

# CONTROLLED HAHN-MAZURKIEWICZ THEOREM AND SOME NEW DIMENSION FUNCTIONS OF PEANO CONTINUA

T.BANAKH AND M.TUNCALI

ABSTRACT. Given a metric Peano continuum  $X$  we introduce and study the Hölder Dimension  $\text{Hö-Dim}(X) = \inf\{d : \text{there is a } \frac{1}{d}\text{-Hölder onto map } f : [0, 1] \rightarrow X\}$  of  $X$  as well as its topological counterpart  $\text{Hö-dim}(X) = \inf\{\text{Hö-Dim}(X, d) : d \text{ is an admissible metric for } X\}$ . We show that for each convex metric continuum  $X$  the dimension  $\text{Hö-Dim}(X)$  equals the fractal dimension of  $X$ . The topological Hölder dimension  $\text{Hö-dim}(M^n)$  of the  $n$ -dimensional universal Menger cube  $M^n$  equals  $n$ . On the other hand, there are 1-dimensional rim-finite Peano continua  $X$  with arbitrary prescribed  $\text{Hö-dim}(X) \geq 1$ .

In this paper we are interested in controlled versions of two important topological results:

**Theorem 1** (Alexandroff-Urysohn, 1929). *Each metric compact space is a continuous image of the Cantor set.*

**Theorem 2** (Hahn-Mazurkiewicz, 1914). *Each metric compact connected locally connected space  $X$  is a continuous image of the arc  $[0, 1]$ .*

In fact, the latter theorem can be derived from the former: given a locally connected metric continuum  $X$  first find a continuous surjective map  $f : C \rightarrow X$  from the Cantor set and then use the (local) path-connectedness to extend the map  $f$  onto the whole interval  $[0, 1] \supset C$ .

The controlled versions of the Alexandroff-Urysohn and Hahn-Mazurkiewicz Theorems we are interested in treat possible continuity moduli of surjective maps from a Cantor set or  $[0, 1]$  onto a given metric space.

By the continuity modulus  $\omega_f$  of a function  $f : X \rightarrow Y$  between metric spaces  $(X, d)$ ,  $(Y, \rho)$  we understand the function  $\omega_f : [0, +\infty) \rightarrow [0, \infty]$  assigning to each non-negative real number  $t \geq 0$  the (finite or infinite) number

$$\omega_f(t) = \sup\{\rho(f(x), f(x')) : d(x, x') \leq t\}.$$

It is clear that  $\omega_f$  is a non-decreasing function with  $\omega_f(0) = 0$ . It is continuous at zero if and only if the function  $f$  is uniformly continuous (the latter happens if  $f$  is continuous and  $X$  is compact). If  $X = [a, b]$  is a closed interval, then  $\omega_f$  is bounded, continuous and *subadditive* in the sense that  $\omega_f(a + b) \leq \omega_f(a) + \omega_f(b)$  for all  $a, b \in [0, +\infty)$ .

Having in mind these properties of continuity moduli, by a *continuity modulus* we shall understand any continuous non-decreasing subadditive function  $\omega : [0, +\infty) \rightarrow [0, \infty)$  such that  $\omega(0) = 0$ . It is easy to see that each continuity modulus  $\omega$  coincides with the continuity modulus of  $\omega$ .

In this paper we consider the following two general problems related to the Hahn-Mazurkiewicz and Alexandroff-Urysohn Theorems.

**Problem 1.** *Given a continuity modulus  $\omega$  describe metric compacta  $X$  which are images of an interval  $[a, b]$  under a surjective map  $f : [a, b] \rightarrow X$  with continuity modulus  $\omega_f \leq \omega$ .*

**Problem 2.** *Given a continuity modulus  $\omega$  describe metric compacta  $X$  which are images of a Cantor set  $C \subset \mathbb{R}$  under a surjective map  $f : C \rightarrow X$  with continuity modulus  $\omega_f \leq \omega$ .*

By a *Cantor set* we shall understand any zero-dimensional compact subset  $C \subset \mathbb{R}$  without isolated point. According to the classical Brouwer Theorem (1910) any Cantor set is homeomorphic to the standard Cantor set in  $[0, 1]$ .

## 1. HÖLDER DIMENSION OF PEANO CONTINUA AND SOME MOTIVATION

Functions  $f$  whose continuity modulus has polynomial decrease are well-known in analysis as Hölder functions. We recall that a function  $f : X \rightarrow Y$  between two metric spaces  $(X, d)$ ,  $(Y, \rho)$  is  $\alpha$ -Hölder for some constant  $\alpha$  if there is a positive  $C$  such that

$$\rho(f(x), f(x')) \leq C \cdot d(x, x')^\alpha \text{ for all } x, x' \in X.$$

Lipschitz functions are particular cases of  $\alpha$ -Hölder functions with Hölder constant  $\alpha = 1$ . Observe that a function  $f$  is  $\alpha$ -Hölder iff there is a constant  $C$  such that  $\omega_f(t) \leq C \cdot t^\alpha$  for all  $t \geq 0$ .

Now it is time for some history. The Hahn-Mazurkiewicz Theorem was a culminating point in investigations initiated by the famous example of G.Peano who constructed a continuous map  $f : [0, 1] \rightarrow [0, 1]^2$  of the interval onto the square. Afterward, compacta that are continuous images of the interval were referred to as *Peano continua*. The Peano map  $f : [0, 1] \rightarrow [0, 1]^2$ , being continuous, cannot be differentiable or Lipschitz. Ya.Mykytyuk, an analyst from Lviv has asked the first author if such a Peano map  $f : [0, 1] \rightarrow [0, 1]^2$  can be  $\alpha$ -Hölder for a suitable Hölder constant  $\alpha$ . Inspecting the classical construction of the Peano map  $f : [0, 1] \rightarrow [0, 1]^2$  the first author invented that indeed, it is  $\frac{1}{2}$ -Hölder and the constant  $\frac{1}{2}$  is the largest possible. Similarly, the  $n$ -dimensional cube  $[0, 1]^n$  is the image of the interval  $[0, 1]$  under a suitable  $\frac{1}{n}$ -Hölder function  $f : [0, 1] \rightarrow [0, 1]^n$  with Hölder constant  $\frac{1}{n}$ , which is the best possible.

These observation led the authors to introducing a new dimension function  $\text{Hö-Dim}(X)$  of a metric Peano continuum  $X$ , called the *Hölder dimension* of  $X$ :

$\text{Hö-Dim}(X) = \inf\{d : \text{there is a surjective } \frac{1}{d}\text{-Hölder map } f : [0, 1] \rightarrow X\}$   
if  $X$  is the image of  $[0, 1]$  under a Hölder map and  $\text{Hö-Dim}(X) = \infty$  otherwise.

Equivalently,  $\text{Hö-Dim}(X)$  can be defined as the smallest number  $d$  such that for any  $\varepsilon > 0$  the space  $X$  is the image of  $[0, 1]$  under a  $\frac{1}{d+\varepsilon}$ -Hölder map.

**Problem 3.** *Investigate Hölder dimension  $\text{Hö-Dim}$  and its interplay with other dimension functions of metric Peano continua.*

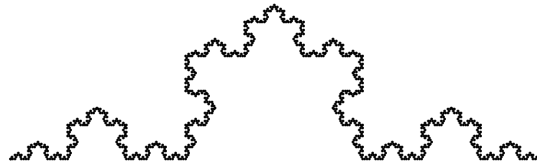
**Problem 4** (of attainability). *Characterize metric Peano continua  $X$  which are images of the interval  $[0, 1]$  under an  $\alpha$ -Hölder map with  $\frac{1}{\alpha} = \text{Hö-Dim}(X)$ .*

The answer to the latter question is known for  $\alpha = 1$ : a metric continuum  $X$  is an image of  $[0, 1]$  under a Lipschitz map iff  $X$  has finite length, see [Fr]. Problem 4 is not trivial as there are metric Peano continua  $X$  with  $\text{Hö-Dim}(X) = 1$  of infinite length, see Remark 1.

