_{fd}$-universal map $\mu : \mathbb{R}^\infty \to Q^\infty$. Afterwards, he proved that this map μ is homeomorphic to a group homomorphism [Za$_5$] and to an affine map between suitable spaces of probability measures [Za$_4$], thus giving an alternative and simpler constructions of the map μ. The Zarichnyi map $\mu : \mathbb{R}^\infty \to Q^\infty$ contains any map $f : A \to B$ from a finite-dimensional metrizable compactum A into a metrizable compactum B in the sense that there are two embeddings $e_A : A \to \mathbb{R}^\infty$ and $e_B : B \to Q^\infty$ such that $\mu \circ e_A = e_B \circ f$.

Characterizing linear operators homeomorphic to the map μ

We define two maps $\pi : X \to Y$ and $\pi' : X' \to Y'$ to be *homeomorphic* if $\pi' \circ h = H \circ \pi$ for some homeomorphisms $h : X \to X'$ and $H : Y \to Y'$. In this section we characterize linear operators homeomorphic to the universal Zarichnyi map and classify $\mathcal{K}_{fd}$-invertible linear operators from a linear topological space $X \in \mathcal{K}^\infty_{fd}$ onto a locally convex space $Y \in \mathcal{K}^\infty$. By a "linear operator" we understand a linear continuous operator between linear topological spaces.

We define a map $\pi : X \to Y$ to be $\mathcal{C}$*-invertible* if for every map $g : A \to Y$ of a space $A \in \mathcal{C}$ there is a map $f : A \to X$ such that $\pi \circ f = g$. It is clear that each strongly $\mathcal{C}$-universal or $\mathcal{C}$-soft map is $\mathcal{C}$-invertible. For linear operators the converse statement is also true. In [Za$_5$, 2.1] M.Zarichnyi proved that a continuous group homomorphism $h : G \to H$ is $\mathcal{K}_{fd}$-soft if and only if h is $\mathcal{K}_{fd}$-invertible and its kernel $\operatorname{Ker} h$ is an $\mathrm{AE}(\mathcal{K}_{fd})$. Since each linear topological space is an $\mathrm{AE}(\mathcal{K}_{fd})$ (see, e.g., [Ba$_2$]), we get

Theorem 1. *A linear operator between linear topological spaces is $\mathcal{K}_{fd}$-soft if and only if it is $\mathcal{K}_{fd}$-invertible.*

Next, we find conditions under which a given linear operator is strongly $\mathcal{K}_{fd}$-universal.

Theorem 2. *A linear operator $T : X \to Y$ from a linear topological space $X \in \mathcal{K}^{\infty}_{fd}$ to a linear topological space Y is strongly $\mathcal{K}_{fd}$-universal if and only if the operator T is $\mathcal{K}_{fd}$-invertible and has infinite-dimensional kernel.*

In the proof we will exploit two lemmas.

Lemma 1. *If $X \in \mathcal{K}^{\infty}_{fd}$ is an infinite-dimensional linear topological space, then for every compactum $K \subset X$ there is a non-zero point $x \in X$ such that $([-1,1]\cdot x) \cap K \subset \{0\}$.*

Proof. Replacing K by $[-1,1]\cdot K$, if necessary, we may assume that $K = [-1,1]\cdot K$. Since K is a compact subset of the space $X \in \mathcal{K}^{\infty}_{fd}$, $\dim K < n$ for some $n \in \mathbb{N}$. The linear space X, being infinite-dimensional, contains an n-dimensional linear space $\mathbb{R}^n$. We claim that there is a point x on the unit sphere S of $\mathbb{R}^n$ such that $([-1,1]\cdot x) \cap K = \{0\}$. Assuming the converse, for every $x \in S$ we would find a number $n(x) \in \mathbb{N}$ such that $([-1,1]\cdot x) \cap K \supset [0, \frac{1}{n(x)}]\cdot x$. It can be shown that for every $n \in \mathbb{N}$ the set $S_n = \{x \in S : n(x) \le n\}$ is closed in S. Since $S = \bigcup_{n\in\mathbb{N}} S_n$, the Baire Theorem guarantees that S_n has non-empty interior in S for some n. Then $\dim S_n = n-1$. Since $K \supset [0,1/n]\cdot S_n$ and $n > \dim K > \dim([0,1/n]\cdot S_n) = n$, we get a contradiction. □

