SELECTION PRINCIPLES AND INFINITE GAMES ON MULTICOVERED SPACES

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Contents

Introduction

This article is a survey of results included in the forthcoming book [9]. The starting impulse for writing this book came from topological algebra. Trying to find an inner characterization of subgroups of σ-compact topological groups, Okunev introduced the concept of an o-bounded group (see [74]): a topological group G is *o-bounded* if for any sequence $(U_n)_{n\in\omega}$ of open neighborhoods of the unit of G there is a sequence $(F_n)_{n\in\omega}$ of finite subsets of G such that $G=\bigcup_{n\in\omega}F_n\cdot U_n$ where $A\cdot B=\{ab: a\in A,\ b\in B\}$ is the product of two subsets A, B in the group G. The class of o-bounded groups turned out to be much wider than the class of subgroups of σ-compact groups: for example the group $\mathbb{R}^\omega$ contains a non-meager dense o-bounded subgroup. To overcome this difficulty M.Tkachenko introduced the narrower class of so-called strictly o-bounded groups, defined with help of the infinite game "Open-Finite" played by two players, I and II, on a topological group G: at the n-th inning the first player selects a neighborhood U_n of the unit in G while the second player responds with a finite subset F_n of G. At the end of the game the second player is declared the winner if $G=\bigcup_{n\in\omega}F_n\cdot U_n$. Following Tkachenko we define a topological group G to be *strictly o-bounded* if the second player has a winning strategy in the game "Open-Finite" on G. The class of strictly o-bounded groups includes all subgroups of σ-compact groups and lies in the class of all o-bounded groups. (Strictly) o-bounded groups were intensively studied last time, see [3], [4], [6], [8], [37], [38], [49], [56], [74], [77], [91].

In spirit, o-boundedness is very close to the Menger property introduced by K. Menger [58] in 1924 and studied in detail by W. Hurewicz [40] who introduced another property known in topology as the Hurewicz property. Recall that a topological space X has the *Menger property* (resp. the *Hurewicz property*) if for any sequence $(u_n)_{n\in\omega}$ of open covers of X there is a sequence $(v_n)_{n\in\omega}$ such that each v_n, $n\in\omega$, is a finite subcollection of u_n and $X=\bigcup_{n\in\omega}\cup v_n$ (resp. $X=\bigcup_{n\in\omega}\bigcap_{m\geq n}\cup v_m$). Our crucial observation is that the o-boundedness of a topological group (G,τ) is noting else but the Menger property applied to the family $\mu_L=\big\{\{g\cdot U:g\in G\}:e\in U\in\tau\big\}$ of open covers of G by left shifts of fixed neighborhoods of the unit e of G. This observation naturally led us to the concept of a *multicovered space*, by which we understand a pair (X,μ) consisting of a set X and a collection μ of covers of X (such a collection μ is called a *multicover* of X).

The category of multicovered spaces seems to be the most natural place for considering various concepts such as the Menger and Hurewicz properties of topological spaces or the (strict) o-boundedness of topological groups. Such an abstract approach allows us to prove general results having applications and interpretations in various fields of mathematics as different as Set Theory, General Topology, Descriptive Set Theory, Topological Algebra or Theory of Uniform Spaces.

1. Multicovered Spaces and their Morphisms

1.1. **Basic definitions and concepts.** By a *multicover* of a set X we understand any family μ of covers of X. A set X endowed with a multicover μ will be called a *multicovered space* (denoted by (X,μ)) or simply X if the multicover is clear from the context.

There are many natural examples of multicovered spaces:

- Each topological space X can be considered as a multicovered space $(X,\mathcal{O})$, where $\mathcal{O}$ denotes the family of all open covers of X;
- Every metric space (X,ρ) carries a natural multicover μ_ρ consisting of covers by ε-balls: $\mu_\rho=\{\{B(x,\varepsilon):x\in X\}:\varepsilon>0\}$, where $B(x,\varepsilon)\stackrel{\text{def}}{=}\{y\in X:\rho(y,x)<\varepsilon\}$;
- Every uniform space $(X,\mathcal{U})$ has a multicover $\mu_{\mathcal{U}}$ consisting of uniform covers, i.e. $\mu_{\mathcal{U}}=\{\{U(x):x\in X\}:U\in\mathcal{U}\}$, where $U(x)=\{y\in X:(x,y)\in U\}$;
- In particular, each Abelian topological group G carries a natural multicover $\mu_G=\{\{g+U:g\in G\}:U\neq\emptyset\text{ is open in }G\}$ corresponding to the uniform structure of G.