Like many metric dimensions, the Hölder dimension is not raised by Lipschitz maps.

**Proposition 1.** *If  $f : X \rightarrow Y$  is a surjective  $\frac{1}{\alpha}$ -Hölder map between metric Peano continua, then  $\text{Hö-Dim}(Y) \leq \alpha \cdot \text{Hö-Dim}(X)$ .*

This simple proposition follows immediately from the definitions but it shows that the Hölder dimension  $\text{Hö-Dim}$  is invariant under bi-Lipschitz homeomorphisms (i.e., homeomorphisms  $h$  such that both  $h$  and its inverse  $h^{-1}$  are Lipschitz). However Hölder dimension is not preserved by arbitrary homeomorphisms (for example, the Koch curve  $K$



is homeomorphic to  $[0, 1]$  but has Hölder Dimension  $\text{Hö-Dim}(K) = \log_3 4 > 1$ ).

On the other hand, the metric invariant  $\text{Hö-Dim}$  has its topological version

$$\text{Hö-dim}(X) = \inf\{\text{Hö-Dim}(X, d) : d \text{ is an admissible metric on } X\}$$

defined for any Peano continuum  $X$  and invariant under homeomorphisms. As we shall see later,  $\text{Hö-Dim}([0, 1]^n) = \text{Hö-dim}([0, 1]^n) = n$  for any  $n \geq 1$ , so  $\text{Hö-dim}$  has the basic feature of the usual topological dimension  $\text{dim}$ . One can guess that  $\text{Hö-dim}X$  equals  $\text{dim} X$ . However this is not so:  $\text{Hö-dim}$

is a new topological invariant which (unlike classical topological dimension functions) can take any real value  $\geq 1$ , see Corollary 5.

We start studying Hö-dim with its metric origin, the Hölder dimension Hö-Dim. For this we shall need a controlled version of Alexandroff-Urysohn Theorem on continuous images of Cantor sets, which could have an independent value.

## 2. FRACTAL DIMENSION AND CONTROLLED VERSION OF ALEXANDROFF-URYSOHN THEOREM

By analogy with the Hölder dimension of metric Peano continua we could introduce a metric dimension function classifying images of Cantors sets under Hölder maps. However such a dimension function coincides with the fractal dimension, well-known in the Theory of Fractals, see [Ba].

The fractal dimension  $\text{Dim}(X)$  of a totally bounded metric space  $X$  is defined with help of the function  $N_X(\varepsilon)$  assigning to each positive real number  $\varepsilon > 0$  the smallest size of a cover of  $X$  by subsets of diameter  $< \varepsilon$ .

We shall write  $N_X(\varepsilon) = O(\varepsilon^{-d})$  for a positive real number  $d$  if there is a constant  $C$  such that  $N_X(\varepsilon) \leq C \cdot \varepsilon^{-d}$  for all  $\varepsilon > 0$ . If, in addition, there is another constant  $c > 0$  such that  $N_X(\varepsilon) \geq c \cdot \varepsilon^{-d}$ , then we write  $N_X(\varepsilon) = O^*(\varepsilon^{-d})$ .

We put

$$\text{Dim}(X) = \inf \{d : N_X(\varepsilon) = O(\varepsilon^{-d})\}$$

if  $N_X(\varepsilon) = O(\varepsilon^{-d})$  for some real  $d$  and  $\text{Dim}(X) = \infty$  otherwise.

For an  $n$ -dimensional cube  $[0, 1]^n$  we get  $N_X(\varepsilon) = O^*(\varepsilon^{-n})$  and thus  $\text{Dim}([0, 1]^n) = n$  (however, the fractal dimension has some counterintuitive properties; for example, the fractal dimension of the convergent sequence  $\{0, \frac{1}{n} : n \in \mathbb{N}\}$  is strictly positive, see [Ed, 6.5.4]).

A map  $f : X \rightarrow Y$  between two topological spaces is called *perfect* if it is continuous and  $f^{-1}(K)$  is compact for each compact subset  $K \subset Y$ .

**Theorem 3.** *For a real number  $d > 0$  and a totally bounded metric space  $X$  the following conditions are equivalent:*

- (1) *the fractal dimension  $\text{Dim}(X) < d$ ;*
- (2)  *$X$  is the image of some subset  $C \subset [0, 1]$  under a  $\frac{1}{\alpha}$ -Hölder map  $f : C \rightarrow X$  with  $\alpha < d$ ;*
- (3)  *$X$  is the image of a Lebesgue null subset  $C \subset [0, 1]$  under a perfect  $\frac{1}{\alpha}$ -Hölder map  $f : C \rightarrow X$  with  $\alpha < d$ .*

We shall derive this theorem from the following two propositions related to the general Problem 2.

**Proposition 2.** *Let  $\omega$  be an increasing continuity modulus and  $X$  be a totally bounded metric space. If  $X$  is the image of some subset  $C$  of an*

interval  $[a, b]$  under a map  $f : C \rightarrow X$  with  $\omega_f \leq \omega$ , then  $N_X(\varepsilon) \leq 1 + \frac{b-a}{\omega^{-1}(\varepsilon)}$  for all  $\varepsilon > 0$ .

*Proof.* Assume that  $X$  is the image of a subset  $C \subset [a, b]$  under a map  $f : C \rightarrow X$  with  $\omega_f \leq \omega$ . Given  $\varepsilon > 0$  consider the smallest integer  $m$  greater than  $\frac{b-a}{\omega^{-1}(\varepsilon)}$  and note that  $m \leq 1 + \frac{b-a}{\omega^{-1}(\varepsilon)}$ . By the continuity of  $\omega^{-1}(\cdot)$ , there is  $\delta > 0$  such that  $\frac{b-a}{\omega^{-1}(\varepsilon-\delta)} \leq m$ . Then  $\omega^{-1}(\varepsilon-\delta) \geq \frac{b-a}{m}$  and thus  $\omega(\frac{b-a}{m}) \leq \varepsilon - \delta$ . Write  $[a, b] = \bigcup_{k=1}^m I_k$  as the union of closed intervals of length  $\frac{b-a}{m}$ . Then  $\text{diam } f(I_k \cap C) \leq \omega(\text{diam } I_k) = \omega(\frac{b-a}{m}) \leq \varepsilon - \delta < \varepsilon$  and thus  $X$  can be covered by  $m$  subsets  $f(C \cap I_k)$ ,  $k \leq m$ , of diameter  $< \varepsilon$ , which proves the inequality  $N_X(\varepsilon) \leq m \leq 1 + \frac{b-a}{\omega^{-1}(\varepsilon)}$ .  $\square$

**Proposition 3.** *Let  $\omega$  be an increasing continuity modulus and  $(X, \rho)$  be a compact metric space of diameter  $\leq 1$ . If  $s = \sum_{n=1}^{\infty} N_X(2^{-n})\omega^{-1}(2^{-n+3}) < \infty$ , then  $X$  is the image of some closed Lebesgue null subset  $C \subset [0, s]$  under a continuous map  $f : C \rightarrow X$  with  $\omega_f \leq \omega$ .*

*Proof.* We shall say that a subset  $S$  of the metric compactum  $(X, \rho)$  is  $\varepsilon$ -separated if  $\rho(x, y) > \varepsilon$  for any distinct  $x, y \in S$ . By induction we can construct an increasing sequence

$$S_0 \subset S_1 \subset S_2 \subset \dots$$

of finite subsets such that for every  $n \geq 0$  the set  $S_n$  is maximal  $2^{-n}$ -separated in  $X$ . Note that  $S_0$ , being 1-separated in the metric space  $X$  of diameter  $\leq 1$ , consists of a unique point. It will be convenient to put  $S_n = \emptyset$  for  $n \leq -1$ . Let also  $S = \bigcup_{n \geq 0} S_n$ .