Lemma 2. *If $T : X \to Y$ is a linear operator with infinite-dimensional kernel from a linear topological space $X \in \mathcal{K}^{\infty}_{fd}$, then for every compact subset $C \subset X$ there exists an embedding $e : C \times [0,1] \to X$ such that $e(c,0) = c$ and $T \circ e(c,t) = T(c)$ for all $c \in C$ and $t \in [0,1]$.*

Proof. Let $L = \operatorname{Ker} T$ and $K = L \cap (C - C)$. By Lemma 1, there exists a non-zero point $x_0 \in L$ such that $([-1,1]\cdot x_0) \cap K \subset \{0\}$. Define the map $e : C \times [0,1] \to X$ letting $e(c,t) = c + tx_0$ for $(c,t) \in C \times [0,1]$. It is clear that $e(c,0) = c$ and $T \circ e(c,t) = T(c)$ for every $c \in C$ and $t \in [0,1]$. To show that the map e is injective, fix two points $(c,t),(c',t') \in C \times [0,1]$ with $e(c,t) = e(c',t')$. Then $c - c' = (t'-t)x_0$ and $T(c-c') = 0$ which implies $c - c' \in K$. Since $K \ni c - c' = (t'-t)x_0 \in [-1,1]x_0$ and $K \cap ([-1,1]x_0) \subset \{0\}$, we get $c - c' = 0$ and $t' - t = 0$, i.e., $(c,t) = (c',t')$. □

Proof of Theorem 2. The "only if" part of Theorem 2 is trivial. To prove the "if" part, assume that $T : X \to Y$ is an $\mathcal{K}_{fd}$-invertible operator, $X \in \mathcal{K}^{\infty}_{fd}$, and $\dim \operatorname{Ker} T = \infty$. To prove the strong $\mathcal{K}_{fd}$-universality of the map T, fix an embedding $f : B \to X$ of a closed subset B of a space $A \in \mathcal{K}_{fd}$ and a map $g : A \to Y$ such that $T \circ f = g|B$. By Theorem

1, the operator T is $\mathcal{K}_{fd}$-soft and hence there is a map $\tilde{f} : A \to X$ such that $\tilde{f}|B = f$ and $T \circ \tilde{f} = g$.

Denote by A/B the quotient space and let $q : A \to A/B$ be the quotient map. It is clear that the space A/B is finite-dimensional and thus admits an embedding $i : A/B \to [0,1]^n$ for some $n \in \mathbb{N}$ such that $e(\{B\}) = 0^n \in [0,1]^n$. Applying Lemma 2 several times, construct an embedding $e : \tilde{f}(A) \times [0,1]^n \to X$ such that $e(x, 0^n) = x$ and $T \circ e(x,t) = T(x)$ for any $x \in \tilde{f}(A)$ and $t \in [0,1]^n$. It is easy to verify that the map $\bar{f} : A \to X$ defined by $\bar{f}(a) = e(\tilde{f}(a), i \circ q(a))$ for $a \in A$ is an embedding satisfying the conditions $\bar{f}|B = f$ and $T \circ \bar{f} = g$. □

We apply Theorem 2 to prove the following theorem characterizing linear operators homeomorphic to the universal Zarichnyi map μ.

