DIVISION AND k-TH ROOT THEOREMS FOR Q-MANIFOLDS

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Abstract. We prove a that a locally compact ANR-space X is a Q-manifold if and only if it has the Disjoint Disk Property (DDP), all points of X are homological Z_∞-points and X has the countable-dimensional approximation property (cd-AP), which means that each map $f : K \to X$ of a compact polyhedron can be approximated by a map with countable-dimensional image. As an application we prove that a space X with DDP and cd-AP is a Q-manifold if some finite power of X is a Q-manifold. If some finite power of a space X with cd-AP is a Q-manifold, then X^2 and $X \times [0, 1]$ are Q-manifolds as well. We construct a countable family $\mathcal{X}$ of spaces with DDP and cd-AP such that no space $X \in \mathcal{X}$ is homeomorphic to the Hilbert cube Q whereas the product $X \times Y$ of any different spaces $X, Y \in \mathcal{X}$ is homeomorphic to Q. We also show that no uncountable family $\mathcal{X}$ with such properties exists.

1. Division and k-Root Theorems for Cantor and Tychonov cubes

It is obvious that the powers X^k, M^k of two homeomorphic topological spaces X, M are homeomorphic as well. For certain "nice" spaces M the converse implication is also true: a space X is homeomorphic to M if for some finite number k the powers X^k and M^k are homeomorphic. Results of this type will be referred as k-th Root Theorems. A typical example of such a theorem is k-th Root Theorem for Cantor and Tychonov cubes.

Theorem 1 (k-th Root Theorem for Cantor and Tychonov cubes)**.** *Let M be either a Cantor cube $\{0, 1\}^\tau$ with $\tau \geq 1$ or a Tychonov cube $[0, 1]^\kappa$ with $\kappa \geq \aleph_1$. A topological space X is homeomorphic to M if and only if for some finite number $k \in \mathbb{N}$ the powers X^k and M^k are homeomorphic.*

By induction this theorem can be easily derived from the following

Theorem 2 (Division Theorem for Cantor and Tychonov cubes)**.** *Let M be either a Cantor cube $\{0, 1\}^\tau$ with $\tau \geq \aleph_0$ or a Tychonov cube $[0, 1]^\kappa$ with $\kappa \geq \aleph_1$. If the product $X \times Y$ of two spaces X, Y is homeomorphic to M, then X or Y is homeomorphic to M.*

This theorem can be easily derived from famous topological characterizations of Cantor and Tychonov cubes, due to Ščepin (see [S1] and [S2]).

Theorem 3 (Ščepin)**.** *Let X be a compact Hausdorff space.*

1. *X is homeomorphic to a Cantor cube $\{0, 1\}^\tau$ of weight $\tau \geq \aleph_0$ if and only if X is a uniform-by-character zero-dimensional* AE(0)*-space of weight $w(X) = \tau$;*
2. *X is homeomorphic to a Tychonov cube $[0, 1]^\tau$ of weight $\tau \geq \aleph_1$ if and only if X is a uniform-by-character* AE*-space of weight $w(X) = \tau$.*

We recall that a topological space X is called an AE-*space* (resp. AE(0)-*space*) if any continuous map $f : B \to X$ defined on a closed subset B of a (zero-dimensional) compact Hausdorff space A can be extended to a continuous map $\bar{f} : A \to X$.

A topological space X is called *uniform-by-character* if the character of X at each point $x \in X$ is equal to some fixed cardinal κ. We recall that the *character* of X at a point $x \in X$ is the smallest size $|\mathcal{B}|$ of a neighborhood base at x.

For $\tau = \aleph_0$ Ščepin's characterization of Cantor cubes turns into the classical Brouwer characterization of the Cantor set: a space X is homeomorphic to the Cantor set $\{0, 1\}^\omega$ if and only if X is a zero-dimensional compact metrizable space without isolated points.

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Now we are able to derive the Division Theorem 2 from Ščepin's characterization theorem. Assume that $M = \{0,1\}^\tau$ is a Cantor cube with $\tau \geq \aleph_0$ and let X, Y be two spaces whose product $X \times Y$ is homeomorphic to M.

Then X, Y are compact zero-dimensional AE(0)-spaces, being retracts of the product $X \times Y$. Since all points of the product $X \times Y$ have character τ, either X or Y contains no point of character $< \tau$. We lose no generality assuming that X is such a space. Then, X, being a uniform-by-character compact zero-dimensional AE(0)-space of weight τ, is homeomorphic to the Cantor cube $\{0,1\}^\tau = M$, according to the Ščepin characterization of $\{0,1\}^\tau$. This completes the proof of Division Theorem for Cantor cubes.

By analogy, the Division Theorem for Tychonov cubes can be derived from Ščepin's characterization of Tychonov cubes. □

2. Division and k-Root Theorems for the Hilbert cube

From now on all topological spaces are separable and metrizable. Observe that Theorems 1 and 2 do not cover the case of the Hilbert cube $Q = I^\omega$, where $I = [0,1]$. This is not incidental because without any restrictions the k-th Root and Division Theorems for the Hilbert cube are not true. A suitable counterexample is due to Singh [Singh] who constructed a compact absolute retract S such that $S \times S$ and $S \times [0,1]$ are homeomorphic to Q but S is not homeomorphic to Q. Singh's space S contains no topological copy of the 2-disk I^2 and hence does not possess the Disjoint Disks Property.