Next, we define some notions related to boundedness in multicovered spaces. A subset B of a set X is defined to be *u-bounded* with respect to a cover u of X if $B\subset\cup v$ for some finite subcollection v of u. A subset B of a multicovered space (X,μ) is defined to be *μ'-bounded*, where $\mu'\subset\mu$, if B is u-bounded for every $u\in\mu'$. If $\mu'=\mu$, then we simply say that B is *bounded* in place of μ-bounded. The notion of a bounded subset is a natural generalization of such concepts as a precompact subset of a regular topological space (i.e., a subset with compact closure) or a totally bounded subset of a uniform space to the realm of multicovered spaces. A multicovered space X is defined to be *σ-bounded* if it is a countable union of its bounded subsets.

On the family $\mathrm{cov}(X)$ of all covers of a set X there is a natural preorder $\succ$: $u \succ v$ if each u-bounded subset is v-bounded. A multicover space (X, μ) is defined to be *centered* if any finite subfamily $\delta \subset \mu$ has an upper bound $u \in \mu$ with respect to $\prec$ (the latter means that each u-bounded subset of X is δ- bounded). For example, all multicovered spaces described before are centered. We shall say that a multicover μ on X is *finer* than a multicover η on X ($\mu \succ \eta$) if for any $u \in \eta$ there is $v \in \mu$ with $v \succ u$. Two multicovers μ, η of a set X will be called *equivalent* if each of them is finer than the other.

Now we are ready to introduce three important cardinal characteristics of a multicovered space (X, μ):

- $b\chi(X, \mu)$, the *boundedness character* of (X, μ), is the smallest size $|\delta|$ of a subcollection $\delta \subset \mu$ such that each δ-bounded subset of X is μ-bounded;
- $\mathrm{cof}(X, \mu)$, the *cofinality* of (X, μ), is the smallest size $|\eta|$ of a subfamily $\eta \subset \mu$, equivalent to μ (which means that for every $u \in \mu$ there exists $v \in \eta$ with $v \succ u$);
- $\mathrm{bc}(X, \mu)$, the *bounded covering number* of (X, μ), is the smallest size of a cover of X by μ-bounded subsets;
- $\mathrm{bc}_\omega(X, \mu) = \sup\{\mathrm{bc}(X, \eta) : \eta \subset \mu \text{ is a countable subcollection}\}$, the *countably-bounded covering number.*

It is clear that $b\chi(X, \mu) \le \mathrm{cof}(X, \mu)$ for every multicovered space (X, μ), and a multicovered space (X, μ) is σ-bounded iff $\mathrm{bc}(X, \mu) \le \omega$. It is also clear that $\mathrm{bc}_\omega(X, \mu) \le \mathrm{bc}(X, \mu)$ and both the cardinals are equal if $b\chi(X, \mu) \le \aleph_0$.

Another important notion related to multicovered spaces is ω-boundedness generalizing the ω-boundedness of topological groups, see [35]. We shall say that a multicovered space (X, μ) is *ω-bounded* if every cover $u \in \mu$ contains a countable subcover u'. Unfortunately, this does not mean that $u' \in \mu$. We define a multicovered space (X, μ) to be *properly ω-bounded* if for every cover $u \in \mu$ there exists a countable subcover $u' \subset u$ and a cover $v \in \mu$ such that $v \succ u'$, i.e. each v-bounded subset is u'-bounded. A multicovered space (X, μ) is said to be *paracompact* if for any cover $u \in \mu$ there is a cover $v \in \mu$ with $St(v) \succ u$. Here, as expected, $St(v) = \{St(V, v) : V \in v\}$ where $St(V, v) = \cup\{U \in v : U \cap V \neq \emptyset\}$.

Proposition 1. *Every ω-bounded paracompact multicovered space is properly ω- bound.*

1.2. **Morphisms between multicovered spaces.** Trying to introduce a proper notion of isomorphic multicovered spaces we came to the conclusion that it is too restrictive to look at morphisms as usual single-valued functions. By a morphism from a multicovered space (X, μ_X) into a multicovered space (Y, μ_Y) we shall understand a multifunction $\Phi : X \Rightarrow Y$, see [11, section 6] for basic information about multifunctions. Such a morphism $\Phi : X \Rightarrow Y$ is defined to be:

- *uniformly bounded*, if for every cover $u \in \mu_Y$ there exists a cover $v \in \mu_X$ such that the image $\Phi(B)$ of any v-bounded subset B of X is u-bounded;
- *perfect*, if Φ^{-1} is uniformly bounded;
- an *isomorphic embedding* if Φ is a perfect uniformly bounded morphism with $\mathrm{Dom}(\Phi) = X$;
- an *isomorphism*, if Φ is a perfect uniformly bounded morphism with $\mathrm{Dom}(\Phi) = X$ and $\mathrm{Im}(\Phi) = Y$ (equivalently, both Φ and Φ^{-1} are isomorphic embeddings).