Note that  $|S_n| \leq N(2^{-n})$ . Indeed, take any cover  $\mathcal{C}$  of  $X$  by subsets of diameter  $< 2^{-n}$  with  $|\mathcal{C}| = N(2^{-n})$  and observe that distinct points of  $S_n$  belong to distinct elements of the cover  $\mathcal{C}$ .

The maximality of  $S_n$  implies that for each  $x \in X$  there is  $y \in S_n$  with  $\rho(x, y) \leq 2^{-n}$ . This allows us for every  $n \geq 0$  choose a retraction  $r_{n+1} : S_{n+1} \rightarrow S_n$  such that  $\rho(x, r_{n+1}(x)) \leq 2^{-n}$  for all  $x \in S_{n+1}$ .

For every  $n \geq 0$  define the retraction  $R_n : S \rightarrow S_n$  letting  $R_n(x) = x$  for  $x \in S_n$  and  $R_n(x) = r_{n+1} \circ \dots \circ r_m(x)$  for  $x \in S_m \setminus S_{m-1}$ ,  $m > n$ . In the latter case we get  $\rho(x, R_n(x)) \leq \sum_{k=n+1}^m 2^{-k+1} < 2^{-n+1}$ .

Endow each set  $S_n$  with a linear order  $\leq_n$  such that  $\leq_n$  extends  $\leq_{n-1}$  and for any  $x <_{n-1} y$  in  $S_{n-1}$  and any  $z \in S_n \setminus S_{n-1}$  with  $r_n(z) = x$  we get  $x <_n z <_n y$ . These (compatible) orders  $\leq_n$  on  $S_n$ 's induce a linear order  $\leq$  on the union  $S = \bigcup_{n \geq 0} S_n$ .

Assign to each point  $x \in S_n \setminus S_{n-1}$ ,  $n \geq 0$ , the weight  $w(x) = \omega^{-1}(2^{-n+3})$ . Now consider the map  $g : S \rightarrow \mathbb{R}$ ,  $g : x \mapsto \sum_{y \leq x} w(y)$  and observe that

$g(S) \subset [a, b]$ , where  $a = \omega^{-1}(2^3)$  and

$$\begin{aligned} b &= \sum_{n \geq 1} \sum_{x \in S_n \setminus S_{n-1}} w(x) = \sum_{n \geq 0} |S_n \setminus S_{n-1}| \cdot \omega^{-1}(2^{-n+3}) \leq \\ &\leq \sum_{n \geq 1} N_X(2^{-n}) \omega^{-1}(2^{-n+3}) = s. \end{aligned}$$

The map  $g$  is increasing with respect to the linear order  $\leq$  on  $S$ . So we can consider the inverse map  $f = g^{-1} : g(S) \rightarrow S$ . We claim that  $f$  is uniformly continuous map with  $\omega_f \leq \omega$ . Given two distinct points  $x, y \in S$  we should prove that  $\rho(x, y) = \rho(f(g(x)), f(g(y))) \leq \omega(|g(x) - g(y)|)$ , which is equivalent to  $|g(x) - g(y)| \geq \omega^{-1}(\rho(x, y))$ .

Without loss of generality we can assume that  $x < y$ . Let  $k \geq 1$  be the largest number such that  $R_k(x) = R_k(y)$ . Then

$$\rho(x, y) \leq \rho(x, R_k(x)) + \rho(R_k(y), y) \leq 2^{-k+1} + 2^{-k+1} = 2^{-k+2}.$$

On the other hand, the maximality of  $k$  means that  $x' = R_{k+1}(x)$  is not equal to  $y' = R_{k+1}(y) \in S_{k+1}$ . Then  $x' \leq x < y' \leq y$  and  $|g(y) - g(x)| = \sum_{x < z \leq y} w(z) \geq w(y') \geq \omega^{-1}(2^{-(k+1)+3}) = \omega^{-1}(2^{-k+2}) \geq \omega^{-1}(\rho(x, y))$ .

Being uniformly continuous, the map  $f : g(S) \rightarrow S$  can be extended to a uniformly continuous map  $f : \overline{g(S)} \rightarrow X$  defined on the closure  $C = \overline{g(S)}$  of  $g(S)$  in  $[a, b]$ . Because of the density of  $S$  in  $X$  the extended map  $f$  we will surjective and will have the same continuity modulus  $\omega_f \leq \omega$ . It remains to note that  $C$  is Lebesgue null set in  $[a, b]$  since its complement is a countable union of open intervals of total length  $b - a$ .  $\square$

*Proof of Theorem 3.* Let  $(X, \rho)$  be a totally bounded metric space and  $d > 0$  be a positive real number. We shall prove the implications (3)  $\Rightarrow$  (2)  $\Rightarrow$  (1)  $\Rightarrow$  (3) from Theorem 3. In fact, the first implication is trivial.

To prove that (2)  $\Rightarrow$  (1) assume that  $X$  is the image of some subset  $C \subset [b, a]$  under a  $\frac{1}{\alpha}$ -Hölder map  $f : C \rightarrow X$  with  $\alpha < d$ . Then  $\omega_f(t) \leq Ct^{1/\alpha}$  for some positive constant  $C$ . Applying Proposition 2 we get  $N_X(\varepsilon) \leq 1 + \frac{b-a}{\omega^{-1}(\varepsilon)}$ , where  $\omega(t) = Ct^{1/\alpha}$ . Since  $\omega^{-1}(\varepsilon) = (\varepsilon/C)^\alpha$ , we get  $N_X(\varepsilon) \leq 1 + (b-a)C^\alpha \varepsilon^{-\alpha}$  and thus  $N_X(\varepsilon) = O(\varepsilon^{-\alpha})$ , which means that  $\text{Dim}(X) \leq \alpha < d$ .

To prove (1)  $\Rightarrow$  (3) consider the completion  $(\bar{X}, \rho)$  of the metric space  $(X, \rho)$ , which can be assumed to have diameter  $\leq 1$  (if necessary, we can replace the metric  $\rho$  by  $\frac{\rho}{\text{diam } X}$ ). Let us show that the functions  $N_{\bar{X}}$  and  $N_X$  coincide. The inequality  $N_X \leq N_{\bar{X}}$  is trivial. To show that  $N_X \leq N_{\bar{X}}$ , given any  $\varepsilon > 0$ , pick an cover  $\mathcal{C}$  of  $X$  by subsets of diameter  $< \varepsilon$  with  $|\mathcal{C}| = N_X(\varepsilon)$ . Without loss of generality, we can assume that each set  $C \in \mathcal{C}$  is open. Then the closure  $\bar{C}$  of  $C$  in  $\bar{X}$  has diameter  $< \varepsilon$  and  $\bar{\mathcal{C}} = \{\bar{C} : C \in \mathcal{C}\}$  is a cover of  $\bar{X}$ . Consequently,  $N_{\bar{X}}(\varepsilon) \leq |\bar{\mathcal{C}}| \leq |\mathcal{C}| = N_X(\varepsilon)$ .