Theorem 3. *A linear operator $T : X \to Y$ between linear topological spaces is homeomorphic to the strongly $\mathcal{K}_{fd}$-universal Zarichnyi map μ if and only if $X \in \mathcal{K}_{fd}^\infty$, Y is homeomorphic to Q^∞, and the operator T is $\mathcal{K}_{fd}$-invertible.*

Proof. This theorem will follow from Theorem 2 and Uniqueness Theorem for strongly $\mathcal{K}_{fd}$-universal map as soon as we prove that each $\mathcal{K}_{fd}$-invertible linear operator $T : X \to Y$ from a linear topological space $X \in \mathcal{K}_{fd}^\infty$ onto a linear topological space Y containing a Hilbert cube has infinite-dimensional kernel. Assume to the contrary that $\operatorname{Ker} T$ is finite-dimensional. By Theorem 1, the operator T, being $\mathcal{K}_{fd}$-invertible, is $\mathcal{K}_{fd}$-soft. Fix a copy $Q \subset Y$ of the Hilbert cube in Y and an open surjective map $g : A \to Q$ of a finite-dimensional compactum A onto Q (such a map exists according to [Dr]). Since the operator T is $\mathcal{K}_{fd}$-invertible, there is a map $f : A \to X$ such that $T \circ f = g$. Let $B \subset \operatorname{Ker} T$ be any compact neighborhood of the origin in the finite-dimensional linear space $\operatorname{Ker} T$. Next, consider the compact set $K = f(A) + B \subset X$. Since $X \in \mathcal{K}_{fd}^\infty$, $\dim K < n$ for some $n \in \mathbb{N}$. Let $e : I^n \to Q \subset Y$ be any embedding of the n-dimensional cube I^n into Q. By the $\mathcal{K}_{fd}$-softness of the map T, there is a map $i : I^n \to X$ such that $T \circ i = e$ and $i(0^n) \in f(A)$. It is clear that i is an embedding.

We claim that K is a neighborhood of the point $x_0 = i(0^n)$ in $i(I^n)$. Assuming that it is not true, we would find a sequence $(x_n)_{n=1}^\infty \in i(I^n) \setminus K$ tending to x_0. Then the sequence $(T(x_n))_{n=1}^\infty$ converges to $T(x_0)$. Let $a_0 \in A$ be any point with $f(a_0) = x_0$. Since the map $g : A \to Q$ is open and $g(a_0) = T \circ f(a_0) = T(x_0) = \lim_{n\to\infty} T(x_n)$, there exists a sequence $(a_n)_{n=1}^\infty \subset A$ tending to a_0 such that $g(a_n) = T(x_n)$ for each n. Then the sequence $(f(a_n))_{n=1}^\infty$ converges to $f(a_0) = x_0$ and has the property: $T \circ f(a_n) = g(a_n) = T(x_n)$ for every n. Hence $x_n - f(a_n) \in \operatorname{Ker} T$ for every n. Since $\lim_{n\to\infty} x_n = x_0 = \lim_{n\to\infty} f(a_n)$, we get $\lim_{n\to\infty}(x_n - f(a_n)) = 0$ and thus $x_m - f(a_m) \in B$ for some m. Then $x_m \in B + f(a_m) \subset B + f(A) = K$, a contradiction with the choice of the sequence $(x_n)_{n=1}^\infty$. Thus K is a neighborhood of the point $x_0 = i(0^n)$ in $i(I^n)$ what is not possible since $\dim K < n = \dim(U)$ for any neighborhood $U \subset i(I^n)$ of $i(0^n)$. This contradiction shows that the kernel of T is infinite-dimensional. □

We remind that a topological space Y is called a *k-space* if a subset $F \subset Y$ is closed in Y if and only if for every compact subset $K \subset Y$ the intersection $F \cap K$ is closed in K.