We recall that a space X has the Disjoint Disks Property (briefly, DDP) if any two maps $f, g : I^2 \to X$ from a 2-dimensional cube can be uniformly approximated by maps with disjoint images. This property was introduced by Cannon [Cannon] and is crucial in the topological characterization of finite-dimensional manifolds (see [Dav]).

In spite of Singh's counterexample, some restricted forms of the k-th Root and Division Theorems still hold for the Hilbert cube. The restrictions involve the Disjoint Disk Property and the countable-dimensional Approximation Property, which is a particular case of $\mathcal{P}$-Approximation Property.

We shall say that a topological space X has a *$\mathcal{P}$-Approximation Property* (briefly, $\mathcal{P}$-AP) where $\mathcal{P}$ is a family of subsets of a space X, if for each map $f : K \to X$ defined on a compact polyhedron K and each open cover $\mathcal{U}$ of X there is a map $f' : K \to X$ such that $f'(K) \in \mathcal{P}$ and f' is $\mathcal{U}$-near f in the sense that for each point $x \in K$ the set $\{f(x), f'(x)\}$ lies in some set $U \in \mathcal{U}$.

If $\mathcal{P}$ is the family of finite-dimensional (resp. countable-dimensional, weakly infinite-dimensional) subspaces of X, then we shall refer to the $\mathcal{P}$-AP as fd-AP (resp. cd-AP, wid-AP). We recall that a space X is *countable-dimensional* if X is the countable union of finite-dimensional subspaces.

All these approximation properties follow from Borsuk's property (Δ) (see [Bor, §VII]. We recall that a space X has the property (Δ) if for any point $x \in X$ and a neighborhood $U \subset X$ of x there is a neighborhood $V \subset U$ of x such that any compact subset $K \subset V$ is contractible in a subset $H \subset U$ having dimension $\dim(H) \leq \dim(K) + 1$. It follows from (the proof of) Theorem VII.2.1 of [Bor] that each metrizable space with the property (Δ) has fd-AP. Therefore these properties are related as follows:

$$(\Delta) \Rightarrow (\text{fd-AP}) \Rightarrow (\text{cd-AP}) \Rightarrow (\text{wid-AP}).$$

It is clear that the Hilbert cube Q has the property (Δ) and consequently all weaker approximation properties.

On the other hand, it is easy to construct a compact AR without wid-AP: just consider the space $I^2 \cup_\varphi Q$ obtained by gluing the 2-disk to the Hilbert cube Q along a surjective map $\varphi : J \to Q$ of an arc $J \subset I^2 \setminus \partial I^2$. Replacing the Hilbert cube by the 4-dimensional cube I^4 we can construct a 4-dimensional absolute retract without property (Δ). Replacing Q by a countable-dimensional infinite-dimensional (resp. weakly infinite-dimensional uncountable-dimensional) absolute retract we can construct a compact absolute retract having cd-AP but not fd-AP (resp. wid-AP but not cd-AP).

It turns out that the k-th Root Theorem for the Hilbert cube holds for spaces possessing DDP and cd-AP. The following four theorems are the main results of the paper and will be proved in Section 6 and 7.

Theorem 4 (k-th Root Theorem for the Hilbert cube)**.** *A topological space X with* DDP *and* cd-AP *is homeomorphic to the Hilbert cube Q if some finite power X^k of X is homeomorphic to Q.*

This theorem will be applied to prove another

Theorem 5. *Let X be a space having* cd-AP. *If some finite power X^k of X is homeomorphic to Q, then both X^2 and $X \times I$ are homeomorphic to Q.*

The situation with Division Theorem for the Hilbert cube is more delicate. On one hand, we have a negative result.

Theorem 6. *There is a countable family $\mathcal{X}$ of spaces possessing* DDP *and* fd-AP *such that*
(1) *the square $X \times X$ of any space $X \in \mathcal{X}$ is not homeomorphic to Q;*
(2) *the product $X \times Y$ of any two different spaces $X, Y \in \mathcal{X}$ is homeomorphic to Q.*

On the other hand we have a positive result showing that the family $\mathcal{X}$ from the preceding theorem cannot be uncountable.

Theorem 7 (Collective Division Theorem for the Hilbert cube)**.** *An uncountable family $\mathcal{X}$ of topological spaces with* DDP *and* cd-AP *contains a space $X \in \mathcal{X}$ homeomorphic to the Hilbert cube Q provided the product $X \times Y$ of any different spaces $X, Y \in \mathcal{X}$ is homeomorphic to Q.*

3. Homological characterizations of the Hilbert cube

The proofs of Collective Division and k-Roots Theorems for the Hilbert cube rely on homological characterizations of Q, due to Daverman and Walsh [DW]. First we recall some notation.

We use singular homology $H_*(X; G)$ with coefficients in an abelian group G. By $\widetilde{H}_*(X; G)$ we shall denote the singular homology of X, reduced in dimension zero. If $G = \mathbb{Z}$, then we omit the symbol of the group and will write $H_*(X)$ in place of $H_*(X; \mathbb{Z})$.

A closed subset A of a space X is called

- a *Z_n-set* if every map $f : I^n \to X$ can be approximated by maps into $X \setminus A$;
- a *homotopical Z_∞-set* if for every open set $U \subset X$ the relative homotopy groups $\pi_k(U, U \setminus A)$ are trivial for all k;
- a *G-homological Z_∞-set* if for every open set $U \subset X$ the relative homology groups $\widetilde{H}_k(U, U \setminus A; G)$ are trivial for all k;
- a *homological Z_∞-set* if it is a $\mathbb{Z}$-homological Z_∞-set in X.