It follows from the equality  $N_{\bar{X}} = N_X$  that  $\text{Dim}(\bar{X}) = \text{Dim}(X)$ . Assuming that  $\text{Dim}(X) < d$ , we get  $\text{Dim}(\bar{X}) = \text{Dim}(X) < d$ . Take any  $\alpha, \beta$  with

$\text{Dim}(\bar{X}) < \alpha < \beta < d$ . It follows that  $N_{\bar{X}}(\varepsilon) \leq C\varepsilon^{-\alpha}$  for some constant  $C$  (not depending on  $\varepsilon$ ). Consider the continuity modulus  $\omega(t) = t^{1/\beta}$  and note that

$$\begin{aligned} s &= \sum_{n \geq 1} N_{\bar{X}}(2^{-n}) \omega^{-1}(2^{-n+3}) \leq \sum_{n \geq 1} C(2^{-n})^{-\alpha} (2^{-n+3})^\beta = \\ &= C 2^{3\beta} \sum_{n \geq 1} \frac{1}{2^{(\beta-\alpha)n}} < +\infty. \end{aligned}$$

Applying Proposition 3, we can find a continuous onto-map  $f : C \rightarrow X$  from a closed Lebesgue null subset  $C \subset [0, s]$  with  $\omega_f \leq \omega$ . Then  $f$  is  $\frac{1}{\beta}$ -Hölder. Restricting  $f$  to  $f^{-1}(X)$ , we get a perfect  $\frac{1}{\beta}$ -Hölder map from the Lebesgue null subset  $f^{-1}(X) \subset [0, s]$  onto  $X$ .

### 3. HÖLDER DIMENSION: SOME LOWER BOUNDS

With Theorem 3 in hands we return to studying the Hölder Dimension. We start with lower bounds for Hö-Dim. Let us note that the implication (2)  $\Rightarrow$  (1) of Theorem 3 yields us a lower bound  $\text{Dim} \leq \text{Hö-Dim}$  for the Hölder dimension. It can be improved by inserting in between of  $\text{Dim}$  and  $\text{Hö-Dim}$  another dimension function  $\text{S-Dim}$  of metric Peano continua, related to the property  $S$ .

We recall that a metric space  $X$  has *the property S* if for any  $\varepsilon > 0$  the space  $X$  can be covered by finitely many connected subsets of diameter  $< \varepsilon$ . The smallest cardinality of such a cover is denoted by  $S_X(\varepsilon)$ . Thus we obtain a function  $S_X$  called the  $S$ -function on  $X$ . Replacing the function  $N_X$  in the definition of the fractal dimension by the  $S$ -function  $S_X$  we will get the definition of  $S$ -dimension. Namely, we put

$$\text{S-Dim}(X) = \inf\{d : S_X(\varepsilon) = O(\varepsilon^{-d})\}$$

if such a  $d$  exists and  $\text{S-Dim}(X) = \infty$ , otherwise. So, the  $S$ -dimension  $\text{S-Dim}$  can be considered as a connected counterpart of the fractal dimension  $\text{Dim}$ .

**Theorem 4.**  $\text{Dim}(X) \leq \text{S-Dim}(X) \leq \text{Hö-Dim}(X)$  for any metric Peano continuum  $X$ .

The first inequality is trivial while the second one follows from the subsequent Proposition that can be proved by analogy with Proposition 2.

**Proposition 4.** *If a metric space  $X$  is the image of an interval  $[a, b]$  under a map  $f : [a, b] \rightarrow X$  with  $\omega_f \leq \omega$  for some increasing convexity modulus  $\omega$ , then  $S_X(\varepsilon) \leq 1 + \frac{b-a}{\omega^{-1}(\varepsilon)}$  for all  $\varepsilon > 0$ .*

## 4. HÖLDER DIMENSION: AN UPPER BOUND

As we saw in the preceding section,  $\text{Dim}(X) \leq \text{Hö-Dim}(X)$  for any metric Peano continuum  $X$ . It turns out that this inequality turns into equality if the metric of  $X$  is convex, or more generally,  $\text{Dim}(X)$ -convex.

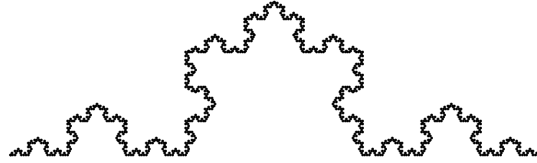
We recall that a metric space  $(X, d)$  is *convex* if for any points  $x, y \in X$  there is an isometric embedding  $\gamma : [0, d(x, y)] \rightarrow X$  with  $\gamma(0) = x$  and  $\gamma(d(x, y)) = y$ . By the famous Bing-Moise Convexification Theorem [Bi], [Mo] each Peano continuum has an equivalent convex metric.

Convex metrics are particular cases of  $d$ -convex metrics, where  $d > 1$  is a real number. We shall say that a metric  $\rho$  on  $X$  is  $d$ -convex if there is a positive constant  $C$  such that for any points  $x, y \in X$  there is a path  $\gamma : [0, 1] \rightarrow X$  such that

$$\gamma(0) = x, \gamma(1) = y \text{ and } \rho(\gamma(t), \gamma(t')) \leq C \cdot \rho(x, y) \cdot |t - t'|^{1/d}$$

for all  $t, t' \in [0, 1]$ .

By the *convexity degree*  $\text{deg}_{\text{conv}}(X)$  of a metric space  $X$  we understand the infimum of real numbers  $d$  such that the metric of  $X$  is  $d$ -convex. We put  $\text{deg}_{\text{conv}}(X) = \infty$  if no such a  $d$  exists. It is easy to see that  $\text{deg}_{\text{conv}}(X) \geq 1$  for any metric space  $X$  containing more than one point, and that each convex metric space  $X$  has  $\text{deg}_{\text{conv}}(X) = 1$ . On the other hand, it can be shown that the famous Koch curve  $K$



has  $\text{deg}_{\text{conv}}(K) = \log_3 4$ .

**Theorem 5.**  $\text{Hö-Dim}(X) \leq \max\{\text{Dim}(X), \text{deg}_{\text{conv}}(X)\}$  for any metric Peano continuum  $X$ .

*Proof.* Given any real number  $d > \max\{\text{Dim}(X), \text{deg}_{\text{conv}}(X)\}$  we should construct a  $\frac{1}{d}$ -Hölder surjective map  $f : [0, 1] \rightarrow X$ . Applying Theorem 3 we can find a surjective  $\frac{1}{d}$ -Hölder map  $g : S \rightarrow X$  defined on a closed Lebesgue null subset  $S$  of some interval  $[a, b]$  with  $a, b \in S$ . Find a constant  $C \geq 1$  such that  $\rho(g(x), g(y)) \leq C|x - y|^{1/d}$  for all  $x, y \in S$ . The constant  $C$  can be taken so large that any two points  $x, y \in X$  can be linked by a path  $\gamma : [0, 1] \rightarrow X$  with

$$\rho(\gamma(t), \gamma(t')) \leq C \cdot \rho(x, y) \cdot |t - t'|^{1/d}$$

for all  $t, t' \in [0, 1]$  (this follows from the inequality  $\text{deg}_{\text{conv}}(X) < d$ ).



Write the complement  $[a, b] \setminus S = \bigcup_{k \in \mathbb{N}} (a_k, b_k)$  as the disjoint union of open intervals. For every  $k \geq 1$  we can find a path  $\gamma_k : [0, 1] \rightarrow X$  such that  $\gamma_k(0) = g(a_k)$ ,  $\gamma_k(1) = g(b_k)$  and  $\rho(\gamma_k(t), \gamma_k(t')) \leq C \cdot \rho(g(a_k), g(b_k)) \cdot |t - t'|^{1/d}$  for all  $t, t' \in [0, 1]$ .