Theorem 4. *A $\mathcal{K}_{fd}$-invertible linear operator $T : X \to Y$ from a linear topological space $X \in \mathcal{K}^\infty_{fd}$ onto a locally convex k-space Y is homeomorphic either to the projection $pr : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ for some $n, m \in \omega \cup \{\infty\}$ or to the strongly $\mathcal{K}_{fd}$-universal Zarichnyi map $\mu : \mathbb{R}^\infty \to Q^\infty$. The latter case occurs if and only if the space Y has uncountable Hamel basis.*

Proof. First, we show that $Y \in \mathcal{K}^\infty$. Since $X \in \mathcal{K}^\infty_{fd}$, the space X contains a countable collection $\{X_n : n \in \omega\}$ of compact subsets, fundamental in the sense that every compact subset of X lies in some X_n. By [En_1, 3.1.22], for every n the image $f(X_n)$ of the compact set X_n is a metrizable compactum. The operator T, being $\mathcal{K}_{fd}$-invertible, is surjective. Then $Y = \bigcup_{n\in\omega} f(X_n)$. We claim that the collection $\{f(X_n) : n \in \omega\}$ generates the topology of Y, i.e., $Y \in \mathcal{K}^\infty$.

Assuming the converse, we would find a non-closed subset $F \subset Y$ such that $F \cap f(X_n)$ is closed in $f(X_n)$ for every n. Since Y is a k-space, there is a compactum $K \subset Y$ such that $F \cap K$ is not closed in K. The compactum $K = \bigcup_{n\in\omega} K \cap f(X_n)$, being the countable union of metrizable compacta, is metrizable. Consequently, there is a sequence $(y_n)_{n=1}^\infty \subset F \cap K$ converging to a point $y_0 \in K \setminus F$. Since the map T is $\mathcal{K}_{fd}$-invertible, there is a sequence $(x_n)_{n=1}^\infty \subset X$ converging to a point $x_0 \in X$ such that $T(x_n) = y_n$ for all $n \in \omega$. The subset $\{x_n : n \in \omega\} \subset X$, being compact, lies in the compactum X_m for some m. Then $\{y_n : n > 0\} \subset f(X_m) \cap F$. Since the intersection $f(X_m) \cap F$ is closed in $f(X_m)$, we get $y_0 = \lim_{n\to\infty} y_n \in f(X_m) \cap F$, a contradiction with $y_0 \notin F$.

Thus the locally convex space Y belongs to the class $\mathcal{K}^\infty$. By [Ba_3], Y is homeomorphic either to Q^∞ or to $\mathbb{R}^n$ for some $n \in \omega \cup \{\infty\}$. Moreover, the last case occurs if and only if the algebraic dimension of Y is at most countable. If Y is homeomorphic to Q^∞, then by Theorem 2, the operator T is homeomorphic to the universal Zarichnyi map $\mu : \mathbb{R}^\infty \to Q^\infty$.

If Y is homeomorphic to $\mathbb{R}^n$ for some $n \in \omega \cup \{\infty\}$, then the algebraic dimension of Y is at most countable and Y carries the strongest linear topology, see [Ba_2]. In this case there is a linear continuous operator $S : Y \to X$ such that $T \circ S = \mathrm{id}$ and the map $h : X \to Y \times \mathrm{Ker}\, T$ defined by $h(x) = (T(x), x - S \circ T(x))$ for $x \in X$ is a linear homeomorphism (with inverse $h^{-1}(y, l) = S(y) + l$) such that $\mathrm{pr} \circ h = T$, where $pr : Y \times \mathrm{Ker}\, T \to Y$ is the projection. Hence the operator T is homeomorphic to the projection $\mathrm{pr} : Y \times \mathrm{Ker}\, T \to Y$. Since $\mathrm{Ker}\, T$ is a linear topological space from the class $\mathcal{K}^\infty_{fd}$ we can apply Corollary 1 (or [Ba_2]) to conclude that $\mathrm{Ker}\, T$ is homeomorphic to $\mathbb{R}^m$ for some $m \in \omega \cup \{\infty\}$. Therefore T is homeomorphic to the projection $pr : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. □

Next, we show that linear operators classifying by Theorem 3 do exist.