In [DW] homological Z_∞-sets are referred to as closed sets with infinite codimension.

A point $x \in X$ is called a (*homotopical, homological*) *Z_∞-point* if the singleton $\{x\}$ is a (homotopical, homological) Z_∞-set in X. The Excision Axiom for singular homology [Hat, 2.20] implies that a point $x \in X$ is a G-homological Z_∞-point if and only if $\widetilde{H}_k(X, X \setminus \{x\}; G) = 0$ for all k. It is well-known that each point of the Hilbert cube is a Z_∞-point and consequently, a G-homological Z_∞-point for any non-trivial abelian group G.

Theorem 2.3 of [To78] implies that a closed subset A of an ANR-space is a Z_∞-set if and only if it is a homotopical Z_∞-set. Also each homotopical Z_∞-set is a homological Z_∞-set. Many examples of homological Z_∞-sets in Q, which are not homotopical Z_∞-sets can be constructed using the following fact proved in Corollary 2.4 of [DW]:

Proposition 1 (Daverman-Walsh)**.** *Assume that X is a locally compact ANR whose any point is a homological Z_∞-point. Then each closed finite-dimensional subset of X is a homological Z_∞-set.*

The proof of this proposition follows from a characterization of G-homological Z_∞-sets proved in Proposition 2.3 of [DW].

Proposition 2 (Daverman-Walsh)**.** *A closed subset A of a locally compact ANR-space X is a G-homological Z_∞-set if each point $a \in A$ is a G-homological Z_∞-point in X and has arbitrarily small neighborhoods $U_a \subset A$ whose relative boundary in A are G-homological Z_∞-sets in X.*

In fact, we shall derive a bit more from this proposition. Namely, that each closed countable-dimensional subset of Q is a homological Z_∞-set. According to [En, 7.1.9] each completely-metrizable countable-dimensional space X has transfinite inductive dimension $\operatorname{trind}(X) \neq \infty$ defined as follows. We put $\operatorname{trind}(X) = -1$ if and only if $X = \emptyset$. Given an ordinal α we write $\operatorname{trind}(X) \le \alpha$ if X has a base of the topology consisting of open sets $U \subset X$ whose boundaries have transfinite dimension $\operatorname{trind}(\partial U) < \alpha$.

The transfinite inductive dimension of a space X equals the smallest ordinal α with $\mathrm{trind}\,(X) \le \alpha$ if such ordinal α exists and $\mathrm{trind}\,(X) = \infty$ otherwise.

Proposition 3. *Assume that X is a locally compact ANR whose points all are G-homological Z_∞-points for some non-trivial abelian group G. Then each closed countable-dimensional subset $A \subset X$ is a G-homological Z_∞-set in X.*

Proof. According to [En, 7.1.9], the space A, being completely-metrizable and countable-dimensional, has transfinite inductive dimension $\mathrm{trind}\,(A) \ne \infty$. So, we shall prove the proposition by transfinite induction on $\alpha = \mathrm{trind}\,(A)$. For $\alpha = -1$ the proposition is trivial. Assume that for some ordinal α the assertion is true for all closed subsets $A \subset X$ with $\mathrm{trind}\,(A) < \alpha$. Assuming that A is a closed subset of X with $\mathrm{trind}\,(A) = \alpha$ we get that A has a base of the topology consisting of open sets $U \subset A$ whose relative boundary in A have transfinite dimension $\mathrm{trind}\,(\partial U) < \alpha$. By inductive hypothesis each set ∂U is a G-homological Z_∞-set in X. Applying Proposition 2 we conclude that A is a G-homological Z_∞-set in X. □

We shall say that a space X has Z-AP (resp. HZ-AP) if it has $\mathcal{P}$-AP for the class $\mathcal{P}$ of (homological) Z_∞-sets in X. The latter means that each map $f : K \to X$ from a compact polyhedron can be approximated by a map whose image is a (homological) Z_∞-set in X.

By a *Q-manifold* we understand a metrizable separable space M such that each point $x \in X$ has an open neighborhood $U \subset M$ homeomorphic to an open subset of Q. It is clear that each Q-manifold is a locally compact ANR. By Theorem 22.1 of [Chap], each compact contractible Q-manifold is homeomorphic to Q.

The following Z-AP characterization of Q-manifolds is due to Toruńczyk [To80].

Theorem 8 (Toruńczyk). *A space X is a Q-manifold if and only if X is a locally compact ANR possessing* Z-AP.

A homological version of this characterization was proved by Daverman and Walsh in [DW].

Theorem 9 (Daverman-Walsh). *A space X is a Q-manifold if and only if X is a locally compact ANR possessing* DDP *and* HZ-AP.

Combining this characterization theorem with Proposition 3 we get a local characterization of Q-manifolds whose fd-AP version can be found in Theorem 6.1 of [DW].