Extend the map  $g$  onto the whole interval  $[a, b]$  letting  $g(t) = \gamma_k\left(\frac{t - a_k}{b_k - a_k}\right)$  for  $t \in (a_k, b_k)$ ,  $k \geq 1$ . We claim that so-extended map  $g$  is still  $\frac{1}{d}$ -Hölder. More precisely,  $\rho(g(t), g(t')) \leq (C + 2C^2) \cdot |t - t'|^{1/d}$  for any points  $t < t'$  in  $[a, b]$ .

We consider 5 cases.

1) If  $t, t' \in S$ , then  $\rho(g(t), g(t')) \leq C|t - t'|^{1/d} \leq (C + 2C^2)|t - t'|^{1/d}$ .

2)  $t, t'$  belong to some interval  $[a_k, b_k]$ ,  $k \geq 1$ . In this case

$$\begin{aligned} \rho(g(t), g(t')) &\leq C \cdot \rho(g(a_k), g(b_k)) \cdot \left(\frac{|t - t'|}{b_k - a_k}\right)^{1/d} \leq \\ &\leq C \cdot (C|a_k - b_k|^{1/d}) \cdot \left(\frac{|t - t'|}{b_k - a_k}\right)^{1/d} = C^2|t - t'|^{1/d}. \end{aligned}$$

3)  $t \in S$ ,  $t' \notin S$ . Then  $t' \in (a_k, b_k)$  for some  $k$  with  $t \leq a_k < t'$  and thus  $\rho(g(t), g(t')) \leq \rho(g(t), g(a_k)) + \rho(g(a_k), g(t')) \leq C|t - a_k|^{1/d} + C^2|t' - a_k|^{1/d} \leq (C + C^2)|t - t'|^{1/d}$ .

4) By analogy, we can show that  $\rho(g(t), g(t')) \leq (C + C^2)|t - t'|^{1/d}$  if  $t \notin S$ ,  $t' \in S$ .

5) Finally, consider the case  $t, t' \notin S$  and find  $k, m$  with  $t \in (a_k, b_k)$  and  $t' \in (a_m, b_m)$ . Then  $t < b_k \leq a_m \leq t'$  and

$$\begin{aligned} \rho(g(t), g(t')) &\leq \rho(g(t), g(b_k)) + \rho(g(b_k), g(a_m)) + \rho(g(a_m), g(t')) \leq \\ &\leq C^2|t - b_k|^{1/d} + C|b_k - a_m|^{1/d} + C^2|a_m - t'|^{1/d} \leq \\ &\leq (2C^2 + C) \cdot |t - t'|^{1/d}. \end{aligned}$$

□

**Corollary 1.**  $\text{Dim}(X) = \text{S-Dim}(X) = \text{Hö-Dim}(X)$  for any metric Peano continuum  $X$  with  $\text{deg}_{\text{conv}}(X) \leq \text{Dim}(X)$ .

**Corollary 2.**  $\text{Dim}(X) = \text{S-Dim}(X) = \text{Hö-Dim}(X)$  for any convex metric continuum  $X$ .

**Corollary 3.**  $\text{Hö-Dim}([0, 1]^n) = \text{Dim}([0, 1]^n) = n$  for any  $n \in \mathbb{N}$ .

**Problem 5.** Is  $\text{S-Dim}(X) = \text{Hö-Dim}(X)$  for any metric Peano continuum?

The answer to this problem is affirmative for metric Peano continua  $X$  with  $\text{deg}_{\text{conv}}(X) \leq \text{S-Dim}(X)$ . One can ask if the latter inequality always hold. However it is not the case. This follows from an easy observation.

**Proposition 5.** *If  $(X, \rho)$  is a metric Peano continuum with  $\deg_{\text{conv}}(X) < \infty$ , then there is a constant  $C$  such that any two points  $x, y \in X$  can be linked by a path of diameter  $\leq C \cdot \rho(x, y)$ .*

Now it is easy to distort the standard metric  $d(x, y) = |x - y|$  on  $I = [0, 1]$  to get a metric  $\rho \leq d$  with  $\deg_{\text{conv}}(I, \rho) = \infty$ . Nonetheless  $\text{Dim}(I, \rho) = \text{S-Dim}(I, \rho) = \text{Hö-Dim}(I, \rho) = 1$  because  $\rho \leq d$ .

## 5. TOPOLOGICAL SETTING

All dimension functions considered in the preceding sections have metric nature and are not preserved by homeomorphisms. However, taking infimum of values of such a function over all admissible metrics on a given continuum, we will get a dimension function invariant under homeomorphisms.

Namely, for a Peano continuum  $X$  denote by  $\mathcal{M}(X)$  the set of all metrics generating the topology of  $X$  and let

$$\text{Hö-dim}(X) = \inf\{\text{Hö-Dim}(X, d) : d \in \mathcal{M}(X)\};$$

$$\text{S-dim}(X) = \inf\{\text{S-Dim}(X, d) : d \in \mathcal{M}(X)\};$$

$$\text{dim}(X) = \inf\{\text{Dim}(X) : d \in \mathcal{M}(X)\};$$

$$\text{conv-dim}(X) = \inf\{\text{Dim}(X) : d \text{ is an admissible convex metric on } X\};$$

It should be mentioned that the dimension  $\text{dim}(X)$ , being defined as the infimum of fractal dimensions, coincides with the usual topological covering dimension, see [PS]. That is why we used the symbol  $\text{dim}(X)$ .

Theorem 4 and Corollary 1 imply

**Corollary 4.**  $\text{dim}(X) \leq \text{S-dim}(X) \leq \text{Hö-dim}(X) \leq \text{conv-dim}(X)$  for any Peano continuum  $X$ .

In the next section we shall construct an example of a Peano continuum  $X$  with  $\text{dim}(X) = 1$  and  $\text{S-dim}(X) = \infty$ .

**Problem 6.** *Is  $\text{S-dim}(X) = \text{Hö-dim}(X) = \text{conv-dim}(X)$  for any Peano continuum  $X$ ?*

## 6. CONVEX DIMENSION OF MENGER CUBES

In this section we shall calculate the convex dimension  $\text{conv-dim}(M^n)$  of the universal  $n$ -dimensional Menger cube  $M^n$  for  $n \geq 1$ . The universality of  $M^n$  means that  $M^n$  contains a topological copy of each  $n$ -dimensional metrizable separable space, so in a sense,  $M^n$  is the largest continuum among  $n$ -dimensional compacta.

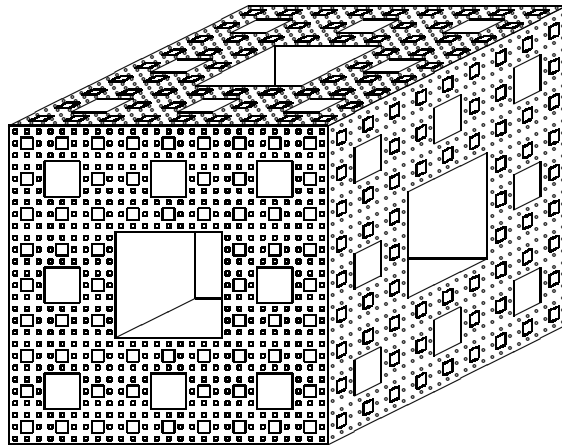
Our main result is

**Theorem 6.** *For every  $n \geq 1$*

$$\text{conv-dim}(M^n) = \text{Hö-dim}(M^n) = \text{S-dim}(M^n) = \text{dim}(M^n) = n.$$

First we prove the theorem for  $n = 1$ . In this case the Menger 1-dimensional cube  $M^1$  is often referred to as the *universal Menger curve*. Let us briefly remind its construction. We start with the unit cube  $[0, 1]^3$  in  $\mathbb{R}^3$  and divide each its edge into three equal subinterval thus dividing the cube into 27 subcubes with edge  $1/3$ . We delete from  $[0, 1]^3$  the 3-dimensional cross consisting of the central subcube and 6 subcubes touching its faces. The remaining 20 subcubes compose the first approximation  $K_1$  of the Menger curve. Then we repeat this procedure with each of these 20 subcubes: divide them into 27 subcubes and delete the 3-dimensional crosses.