Theorem 5. *For every linear topological space $Y \in \mathcal{K}^\infty$ there is a $\mathcal{K}_{fd}$-invertible linear operator $T : X \to Y$ from a linear topological space $X \in \mathcal{K}^\infty_{fd}$.*

Proof. Since $Y \in \mathcal{K}^\infty$, the space Y possesses a countable family $\{F_n : n \in \omega\}$ of compact subsets such that every compact subset of Y lies in some F_n. For every $n \in \omega$ denote by $\mathcal{K}_n$ the subclass of $\mathcal{K}_{fd}$ consisting of all at most n-dimensional compacta. Using the Dranishnikov Theorem [Dr], for every n we can find an $\mathcal{K}_n$-invertible map $f_n : K_n \to F_n$ of a finite-dimensional metrizable compactum K_n onto F_n. Now consider the discrete sum $K = \sqcup_{n\in\omega} K_n$ and the map $f = \sqcup_{n\in\omega} f_n : K \to Y$. It is clear that $K \in \mathcal{K}^\infty_{fd}$ and the map f is $\mathcal{K}_{fd}$-invertible. Let $L(K) \supset K$ be the free linear topological space over K and $T : L(K) \to Y$ unique linear operator extending the $\mathcal{K}_{fd}$-invertible map f. It is clear that the operator T is $\mathcal{K}_{fd}$-invertible. By [Ba_2], $L(K) \in \mathcal{K}^\infty_{fd}$. □

Note that Theorems 4 and 5 allow us to give an alternative construction of the universal map of Zarichnyi.

Question. *Let $h : G \to H$ be a $\mathcal{K}_{fd}$-invertible continuous homomorphism between topological groups. Is h strongly $\mathcal{K}_{fd}$-universal if its kernel $h^{-1}(1)$ is strongly $\mathcal{K}_{fd}$-universal?*

The universal Zarichnyi map is not self-similar

We define a space X to be *locally self-similar* if every point of X has a basis of neighborhoods homeomorphic to X. It is well-known that the spaces $\mathbb{R}^\infty$ and Q^∞ (like many other model spaces of infinite-dimensional topology) are locally self-similar.

In [Za_5] M.Zarichnyi extended the notion of the local self-similarity onto maps and asked if the map $\mu : \mathbb{R}^\infty \to Q^\infty$ is locally self-similar. He defined a map $\pi : X \to Y$ to be *locally self-similar* if for every point $x \in X$ and every neighborhood $U \subset X$ of x there is a neighborhood $V \subset U$ of x such that the map $\pi|V : V \to \pi(V)$ is homeomorphic to π. Observe that for locally self-similar spaces X, Y the projection $\mathrm{pr} : X \times Y \to Y$ is locally self-similar.

The following result shows that the strongly $\mathcal{K}_{fd}$-universal Zarichnyi map is close to being locally self-similar.

Theorem 6. *If $T : X \to Y$ is a linear operator homeomorphic to the strongly $\mathcal{K}_{fd}$-universal Zarichnyi map $\mu : \mathbb{R}^\infty \to Q^\infty$, then for every nonempty open convex subset $U \subset X$ the map $T|U : U \to T(U)$ is homeomorphic to μ.*

Proof. Fix any non-empty open convex subset $U \subset X$. Since X is homeomorpic to $\mathbb{R}^\infty$, U is homeomorphic to $\mathbb{R}^\infty$ too. Next, $T(U)$ is homeomorphic to Q^∞, being open contractible subspace of Y, the topological copy of Q^∞. By Uniqueness Theorem, to show that the map $T|U : U \to T(U)$ is homeomorphic to μ it suffices to verify that $T|U$ is a strongly $\mathcal{K}_{fd}$-universal map.