Theorem 10. *A space X is a Q-manifold if and only if*

(1) *X has* DDP*;*
(2) *X has* cd-AP*; and*
(3) *each point of X is a homological Z_∞-point.*

4. On Cell-like maps between Q-manifolds

In this section we shall apply the Characterization Theorem 10 to obtain some new characterizations of Q-manifolds, involving cell-like maps. We recall that a map $\pi : X \to Y$ is called

- *proper* if the preimage $\pi^{-1}(K)$ of every compact set is compact;
- *cell-like* if π is proper and the preimage $\pi^{-1}(y)$ of every point $y \in Y$ has trivial shape;
- *countable-dimensional* if the preimage $\pi^{-1}(y)$ of every point $y \in Y$ is countable-dimensional.

Theorem 11. *A space X is a Q-manifold if and only if X has* DDP*,* cd-AP *and X is the image of a Q-manifold M under a countable-dimensional cell-like map $\pi : M \to Q$.*

This theorem can be easily derived from Theorem 10 and

Proposition 4. *Let $\pi : M \to X$ be a cell-like map between locally compact ANRs and $N_\pi = \{x \in X : |\pi^{-1}(x)| > 1\}$ be the nondegeneracy set of π. Then*

(1) *a point $x \in X$ is a homological Z_∞-point in X if its preimage $\pi^{-1}(x)$ is a homological Z_∞-set in M;*
(2) *a point $x \in X$ is a homological Z_∞-point in X if its preimage $\pi^{-1}(x)$ is countable-dimensional and each point $z \in \pi^{-1}(x)$ is a homological Z_∞-point in M.*

(3) *If the space N_π is finite-dimensional (resp. countable-dimensional) and the space M has* fd-AP *(resp.* cd-AP*), then X has* fd-AP *(resp.* cd-AP*);*

Proof. (1) The first item is well-known and easily follows from the Approximate Lifting Theorem 16.7 [Dav] for cell-like maps.

(2) Assume that for some point $x \in X$ the preimage $\pi^{-1}(x)$ is countable-dimensional and each point $z \in \pi^{-1}(x)$ is a homological Z_∞-point. By Proposition 3, the set $\pi^{-1}(x)$ is a homological Z_∞-set in M and by the preceding item the point x is a homological Z_∞-point in X.

(3) Assume that N_π is finite-dimensional and M has fd-AP. To show that X has the fd-AP, fix a map $f : K \to X$ of a compact polyhedron and an open cover $\mathcal{U}$ of X. Let $\mathcal{V}$ be an open cover of X whose star $\mathcal{S}t(\mathcal{V})$ refines $\mathcal{U}$. By Approximate Lifting Theorem 16.7 [Dav] for cell-like maps, there exists a map $g : K \to M$ such that $\pi \circ g$ is $\mathcal{V}$-near f. Since M has fd-AP, the map g can be approximated by a map $g' : K \to M$ such that $g'(K)$ is finite-dimensional and g' is $\pi^{-1}(\mathcal{V})$-near g. Then the map $f' = \pi \circ g'$ is $\mathcal{S}t(\mathcal{V})$-near f. It remains to show that $f'(K)$ is finite-dimensional. Write $f'(K)$ as the union $f'(K) = \big(f'(K) \cap N_\pi\big) \cup \big(f'(K) \setminus N_\pi\big)$. The space $f'(K) \cap N_\pi$ is finite-dimensional. On the other hand, the restriction of $\pi|M \setminus \pi^{-1}(N_\pi) : M \setminus \pi^{-1}(N_\pi) \to X \setminus N_\pi$ of π, being a proper injective map, is an embedding and thus $\dim(f'(K) \setminus N_\pi) = \dim\big(\pi(g'(K) \setminus \pi^{-1}(N_\pi)\big) = \dim(g'(K) \setminus \pi^{-1}(N_\pi)) \le \dim(g'(K)) < \infty$. Then $f'(K)$ is finite-dimensional, being the union of two finite-dimensional subspaces (see Theorem 1.5.8 [En]).

The cd-case of (3) can be proved analogously. □

Theorem 10 combined with Proposition 4(3) implies another cell-like characterization of Q-manifolds.

Theorem 12. *A space X with DDP is a Q-manifold if and only if X is the image of a Q-manifold M under a countable-dimensional cell-like map $\pi : M \to X$ whose non-degeneracy set $N_\pi = \{x \in X : |\pi^{-1}(x)| > 1\}$ is countable-dimensional.*

5. Disjoint Disk-Arc Property

Singh's example of a fake Hilbert cube [Singh] shows that HZ-AP does not imply DDP and hence DDP cannot be omitted from Theorems 9–12. Nevertheless, HZ-AP implies DDAP, a bit weaker property than DDP.

Following [Dav] we say that a space X has the *Disjoint Disk-Arc Property* (briefly DDAP) if any maps $f : I^2 \to X$, $g : I \to X$ can be approximated by maps $f' : I^2 \to X$ and $g' : I \to X$ with $f'(I^2) \cap g'(I) = \emptyset$. The following lemma linking DDP with DDAP can be proved by the argument of Proposition 26.6 of [Dav].

Proposition 5. *If an ANR-space X has* DDAP*, then for any ANR-space Y having no isolated point the product $X \times Y$ has* DDP*.*

Proposition 6. *Each space X with* HZ-AP *has* DDAP*.*

Proof. Take any maps $f : I^2 \to X$ and $g : I \to X$. Since X has HZ-AP, the map f can be approximated by a map $f' : I^2 \to X$ whose image $Z = f'(I^2)$ is a homological Z_∞-set in X. Next, we shall approximate the map g'. Given an open cover $\mathcal{U}$ of X we will construct a map $g' : I \to X \setminus Z$ which is $\mathcal{U}$-near g in the sense that for any point $t \in I$ the set $\{g(t), g'(t)\}$ lies in some $U \in \mathcal{U}$. By the compactness of the interval $[0,1]$ there is a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that for every $i \le n$ the image $g([t_{i-1}, t_i])$ lies in some set $U_i \in \mathcal{U}$.