As a result we obtain a compactum  $K_2$  with is the union of  $20^2$  cubes with edge  $\frac{1}{9}$ . Proceeding in this fashion we construct a sequence  $(K_n)$  of compact subsets of  $[0, 1]^3$  whose intersection  $\bigcap_{n \geq 1} K_n$  is the universal Menger curve:



The Menger curve can be viewed as a (self-similar) fractal consisting of 20 copies of itself of 3 times smaller size. This allows us to calculate easily the fractal dimension  $d = \text{Dim}(M^1)$  of  $M^1$ : assuming that  $V$  is the  $d$ -dimensional “volume” of  $M^1$ , we get  $V = 20\left(\frac{V}{3}\right)^d$  and thus  $d = \frac{\ln 20}{\ln 3} > 2$ , see [Ba] for justification of such a computation. Nonetheless homeomorphic copies of  $M^1$  can have fractal dimension as close to 1 as we wish. To find such copies, we modify a little the construction of the Menger curve.

Given a positive integer  $a \geq 3$  we shall divide the edges of the initial cube  $[0, 1]^3$  into  $a$  equal subintervals thus obtaining  $a^3$  subcubes. Then we withdraw a “fat” 3-dimensional cross consisting of all subcubes missing the edges of  $[0, 1]^3$ . After such a withdrawal we will get a compact subset  $K_1(a)$  consisting of  $8 + 12(a - 2) = 12a - 16$  subcubes with edge  $1/a$ . Then we repeat this procedure with the remaining subcubes. At the end we will obtain a self-similar fractal  $M^1(a)$  consisting of  $12a - 16$  copies of itself of  $a$  times smaller size. By analogy with  $M^1$  we can calculate the fractal dimension of

$M^1(a)$ :

$$\text{Dim}(M^1(a)) = \frac{\ln(12a - 16)}{\ln a} \leq \frac{\ln 12 + \ln a}{\ln a}.$$

This dimension tends to 1 as  $a \rightarrow \infty$ .

In fact, the space  $M^1(a)$  is homeomorphic to the Menger curve  $M_1 = M^1(3)$ . This follows from the Anderson's Characterizing Theorem [An] asserting that a 1-dimensional Peano continuum  $X$  is homeomorphic to the Menger curve  $M^1$  iff it has no local cut points and no open subset that embeds into the plane.

Now observe that the modified Menger curve  $M^1(a)$  is bi-Lipschitz equivalent to the convex metric space (this follows from the fact that any two points  $x, y \in M^1(a)$  can be linked by a path of length  $\leq \frac{12a}{1-a}d(x, y)$ ). Consequently,

$$1 = \dim(M^1) \leq \text{conv-dim}(M^1) \leq \inf\{\text{Dim}(M^1(a)) : a \geq 3\} = 1.$$

This finishes the proof of the partial case  $n = 1$ . The proof of the general case is analogous and uses the Bestvina's Characterization of Menger cubes in place of the Anderson's Characterization of the Menger curve. We start with the unit cube  $[0, 1]^{2n+1}$  in the  $(2n + 1)$ -dimensional Euclidean space. Given a positive integer  $a \geq 3$  we divide this cube into  $a^{2n+1}$  subcubes of  $a$  times smaller size and then delete all the subcubes which do not intersect  $n$ -dimensional faces of  $[0, 1]^{2n+1}$ . An easy calculations show us that  $2^{2n+1} \sum_{k=0}^n C_k^{2n+1} \left(\frac{a-2}{2}\right)^k$  subcubes will remain (here  $C_n^m = \frac{m!}{n!(m-n)!}$ ). Iterating this procedure we will get a fractal  $M^n(a) \subset [0, 1]^{2n+1}$  coinciding with the standard Menger cube  $M^n$  for  $a = 3$ .

The fractal  $M^n(a)$  consists of  $2^{2n+1} \sum_{k=0}^n C_k^{2n+1} \left(\frac{a-2}{2}\right)^k$  copies of itself of  $a$  times smaller size, which allows us to calculate the fractal dimension of  $M^n(a)$ :

$$\begin{aligned} \text{Dim}(M^n(a)) &= \ln \left( 2^{2n+1} \sum_{k=0}^n C_k^{2n+1} \left(\frac{a-2}{2}\right)^k \right) / \ln a = \\ &= n + (2n + 1) \ln 2 / \ln a + \ln \left( \sum_{k=0}^n C_k^{2n+1} \left(\frac{a-2}{2}\right)^k a^{k-n} \right) / \ln a \end{aligned}$$

which tends to  $n$  as  $a \rightarrow \infty$ .

All the spaces  $M^n(a)$  are homeomorphic by the Bestvina characterizing Theorem asserting that an  $n$ -dimensional continuum  $X$  is homeomorphic to the Menger cube  $M^n$  iff  $X$  is (locally)  $(n - 1)$ -connected and has the disjoint  $n$ -cells property, see [Be].

Finally observing that each space  $M^n(a)$  is bi-Lipschitz homeomorphic to a convex metric space, we conclude that  $\text{conv-dim}(M^n) = \inf\{\text{Dim}(M^n(a)) : a \geq 3\} = n$ .

## 7. SHARK TEETH

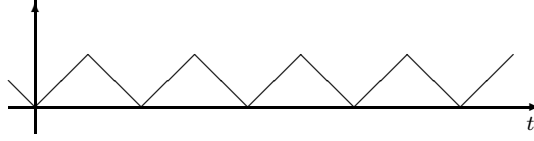
In spite of the fact that the “largest” 1-dimensional compactum  $M^1$  has S-dim equal to 1, there are examples of 1-dimensional Peano continua of arbitrary prescribed S-dimension S-dim. Such spaces exists even among rim-finite continua.

We recall that a topological space  $X$  is called *rim-finite* if open sets with finite boundary form a base of the topology of  $X$ . It is clear that each rim-finite space is at most 1-dimensional.

Our examples are constructed with help of the piecewise linear periodic function

$$\varphi(t) = \begin{cases} t - n, & \text{if } t \in [n, n + \frac{1}{2}] \text{ for some } n \in \mathbb{Z}; \\ n - t, & \text{if } t \in [n - \frac{1}{2}, n] \text{ for some } n \in \mathbb{Z} \end{cases}$$

whose graph looks as follows



Our example  $W_{\vec{n}}$  called “shark teeth” lives in the Banach space  $\mathbb{R} \times l_1$  and is parametrized by an infinite number vector  $\vec{n} = (n_k)_{k=1}^{\infty}$ . Let  $(\mathbf{e}_k)_{k=1}^{\infty}$  stand for the standard basis of the Banach space  $l_1 = \{(x_k)_{k=1}^{\infty} : \sum_{k=1}^{\infty} |x_k| < \infty\}$ .