First we show that the map $T|U : U \to T(U)$ is $\mathcal{K}_{fd}$-invertible. Fix any map $g : A \to T(U)$ from a compactum $A \in \mathcal{K}_{fd}$. Since the map T is $\mathcal{K}_{fd}$-invertible, there is a map $f : A \to X$ such that $T \circ f = g$. For every point $x \in K$ find a point $a_x \in \mathrm{Ker}\, T$ such that $f(x) + a_x \in U$. Next, let $U(x) = f^{-1}(U - a_x)$. Using the compactness of A find a finite subcover $\{U(x_1), \dots, U(x_n)\}$ of the open cover $\{U(x) : x \in A\}$ and let $\{\lambda_i : A \to [0,1]\}_{i=1}^n$ be a partition of unity such that $\lambda_i^{-1}(0,1] \subset U(x_i)$ for every $i \le n$. Consider the map $\alpha : A \to \mathrm{Ker}\, T$ defined by $\alpha(x) = \sum_{i=1}^n \lambda_i(x) a_{x_i}$ for $x \in K$. It can be shown that the map $h = f + \alpha : A \to X$ has the properties: $T \circ h = T \circ f = g$ and $h(A) \subset U$. Therefore, the map $T|U$ is $\mathcal{K}_{fd}$-invertible.

To show that it is strongly $\mathcal{K}_{fd}$-universal, fix an embedding $f : B \to U$ of a closed subset B of a space $A \in \mathcal{K}_{fd}$ and a map $g : A \to T(U)$ such that $T \circ f = g$. Since the map $T|U : U \to T(U)$ is $\mathcal{K}_{fd}$-invertible, there is a map $h_1 : K \to U$ such that $T \circ h_1 = g$. Next, using the strong $\mathcal{K}_{fd}$-universality of the operator T, find a map $h_2 : A \to X$ such that $h_2|B = f$ and $T \circ h_2 = g$. Let $\lambda : A \to [0,1]$ be a continuous map such that $\lambda(B) = \{0\}$

and $\lambda(A \setminus W) = \{1\}$, where $W = h^{-1}(U)$. Next, consider the map $h : A \to X$ defined by $h(a) = \lambda(a)h_1(a) + (1 - \lambda(a))h_2(a)$ for $a \in A$. It is easy to see that $T \circ h = g$ and $h(A) \subset U$.

Let $q : A \to A/B$ be the quotient map. Since the quotient space A/B is finite-dimensional, there is an embedding $i : A/B \to [0,1]^n$ for some $n \in \omega$ such that $i(\{B\}) = 0^n$. The space $h(A)$ belongs to the class $\mathcal{K}_{fd}$, being a compact subset of the space $X \in \mathcal{K}_{fd}^\infty$. Then by the strong $\mathcal{K}_{fd}$-universality of the operator T, there is an embedding $e : h(A) \times [0,1]^n \to X$ such that $e(x, 0^n) = x$ and $T \circ e(x,t) = T(x)$ for every $x \in h(A)$ and $t \in [0,1]^n$. Since $h(A) \subset U$, there exists $\varepsilon > 0$ such that $e(h(A) \times [0,\varepsilon]^n) \subset U$. Finally consider the map $\bar{f} : A \to X$ defined by $\bar{f}(a) = e(h(a), \varepsilon \cdot i \circ q(a))$ for $a \in A$. It can be easily shown that $\bar{f}|B = f$, $\bar{f}(A) \subset U$ and $T \circ \bar{f} = g$. Therefore, the map $T|U$ is strongly $\mathcal{K}_{fd}$-universal. The space U, being an open subset of the space $X \in \mathcal{K}_{fd}^\infty$, belongs to the class $\mathcal{K}_{fd}^\infty$. If the space $T(U)$ is homeomorphic to Y, then by Uniqueness Theorem the maps $T|U$ and T are homeomorphic. □

Thus the local self-similarity of the Zarichnyi map μ would be proven if we would find a linear operator between locally convex spaces, homeomorphic to μ. Unfortunately, no such an operator exists. This is so because each locally convex space $X \in \mathcal{K}_{fd}^\infty$ has at most countable Hamel basis, see [Ba$_2$]. Consequently, each linear image of X also has at most countable Hamel basis and thus can not be homeomorphic to Q^∞. But this is not a unique reason why we can not find a linear operator between locally convex spaces, homeomorphic to the universal Zarichnyi map μ.