Since $\widetilde{H}_0(U_i, U_i \setminus Z) = 0$, the path-connected component of U_i containing the point $g(t_i)$ meets the set $U_i \setminus Z$ at some point x_i. We claim that the points x_{i-1}, x_i lie in the same path-connected component of $U_i \setminus Z$. If the converse were true, then we would get a nontrivial cycle $\alpha = x_{i-1} - x_i$ in $H_0(U_i \setminus Z)$. On the other hand, this cycle is the boundary of an obvious 1-chain β in U_i and thus vanishes in the homology group $H_0(U_i)$. But this contradicts the exactness of the following sequence

$$0 = H_1(U_i, U_i \setminus Z) \to H_0(U_i \setminus Z) \to H_0(U_i)$$

for the pair $(U_i, U_i \setminus Z)$.

Therefore x_{i-1}, x_i lie in the same path-connected component of $U_i \setminus Z$, ensuring the existence of a continuous map $g_i : [t_{i-1}, t_i] \to U_i \setminus Z$ with $g_i(t_{i-1}) = x_{i-1}$ and $g_i(t_i) = x_i$. The maps g_i, $i \le m$, compose a single continuous map $g' : [0,1] \to X \setminus Z = X \setminus f'(I^2)$ which is $\mathcal{U}$-near to g, confirming the DDAP of X. $\square$

Therefore for a locally compact ANR-space whose points all are homological Z_∞-points we have the following implications between different $\mathcal{P}$-Approximation Properties:

$$\begin{array}{ccccccc}
(\text{wid-AP}) & \longleftarrow & (\text{cd-AP}) & \longrightarrow & (\text{HZ-AP}) & \longrightarrow & (\text{DDAP}) \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & (\Delta) & \longleftarrow & (\text{Z-AP}) & \longrightarrow & (\text{DDP})
\end{array}$$

6. Division and kth Root Theorems for Q-manifolds

In this section we shall prove kth Root and Collective Division Theorems for Q-manifolds, whose partial cases are Theorems 4 and 7. The proofs of these theorems essentially rely on the characterization Theorem 10 and the Künneth formula expressing homology of the product $X \times Y$ of two spaces via homology of the factors X, Y. We shall use the following relative version of the Künneth formula that can be found in Theorem 10 [Spa, §5.3].

Relative Künneth Formula: *For open sets $U \subset X$, $V \subset Y$ in topological spaces X, Y and a non-negative integer n the following exact sequence holds:*

$$0 \to [H(X,U) \otimes H(Y,V)]_n \to H_n(X \times Y, X \times V \cup U \times Y) \to [H(X,U) * H(Y,V)]_{n-1} \to 0$$

Here

$$\begin{aligned}
[H(X,U) \otimes H(Y,V)]_n &= \oplus_{i+j=n} H_i(X,U) \otimes H_j(Y,V), \\
[H(X,U) * H(Y,V)]_{n-1} &= \oplus_{i+j=n-1} H_i(X,U) * H_j(Y,V)
\end{aligned}$$

where $G \otimes H$ and $G * H$ stand for the tensor and torsion products of abelian groups G, H, respectively (see [Spa]). We need three elementary properties of tensor and torsion products:

- $G \otimes \mathbb{Z}$ is isomorphic to G;
- $G \otimes H \neq 0$ if both G and H contain elements of infinite order; and
- $G * H$ contains an element of finite order n if and only if both G and H contain such an element (see Exercise 6 on [Hat, p. 267]).

We shall apply Künneth Formula to prove:

Lemma 1. *If x is a homological point of a space X, then for any point y of a space Y the pair (x, y) is a homological Z_∞-point in $X \times Y$.*

Proof. We need to check that the groups $H_k(X \times Y, X \times Y \setminus \{(x, y)\})$ are trivial for all k. This trivially follows from the Relative Künneth Formula and the triviality of the homology groups $H_i(X, X \setminus \{x\})$. $\square$

Our next corollary of the Künneth formula is less trivial.

Proposition 7. *A closed subset A of a space X is a homological Z_∞-set in X provided A^k is a homological Z_∞-set in X^k for some finite number k.*

Proof. First we show that the groups $H_n(U, U \setminus A)$ are torsion groups for all $n \in \omega$ and all open sets $U \subset X$. Otherwise, for some n we can find an element $\alpha \in H_n(U, U \setminus A)$ of infinite order. Then $\alpha \otimes \alpha$ is a non-zero element of infinite order in $H_n(U, U \setminus A) \otimes H_n(U, U \setminus A)$. Now the Künneth formula implies that the homology group $H_{2n}(U^2, U^2 \setminus A^2)$ has non-zero element of infinite order. Proceeding by induction we can show that for each $i \in \mathbb{N}$ the homology group $H_{in}(U^i, U^i \setminus A^i)$ contains a non-zero element of infinite order which is not possible as A^k is a homological Z_∞-set in X^k.