Given a sequence  $\vec{n} = (n_k)_{k=1}^{\infty}$  of positive integers, consider the subspace  $W_{\vec{n}} = \{(t, 0), (t, \frac{\varphi(n_k t)}{n_k} \mathbf{e}_k) : t \in [0, 1], k \in \mathbb{N}\}$  of the product  $\mathbb{R} \times l_1$  endowed with the max-metric. Thus  $W_{\vec{n}} = I \cup \bigcup_{k=1}^{\infty} M_k$ , were  $I = [0, 1] \times \{0\}$  is the “bone” of the “shark teeth”  $W_{\vec{n}}$  and  $M_k = \{(t, \frac{\varphi(n_k t)}{n_k} \mathbf{e}_k) : t \in [0, 1]\}$  is the  $k$ -th generation of “teeth”.

Some properties of the “shark teeth” space  $W_{\vec{n}}$  are described in

**Theorem 7.** *Let  $W_{\vec{n}}$  be the “shark teeth” space parametrized by a number sequence  $\vec{n} = (n_k)$  tending to infinity, and let  $\alpha > 0$  be a positive real number.*

- (1) *The space  $W_{\vec{n}}$  is a 1-dimensional Peano continuum.*
- (2)  *$W_{\vec{n}}$  is rim-finite if for any  $n \in \mathbb{N}$  there are  $k_0 \in \mathbb{N}$  and a number  $m \geq n$  dividing each  $n_k$  for  $k \geq k_0$ .*
- (3)  *$W_{\vec{n}}$  is bi-Lipschitz homeomorphic to a convex metric space.*
- (4)  *$\text{Dim}(W_{\vec{n}}) = \text{S-Dim}(W_{\vec{n}}) = \text{Hö-Dim}(W_{\vec{n}})$ .*
- (5) *If  $\inf_{k \geq 1} \frac{n_k}{k^\alpha} > 0$ , then  $\text{Dim}(X) \leq 1 + 1/\alpha$ .*
- (6) *If  $\sup_{k \geq 1} \frac{n_k}{k^\alpha} < \infty$ , then  $\text{S-dim}(X) \geq 1 + 1/\alpha$ .*
- (7) *If  $0 < \inf_{k \geq 1} \frac{n_k}{k^\alpha} \leq \sup_{k \geq 1} \frac{n_k}{k^\alpha} < +\infty$ , then*

$$\begin{aligned} 1 + 1/\alpha = \text{Dim}(W_{\vec{n}}) &= \text{S-Dim}(W_{\vec{n}}) = \text{Hö-Dim}(W_{\vec{n}}) = \\ &= \text{S-dim}(W_{\vec{n}}) = \text{Hö-dim}(W_{\vec{n}}) = \text{conv-dim}(W_{\vec{n}}). \end{aligned}$$

*Proof.* 1,2) The first two properties of  $W_{\bar{n}}$  follow immediately from the definition of “shark teeth”.

3) Define a convex metric  $d$  on  $W_{\bar{n}}$  letting  $d(x, y)$  be the smallest length of a path linking  $x$  and  $y$ . It is easy to see that  $\|x - y\| \leq d(x, y) \leq 2\|x - y\|$  which means that  $W_{\bar{n}}$  is bi-Lipschitz homeomorphic to the convex metric space  $(W_{\bar{n}}, d)$ .

4) The equalities  $\text{Dim}(W_{\bar{n}}) = \text{S-Dim}(W_{\bar{n}}) = \text{Dim}(W_{\bar{n}})$  follow from the previous item and Corollary 2.

5) Assume that  $\inf_{k \geq 1} \frac{n_k}{k^\alpha} = c > 0$ . Take any  $\varepsilon > 0$  any cover the interval  $I = [0, 1] \times \{0\}$  by  $\lceil \frac{3}{\varepsilon} \rceil$  balls of radius  $\varepsilon/3$  in  $\mathbb{R} \times \ell_1$  centered at points of  $I$ . The union of these balls form a tube  $\{(t, x) \in W_{\bar{n}} : \|x\| < \varepsilon/3\}$  covering the “teeth” of  $k$ -th generation  $M_k$  with  $\frac{1}{n_k} < \varepsilon/3$ . So it remains to cover  $k_0 = |\{k : \frac{1}{n_k} \geq \varepsilon/3\}|$  curves  $M_k$ ,  $k \leq k_0$ , each of length 1. This can be done with help of  $\leq k_0 \lceil \frac{3}{\varepsilon} \rceil$  balls of radius  $\varepsilon/3$ . So totally it suffices  $(k_0 + 1) \lceil \frac{3}{\varepsilon} \rceil$  balls of radius  $\varepsilon/3$  to cover all the set  $W_{\bar{n}}$ . Consequently,  $N(\varepsilon) \leq (k_0 + 1) \lceil \frac{3}{\varepsilon} \rceil \leq (k_0 + 1)(1 + 3/\varepsilon)$ .

Observe that  $k_0 = |\{k : n_k \leq 3/\varepsilon\}| \leq |\{k : ck^\alpha \leq 3/\varepsilon\}| \leq (\frac{3}{c\varepsilon})^{1/\alpha}$  and thus  $N(\varepsilon) \leq (1 + 3/\varepsilon)(1 + (3c/\varepsilon)^{1/\alpha}) = O^*(\varepsilon^{-(1+1/\alpha)})$ , which means that  $\text{Dim}(W_{\bar{n}}) \leq 1 + 1/\alpha$ .

6. Now assume that  $\sup_{k \geq 1} \frac{n_k}{k^\alpha} = A > 0$ . To prove that  $\text{S-dim}(W_{\bar{n}}) \geq 1 + 1/\alpha$ , fix any admissible metric  $\rho$  on  $W_{\bar{n}}$ . We should show that  $\text{S-Dim}(W_{\bar{n}}, \rho) \geq 1 + 1/\alpha$ . Let  $R = \rho((0, 0), (1, 0))$  be the  $\rho$ -distance between the end-points of the “bone”  $I = [0, 1] \times \{0\} \subset W_{\bar{n}}$ .

Given  $\varepsilon > 0$ , consider any cover  $\mathcal{C}$  of  $X$  by connected subsets of diameter  $< \varepsilon$  with  $|\mathcal{C}| = S(\varepsilon)$ . For every  $k \geq 1$  let  $\mathcal{C}_k = \{C \in \mathcal{C} : C \cap M_k \neq \emptyset \text{ and } C \cap I = \emptyset\}$ . It is easy to see that each  $C \in \mathcal{C}_k$  lies in  $M_k \setminus I$  and hence the families  $\mathcal{C}_k$ ,  $k \geq 1$ , are disjoint.