Theorem 6. *Each open $\mathcal{K}_{fd}$-invertible map $f : \mathbb{R}^\infty \to Q^\infty$ is not self-similar.*

Let us define a map $\pi : X \to Y$ to be *locally $\mathcal{K}_{fd}$-invertible* if for every point $x \in X$ and every neighborhood $U \subset X$ of x there is a neighborhood $V \subset U$ of x such that the map $\pi|V : V \to \pi(V)$ is $\mathcal{K}_{fd}$-invertible. It is clear that each locally self-similar $\mathcal{K}_{fd}$-invertible map is locally $\mathcal{K}_{fd}$-invertible.

We define a space X to be *almost finite-dimensional* if there is $n \in \mathbb{N}$ such that $\dim F \le n$ for every finite-dimensional closed subset F of X. Let us note that there exist infinite-dimensional almost finite-dimensional compact spaces, see [En$_2$, 5.2.23].

Theorem 6 will be derived from

Lemma 3. *Each compactum K admitting a surjective locally $\mathcal{K}_{fd}$-invertible map $f : X \to K$ from a space $X \in \mathcal{K}_{fd}^\infty$, is almost finite-dimensional.*

To prove this lemma we need

Lemma 4. *For every compact space K and a closed subset $A \subset K \times Q$ with $\dim A < \dim K$ there is a closed subset $F \subset K \times Q$ such that $F \cap A = \emptyset$ but $F \cap s(K) \ne \emptyset$ for every section $s : K \to K \times Q$ of the projection $\mathrm{pr} : K \times Q \to K$.*

Proof. Find any $n \in \omega$ with $\dim A \le n < \dim K$. By Hurewicz-Wallman Theorem [En$_2$, 1.9.3], there exists a map $f : L \to S^n$ from a closed subset L of K into the n-dimensional sphere which has no continuous extension $\bar{f} : K \to S^n$.

Since S^n is an ANR, we can extend f to a continuous map $\tilde{f} : \bar{O}(L) \to S^n$ defined on the closure of an open neighborhood $O(L)$ of L in K. Let $B = L \times Q$ and $U = O(L) \times Q$.

Since $\dim A \le n$, we can apply Hurewicz-Wallman Theorem again to find a continuous map $p : A \to S^n$ such that $p|A \cap \bar{U} = \tilde{f} \circ pr|A \cap \bar{U}$. Next, since S^n is an ANR, there is a continuous map $\bar{p} : V \to S^n$ defined on an open neighborhood V of the closed set $A \cap \bar{U}$ in $K \times Q$ such that $\bar{p}|A = p$ and $\bar{p}|\bar{U} = \tilde{f} \circ \mathrm{pr}\,|\bar{U}$. Consider the open set $O(A) = O(B) \cup (V \setminus B)$, where $O(B) = \{x \in U : d(\bar{p}(x), \tilde{f} \circ \mathrm{pr}(x)) < 1\}$ and d stands for the standard Euclidean metric of $\mathbb{R}^{n+1} \supset S^n$.