This proves that all the homology groups $H_n(U, U \setminus A)$ are torsion groups. Assuming that A is not a homological Z_∞-point, we can find $n \in \omega$ and an open set $U \subset X$ such that $H_n(U, U \setminus A)$ is not trivial and thus contains an element of a prime order p. Then the torsion product $H_n(U, U \setminus A) * H_n(U, U \setminus A)$ also contains an element of order p. The exact sequence

$$0 \to [H(U, U \setminus A) \otimes H(U, U \setminus A)]_{2n+1} \to H_{2n+1}(U^2, U^2 \setminus A^2) \to$$
$$\to [H(U, U \setminus A) * H(U, U \setminus A)]_{2n} \to 0$$

from the Künneth formula implies that the group $H_{2n+1}(U^2, U^2 \setminus A^2)$ contains an element of order p (here we aso use the fact that the tensor product $[H(U, U \setminus A) \otimes H(U, U \setminus A)]_{2n+1}$ is a torsion group). Repeating this argument again, we can prove that the group $H_{3n+2}(U^3, U^3 \setminus A^3)$ contains an element of order p. Proceeding by induction we can prove that for any $i \in \mathbb{N}$ the group $H_{in+i-1}(U^i, U^i \setminus A^i)$ contains a non-zero element of order p which is not possible as A^k is a homological Z_∞-set in X^k. □

Combining Theorem 10 with Proposition 7 we obtain the k-th Root Theorem for Q-manifolds.

Theorem 13. *A space X with* DDP *and* cd-AP *is a Q-manifold if and only if the power X^k is a Q-manifold for some finite k.*

Since each compact contractible Q-manifolds is homeomorphic to Q, this theorem implies the k-Root Theorem 4 for the Hilbert cube. For the same reason Theorem 5 follows from

Theorem 14. *If some finite power a space X with* cd-AP *is a Q-manifold, then both X^2 and $X \times I$ are Q-manifolds.*

Proof. Assuming that X^k is a Q-manifold for some finite k, we conclude that X is a locally compact ANR and each point of X is a homological Z_∞-point, see Proposition 7. This property combined with cd-AP of X implies HZ-AP by Proposition 3. In its turn, HZ-AP of X implies DDAP for X by Proposition 6 while DDAP of X implies DDP for X^2 and $X \times I$ according to Proposition 5. By Lemma 1, all points in the spaces X^2 and $X \times I$ are homological Z_∞-points. Therefore X^2 and $X \times I$ are locally compact ANR-spaces possessing DDP, cd-AP, and having all points as homological Z_∞-points. By Theorem 10, these spaces are Q-manifolds. □

Since each compact contractible Q-manifold is homeomorphic to Q, Collective Division Theorem 7 for the Hilbert cube follows from

Proposition 8. *An uncountable family $\mathcal{X}$ of locally compact ANR-spaces contains a Q-manifold provided*

(1) *each space X has* DDP *and* cd-AP *and*
(2) *the product $X \times Y$ of any different spaces $X, Y \in \mathcal{X}$ is a Q-manifold.*

Proof. Suppose to the contrary that none of the spaces $X \in \mathcal{X}$ is a Q-manifold and apply the Characterizing Theorem 10 to find a point $a_X \in X$ which fails to be a homological Z_∞-point in X. This means that the homology group $H_k(X, X \setminus \{a_X\})$ is not trivial for some $k = k(X)$. Since the family $\mathcal{X}$ is uncountable there are two different spaces $X, Y \in \mathcal{X}$ and two numbers k, n such that the groups $H_k(X, X \setminus \{a_X\})$ and $H_n(Y, Y \setminus \{a_Y\})$ contain elements of the same order p, where either $p = \infty$ or p is a prime number. If $p = \infty$, then the tensor product $H_k(X, X \setminus \{a_X\}) \otimes H_n(Y, Y \setminus \{a_Y\})$ is not trivial and hence the group $H_{k+n}(X \times Y, X \times Y \setminus \{(a_X, a_Y)\})$ is not trivial by the Künneth Formula, which is impossible since (a_X, a_Y) is a (homological) Z_∞-point in the Q-manifold $X \times Y$.

If p is a prime number, then the torsion product $H_k(X, X \setminus \{a_X\}) * H_n(Y, Y \setminus \{a_Y\})$ is not trivial and hence the group $H_{k+n+1}(X \times Y, X \times Y \setminus \{(a_X, a_Y)\})$ is not trivial by the Künneth Formula, which is impossible since (a_X, a_Y) is a (homological) Z_∞-point in the Q-manifold $X \times Y$.

The obtained contradiction shows that some space $X \in \mathcal{X}$ must be a Q-manifold. □

7. Examples of fake Hilbert cubes

In this section we survey some known examples of fake Hilbert cubes. The first example is due to Singh [Singh].

Example 1 (Singh)**.** There exists a compact space S possessing the following properties:

(1) S is a compact AR;
(2) S is the image of Q under a cell-like map $\pi : Q \to S$ such that the set $N_\pi = \{x \in X : |\pi^{-1}(x)| > 1\}$ is countable and the preimage $\pi^{-1}(y)$ of every point $y \in N_\pi$ is an arc;
(3) Each compact ANR-subspace of dimension ≥ 2 in S coincides with S;
(4) Each point of S is a homological Z_∞-point and each point $x \in S \setminus N_\pi$ is a Z_∞-point;
(5) S has fd-AP and consequently has HZ-AP and DDAP;
(6) S fails to have DDP and hence is not homeomorphic to Q; and
(7) S^2 and $S \times I$ are homeomorphic to Q.