We claim that  $|\mathcal{C}_k| \geq (R - 2n_k\varepsilon)/\varepsilon$  for every  $k \geq 1$ . Indeed, note that each element  $C \in \mathcal{C}$  meeting the set  $M_k \cap I$  at some point  $x \in M_k \cap I$  lies in the  $\varepsilon$ -ball  $B_\varepsilon(x) = \{y \in X : \rho(x, y) < \varepsilon\}$ . Then the family  $\mathcal{C}_k \cup \{B_\varepsilon(x) : x \in M_k \cap I\}$  covers the  $k$ -th generation of “teeth”  $M_k$  and

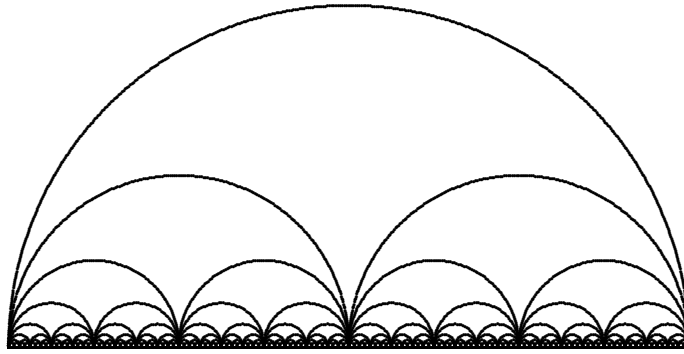
$$R \leq \text{diam } M_k \leq \sum_{C \in \mathcal{C}_k} \text{diam } C + \sum_{x \in M_k \cap I} \text{diam } B_\varepsilon(x) \leq \varepsilon |\mathcal{C}_k| + 2\varepsilon(n_k + 1).$$

Consequently,  $|\mathcal{C}_k| \geq \frac{R}{\varepsilon} - 2(n_k + 1) \geq R/\varepsilon - 2(Ak^\alpha + 1)$ .

Let  $k_0 = (\frac{R-4\varepsilon}{4A\varepsilon})^{1/\alpha}$  and note that for any  $k \leq k_0$ , we get  $|\mathcal{C}_k| \geq R/\varepsilon - 2(Ck_0^\alpha + 1) = R/2\varepsilon$ . Then  $S(\varepsilon) = |\mathcal{C}| \geq \sum_{k \leq k_0} |\mathcal{C}_k| \geq [k_0] \cdot \frac{R}{2\varepsilon} \geq \frac{R}{2\varepsilon}(k_0 - 1) \geq \frac{R}{2\varepsilon} (\frac{R}{4A\varepsilon} - \frac{1}{A})^{1/\alpha} = O^*(\varepsilon^{-(1+1/\alpha)})$  which just implies that  $\text{S-Dim}(W_{\bar{n}}, \rho) \geq 1 + 1/\alpha$ .

7. The final statement follows from the preceding items.  $\square$

**Remark 1.** Rim-finite continua often are referred to as *regular curves*. A continuum  $X$  is defined to be *totally regular* if for any countable subset  $C \subset X$  the family of open sets with finite boundaries missing the set  $C$  forms a base of the topology of  $X$ . Totally regular continua were characterized by S.Eilenberg and O.Harrod [EH] as continua of finite length with respect to some admissible metric, see also [BNT]. Being images of the interval under Lipschitz maps, such continua have Hölder dimension Hö-dim equal to 1. A classical example of a regular continuum which is not totally regular looks as follows:



Let us observe that this space is homeomorphic to the “shark teeth”  $W_{\vec{n}}$  with  $\vec{n} = (2^k)_{k=0}^{\infty}$ . In fact, any “shark teeth” space  $W_{\vec{n}}$  has infinite length for any admissible metric and thus is not totally regular. Nonetheless, among the “shark teeth”  $W_{\vec{n}}$  there are spaces with Hö-Dim( $W_{\vec{n}}$ ) = 1. In particular, so is the space  $W_{\vec{n}}$  with  $\vec{n} = (2^k)$ .

**Corollary 5.** *For any  $d \in [1, +\infty]$  there is a number sequence  $\vec{n} = (n_k)_{k=1}^{\infty}$  such that  $W_{\vec{n}}$  is a rim-finite metric continuum  $X$  with  $d = \text{S-dim}(X) = \text{S-Dim}(X) = \text{Hö-dim}(X) = \text{Hö-Dim}(X) = \text{conv-dim}(X)$ .*

This corollary follows from Theorem 7 and

**Lemma 1.** *Let  $(x_k) \leq (y_k)$  be two non-decreasing number sequences such that  $\lim_{k \rightarrow \infty} y_k - x_k = \infty$ . Then there is a number sequence  $(n_k)$  with  $(x_n) \leq (n_k) \leq (y_k)$  such that for any  $n \in \mathbb{N}$  there is  $k_0$  such that  $n$  divides all  $n_k$  for  $k \geq k_0$ .*

*Proof.* For every  $k \in \mathbb{N}$  take any  $n_k \in [x_k, y_k] \cap (m_k!) \mathbb{Z}$ , where  $m_k$  is the largest number such that  $[x_k, y_k] \cap (m_k!) \mathbb{Z} \neq \emptyset$ . It follows from  $y_k - x_k \rightarrow \infty$  that  $m_k \rightarrow \infty$  and consequently, each  $m$  divides all but finitely many of  $n_k$ 's.  $\square$

Since the “Shark teeth”  $W_{\vec{n}}$ , being 1-dimensional, are subspaces of the Menger curve  $M^1$ , we get

**Corollary 6.** *The  $S$ -dimension  $S\text{-dim}$  is not monotone with respect to taking (Peano) subspaces.*

“Shark teeth” spaces are locally connected but not locally contractible.

**Question 1.** *Is there a locally contractible continuum  $X$  with  $\dim(X) < S\text{-dim}(X)$ ?*

**Remark 2.** The problems considered in this paper trace their history back to A.Besicovitch and H. Ursell who studied the interplay between Hölder functions and the fractal dimension of their graphs. In the paper [BU], given a positive  $\alpha \leq 1$ , they constructed an  $\alpha$ -Hölder function  $f : [0, 1] \rightarrow \mathbb{R}$  whose graph has fractal dimension  $2 - \alpha$ .

Among modern works treating fractal Peano curves let us mention recent papers of E. Shchepin [S1], [S2] describing unexpected applications of such curves in radio-electronics, see his web-site:

[www.mi.ras.ru/~scep](http://www.mi.ras.ru/~scep)

#### REFERENCES

- [An] R. D. Anderson, *One-dimensional continuous curves and a homogeneity theorem*, Ann. Math., II. Ser. **68** (1958), 1-16.
- [Ba] M.Barnsley, *Fractals Everywhere*, Academic Press, 1988.
- [BU] A. Besicovitch, H. Ursell, *Sets of fractional dimension (V): On dimensional numbers of some continuous curves*, Jour. London Math. Soc. **12**:1 (1937), 18–25.
- [Be] M.Bestvina, *Characterizing  $k$ -dimensional universal Menger compacta*, Mem. Am. Math. Soc. **71**, No.380, 110 p. (1988).
- [Bi] R. Bing, *Partitioning a set*, Bull. Amer. Math. Soc. **55** (1949), 1101–1110.
- [BNT] R. Buskirk, J. Nikiel, E. Tymchatyn, *Totally regular curves as inverse limits*. Houston J. Math. **18**:3 (1992), 319–327.
- [Ed] G.Edgar, *Measure, Topology and Fractal Geometry*, Springer, 1990.
- [EH] S. Eilenberg, O. Harrold, *Continua of finite linear measure. I*. Amer. J. Math. **65** (1943), 137–146.
- [Fr] D.H.Fremlin, *Spaces of finite length*, Proc. London Math. Soc. **64** (1992) 449–486.
- [Mo] E. Moise, *Grille decomposition and convexification theorems for compact metric locally connected continua*, Bull. Amer. Math. Soc. **55** (1949), 1111–1121.
- [PS] L. Pontrjagin, L. Schnirelmann, *Sur une propriété métrique de la dimension*, Ann. of Math. (2) **33**:1 (1932), 156–162.
- [S1] E.Shchepin, *Dimension raising maps and continuous information distribution //* Problems of pure and applied mathematics, Vol.1, Tula, 1987, P.148-155.
- [S2] E. Shchepin, *On fractal Peano curves*, see: [www.mi.ras.ru/~scep/#materials](http://www.mi.ras.ru/~scep/#materials)