We claim that the closed set $F = (K \times Q) \setminus O(A)$ misses A but meets the image $s(K)$ of every any section $s : K \to K \times Q$ of the projection pr. Assuming the converse, we would find a section $s : K \to K \times Q$ of the projection pr such that $s(Q) \cap F = \emptyset$. Then $s(K) \subset O(A)$ and we can consider the map $g = \bar{p} \circ s : K \to S^n$. Observe that $s(L) \subset B \cap O(A) = O(B)$ and hence $d(g(x), f(x)) = (d(\bar{p} \circ s(x), \bar{f} \circ \mathrm{pr} \circ s(x)) < 1$ for all $x \in L$ which yields that the maps $g|L$ and f are homotopic. Since $g|L$ has the extension $g : K \to S^n$, we may apply the Borsuk Extension Theorem [En$_2$, 1.9.7] to conclude that the map f has a continuous extension $\bar{f} : K \to S^n$, which contradicts to the choice of f. $\square$

Proof of Lemma 3. Suppose a compactum K admits an open surjective locally $\mathcal{K}_{fd}$-invertible map $\pi : X \to K$ of a space $X \in \mathcal{K}_{fd}^\infty$. Assume that the compactum K is not almost finite-dimensional. The compactum K, being a countable union of metrizable compacta, is metrizable. By the compactness of K there is a point $y \in K$ having no almost infinite dimensional neighborhood in K. Fix a countable base $\{U_n : n \in \omega\}$ of neighborhoods of y in K and for every $n \in \omega$ find a finite dimensional compactum $K_n \subset U_n$ with $\dim K_n > n$.

Fix any point $x \in X$ with $\pi(x) = y$ and let $\{X_n : n \in \omega\}$ be an increasing collection of finite-dimensional compact subsets of X generating its topology. Without loss of generality, $x \in X_0$ and $\dim X_n \le n$ for every $n \in \omega$. The space X, having a countable network of the topology, admits an injective continuous map $i : X \to Q$ into the Hilbert cube. Now consider the injective map $e = (\pi, i) : X \to K \times Q$ defined by $e(x) = (f(x), i(x))$ for $x \in X$. For every $n \in \omega$ let $A_n = i(X_n) \cap (K_n \times Q)$. Since $\dim A_n \le \dim X_n \le n < \dim K_n$, we can apply Lemma 4 to find a closed subset $F_n \subset K_n \times Q$ such that $A_n \cap F_n = \emptyset$ but $F_n \cap s(K_n) \ne \emptyset$ for every section $s : K_n \to K_n \times Q$ of the projection $\mathrm{pr} : K_n \times Q \to K_n$.

Consider the set $F = \bigcup_{k\in\omega} e^{-1}(F_k)$. Since each set F_k is closed in $K \times Q$ and $X_n \cap F = X_n \cap (\bigcup_{k=0}^{n-1} e^{-1}(F_k))$ for each $n \in \omega$, we get that F is a closed subset of X. Next, since $X_0 \cap F = \emptyset$, the set $U = X \setminus F$ is an open neighborhood of the point x in X.

Let us show that for every open neighborhood $V \subset U$ of x the map $\pi|V : V \to \pi(V)$ is not $\mathcal{K}_{fd}$-invertible. Indeed, since the map π is open, $\pi(V)$ is an open neighborhood of the point $\pi(x) = y$ and thus $f(V) \supset K_n$ for some $n \in \omega$. Assuming that the map $\pi|V$ is $\mathcal{K}_{fd}$-invertible we would find a map $g : K_n \to V \subset U$ such that $\pi \circ g = \mathrm{id}$. Then the map $s = e \circ g : K_n \to K \times Q$ has the properties: $\mathrm{pr} \circ s = \mathrm{id}$ and $s(K_n) \cap F_n = \emptyset$, a contradiction with the choice of the set F_n. $\square$

Proof of Theorem 6. Assume that $f : \mathbb{R}^\infty \to Q^\infty$ is an open $\mathcal{K}_{fd}$-invertible locally self-similar map. Then f is locally $\mathcal{K}_{fd}$-invertible. Let $K \subset Q^\infty$ be a topological copy of the Hilbert cube and $X = f^{-1}(K)$. Then $X \in \mathcal{K}_{fd}^\infty$ and $f|K : X \to K$ is an open surjective locally $\mathcal{K}_{fd}$-invertible map, a contradiction with Lemma 3. $\square$

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