The items (1)–(4) were established by Singh in [Singh] while (5)–(7) follow from the preceding items and Propositions 4–5.

Another example of a fake Hilbert cube was constructed by Daverman and Walsh in [DW, 9.3].

Example 2 (Daverman-Walsh)**.** There exists a compact space X possessing the following properties:

(1) X is a compact AR;
(2) X is the image of Q under a cell-like map $\pi : Q \to X$ whose non-degeneracy set N_π is countable;
(3) Each point of X is a Z_∞-point and thus a homological Z_∞-point;
(4) X has fd-AP and consequently has HZ-AP and DDAP;
(5) X fails to have DDP and hence is not homeomorphic to Q; and
(6) X^2 and $X \times I$ are homeomorphic to Q.

Our last example yields a bit more than is required in Theorem 6. We shall construct a series of compact absolute retracts (Λ_p) parameterized by prime numbers p such that no finite power of Λ_p is homeomorphic to Q while all the products $\Lambda_p \times \Lambda_q$ for distinct $p \neq q$ are homeomorphic to the Hilbert cube.

Below

$$\mathbb{Z}_{p^\infty} = \{z \in \mathbb{C} : z^{p^k} = 1 \text{ for some } k \in \omega\}$$

denotes the quasicyclic p-group.

Example 3. *There is a family of pointed compact absolute retracts $(\Lambda_p, *_p)$ indexed by prime numbers p such that*

(1) *$\Lambda_p \setminus *_p$ is a Q-manifold with the unique non-trivial homology group $H_1(\Lambda_p \setminus *_p) = \mathbb{Z}_{p^\infty}$;*
(2) *the point $*_p$ is not a homological Z_∞-point in Λ_p;*
(3) *no finite power Λ_p^k is a Q-manifold;*
(4) *the space Λ_p has the property (Δ) and consequently has cd-AP;*
(5) *the space Λ_p has DDAP and the square Λ_p^2 has DDP; and*
(6) *$\Lambda_p \times \Lambda_q$ is homeomorphic to the Hilbert cube for any prime numbers $p \neq q$.*

Proof. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ stand for the unit circle in the complex plane. Given a prime number p consider the space $X_p = \omega \times [0,1] \times \mathbb{T}$ and its quotient space $Y_p = X_p/_\sim$ by the equivalence relation $\sim$ defined as follows: $(n,t,x) \sim (m,\tau,y)$ if and only if one of the following conditions holds:

- $(n,t,x) = (m,\tau,y)$;
- $m = n$, $t = \tau = 1$ and $x^p = y^p$;
- $m = n+1$, $t = 1$, $\tau = 0$, and $x^p = y$; or
- $n = m+1$, $t = 0$, $\tau = 1$, and $y^p = x$.

Thus the space Y_p consists of an infinite sequence of cylinders of the map $z^p : \mathbb{T} \to \mathbb{T}$, glued together. It is routine to check that the higher homology groups $H_k(Y_p)$, $k > 1$, of the space Y_p are trivial while $H_1(Y_p) = \mathbb{Z}_{p^\infty}$. It is easy to see that the one-point compactification $\widetilde{Y}_p = Y_p \cup \{\infty\}$ is a two-dimensional absolute retract. Then the quotient space $\Lambda_p = \widetilde{Y}_p \times Q/\{\infty\} \times Q$ is an absolute retract, too. The point $*_p = \{\{\infty\} \times Q\}$ is the distinguished point of Λ_p.

We now check that the pointed spaces $(\Lambda_p, *_p)$ satisfy the conditions (1)–(6):

(1) By Edwards' ANR-Theorem [Chap, 44.1], the complement $\Lambda_p \setminus \{*_p\} = Y_p \times Q$ is a Q-manifold. Being homotopy equivalent to Y_p, it has a unique non-trivial homology group $H_1(\Lambda_p \setminus \{*_p\}) = H_1(Y_p) = \mathbb{Z}_{p^\infty}$.

(2) The exact sequence of the pair $(\Lambda_p, \Lambda_p \setminus \{*_p\})$

$$0 = H_2(\Lambda_p) \to H_2(\Lambda_p, \Lambda_p \setminus \{*_p\}) \to H_1(\Lambda_p \setminus \{*_p\}) \to H_1(\Lambda_p) = 0$$

implies that $H_2(\Lambda_p, \Lambda_p \setminus \{*_p\}) = \mathbb{Z}_{p^\infty} \neq 0$ and thus $*_p$ fails to be a homological Z_∞-point in Λ_p.

(3) By Proposition 7, the singleton $\{*_p\}^k$ fails to be a homological Z_∞-set in Λ_p^k for any finite k, and hence Λ_p^k cannot be a Q-manifold.

(4) The first item implies that Λ_p has the property (Δ) at each point $x \neq *_p$. To check that property at the point $*_p$, take any neighborhood $U \subset \Lambda_p$ of $*_p$ and find a neighborhood $V \subset U$ of $*_p$ that is contractible in U (such a neighborhood exists because Λ_p is an AR). Given any compact set $K \subset V$ let $h : K \times [0,1] \to U$ be a map contracting K to $*_p$ (in the sense that $h(x,0) = x$ and $h(x,1) = *_p$ for

all $x \in K$. Consider the closed set $F = h^{-1}(*_p) \subset K \times [0,1]$ and the restriction $g = h|K \times I \setminus F$ of h, mapping the locally compact space $K \times I \setminus F$ into the Q-manifold $U \setminus \{*_p\}$. Applying Theorem 18.2 of [Chap], approximate g by an embedding $g' : K \times I \setminus F \to U \setminus \{*_p\}$ so near the map g that the map $\tilde{h} : K \times I \to U$ defined by $\tilde{h}|K \times I \setminus F = g'$ and $\tilde{h}|F = h|F$ is continuous. Then $\tilde{h} : K \times I \to U$ is a contraction of K in U such that $\dim(\tilde{h}(K \times I)) \le \dim(K \times I) \le \dim(K) + 1$, which means that Λ_p has the property (Δ). By Theorem VII.2.1 of [Bor] this space has fd-AP and hence cd-AP.

(5) To prove the DDAP of Λ_p, fix an open cover $\mathcal{U}$ of Λ_p and two maps $f : I^2 \to \Lambda_p$ and $g : I \to \Lambda_p$. Repeating the argument of the preceding item, we can approximate f by a map $f' : I^2 \to \Lambda_p$ such that $f'(I^2) \setminus \{*_p\}$ is a Z_∞-set in the Q-manifold $\Lambda_p \setminus \{*_p\}$. Because the point $*_p$ is not locally separating in Λ_p the map g can be approximated by a map $g' : I \to \Lambda_p \setminus \{*_p\}$. Moreover, since $f'(I^2) \setminus \{*_p\}$ is a Z_∞-set in $\Lambda_p \setminus \{*_p\}$, we can additionally assume that $g'(I) \cap f'(I^2) = \emptyset$, which means that Λ_p has DDAP. By Proposition 5, the square Λ_p^2 has DDP.

(6) Finally, we shall prove that for distinct prime numbers $p \neq q$ the product $\Lambda_p \times \Lambda_q$ is homeomorphic to the Hilbert cube Q. Being the product of two spaces with cd-AP, this space has cd-AP. By Proposition 5 this product has DDP. According to Theorem 10 it remains to check that each point (x, y) of $\Lambda_p \times \Lambda_q$ is a homological Z_∞-point. This is trivial if $(x, y) \neq (*_p, *_q)$. In case $(x, y) = (*_p, *_q)$ we may use item (1), the equality $\mathbb{Z}_{p^\infty} \otimes \mathbb{Z}_{q^\infty} = 0 = \mathbb{Z}_{p^\infty} * \mathbb{Z}_{q^\infty}$, and the Künneth Formula, to show that $(*_p, *_q)$ is a homological Z_∞-point in $\Lambda_p \times \Lambda_q$. □

8. Some Open Problems

Problem 1. *Can cd-AP in characterization Theorem 10 be replaced by wid-AP?*

Problem 1 is related to another

Problem 2. *Is each closed weakly-infinite dimensional subset A of Q a homological Z_∞-set in Q?*

Note that the k-Root and Division Theorems hold also for some non-locally compact spaces, for example for the Baire space $\mathbb{N}^\omega$.

Theorem 15 (Division Theorem for the Baire space)**.** *If the product $X \times Y$ of two spaces is homeomorphic to $\mathbb{N}^\omega$, then X or Y is homeomorphic to $\mathbb{N}^\omega$.*

This theorems easily follows from a topological characterization of the Baire space $\mathbb{N}^\omega$ due to Aleksandrov and Urysohn (see [Ke, 7.7]): *A topological space X is homeomorphic to $\mathbb{N}^\omega$ if and only if X is a Polish zero-dimensional nowhere locally compact space.*

In light of this result it is natural to ask if the k-th Root and Division Theorems are true for the countable product $s = (0,1)^\omega$ of open intervals. As expected, the answer is negative.

Example 4. Take an arc $J \subset Q$ which is not a Z_∞-set in Q and consider the quotient map $\pi : Q \to Q/J$. Then $X = \pi(s)$ is not homomorphic to s but its square X^2 is homeomorphic to s. This can be proved by the argument of [Bow].

Nonetheless it may happen that the k-th Root and Division Theorems for s hold in some restricted form.

Problem 3. *Find conditions on a space X guaranteeing that X is homeomorphic to s if some finite power X^k is homeomorphic to s.*

Observe that the finite power is an example of a normal functor on the category of compact Hausdorff spaces, see [TZ]. Can the k-Root Theorem for the Hilbert cube be extended to some functor distinct for the functor of finite power?

Problem 4. *Let $F : \mathcal{C}omp \to \mathcal{C}omp$ be a functor such that a compact space X with* DDP *and* cd-AP *is homeomorphic to Q if $F(X)$ is homeomorphic to Q. Is F isomorphic to a power functor?*

Even for the functor $F = SP^2$ of symmetric square the answer is unknown. Let us recall that the symmetric square of a compact space X is the quotient space $X^2/_\sim$ by the equivalence relation $(x, y) \sim (y, x)$.

Problem 5. *Is a compact AR-space X with* DDP *and* cd-AP *homeomorphic to Q if its symmetric square $SP^2(X)$ is homeomorphic to Q.*

In light of this problem let us mention that the quotient space $X = Q\times[-1,1]/Q\times\{0\}$ is an AR-space with the property (Δ) whose symmetric square $SP^2(X)$ is homeomorphic to Q, see [IMN]. However the space X contains a separating point and hence fails to have DDP and be homeomorphic to Q.

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