The above three cardinal characteristics of multicovered spaces as well as the (proper) ω-boundedness are preserved by isomorphisms between multicovered spaces. Moreover, a multicovered space (X, μ) is properly ω-bounded iff it is isomorphic to some multicovered space (Y, ν), where ν consists of countable covers.

Finally we explain the nature of perfect morphisms between multicovered spaces of the form $(X, \mathcal{O})$ where X is a topological space and $\mathcal{O}$ is the multicover consisting of all open covers of X.

Theorem 2. *For a multifunction $\Phi : X \Rightarrow Y$ from a regular topological space X into a k-space Y the morphism $\Phi : (X, \mathcal{O}) \Rightarrow (Y, \mathcal{O})$ is perfect iff its closure in $X \times Y$ considered as a morphism is perfect iff the projection $\mathrm{pr}_Y : \overline{\Phi} \to Y$ is perfect in the usual topological sense.*

We recall from [30, p. 277] that a continuous closed map f from a Hausdorff topological space X into a topological space Y is *perfect*, if $f^{-1}(y)$ is compact for every $y \in Y$.

1.3. **Operations over multicovered spaces.** Each subset Y of a multicovered space (X, μ) carries the induced multicover $\mu|Y$ consisting of all covers $\{U \cap Y : U \in u\}$ where $u \in \mu$. Having this multicover $\mu|Y$ in mind we shall make remarks about subspaces of multicovered spaces. It is immediate that the σ- and ω-boundedness are preserved by countable unions of subspaces.

Note that for each subspace Y of a multicovered space X the identity inclusion $\mathrm{id} : Y \to X$ is an isomorphic embedding. This implies that $b\chi(Y) \le b\chi(X)$, $\mathrm{bc}(Y) \le \mathrm{bc}(X)$, and $\mathrm{cof}(Y) \le \mathrm{cof}(X)$ for every subspace Y of a multicovered space (X, μ).

It should be also stressed here that for a subspace Y of a topological space X the identity inclusion $\mathrm{id} : (Y, \mathcal{O}(Y)) \to (X, \mathcal{O}(X))$ is uniformly bounded but can fail to be an isomorphic embedding of multicovered spaces. Take for example any compact topological space K and its non-compact subspace X.

Let (X, μ_X) and (Y, μ_Y) be multicovered spaces. By their product we understand the set $X \times Y$ endowed with the multicover $\mu_X \cdot \mu_Y = \{u \cdot v : u \in \mu_X, v \in \mu_Y\}$, where $u \cdot v = \{U \times V : U \in u, V \in v\}$. The product of a finite collection of multicovered spaces can be naturally defined by induction. We usually identify $\mu_X \cdot \mu_Y$ with the Cartesian product $\mu_X \times \mu_Y$ and write (X^n, μ^n) for the n-th power of the space (X, μ). The boundedness, σ-boundedness, and (proper) ω-boundedness are preserved by finite products.

Another important operation on multicovered spaces is *centralization*, which assigns to each multicovered space (X, μ) the centered multicovered space $(X, \mathsf{cen}(\mu))$ endowed with the multicover $\mathsf{cen}(\mu) = \{u_1 \wedge \cdots \wedge u_n : u_1, \ldots, u_n \in \mu\}$, where $u_1 \wedge \cdots \wedge u_n = \{U_1 \cap \cdots \cap U_n : U_i \in u_i \text{ for } 1 \le i \le n\}$. The multicover $\mathsf{cen}(\mu)$ has the following universality property.

Proposition 3. *For any multicover μ on a set X the identity morphism $\mathrm{id}_X : (X, \mu) \to (X, \mathsf{cen}(\mu))$ is both perfect and bounded. Moreover for any uniformly bounded morphism $\Phi : Y \Rightarrow (X, \mu)$ from a* centered *multicovered space Y the composition $\mathrm{id}_X \circ \Phi : Y \Rightarrow (X, \mathsf{cen}(\mu))$ is uniformly bounded.*

1.4. **Universal multicovered spaces.** We define a multicovered space X to be *universal in a class $\mathcal{M}$* of multicovered spaces if $X \in \mathcal{M}$ and each multicovered space $M \in \mathcal{M}$ admits an isomorphic embedding $\Phi : M \Rightarrow X$. Up to isomorphism, *each class of centered multicovered spaces contains at most one universal space*. But we do not know whether the centeredness can be dropped in this result.

For every cardinal κ the class of properly ω-bounded multicovered spaces X with $\mathrm{cof}(X, \mu) \le \kappa$ has a universal element: the space (ω^κ, μ_u) endowed with the multicover $\mu_u = \{u_\alpha : \alpha \in \kappa\}$, where $u_\alpha = \{\mathrm{pr}_\alpha^{-1}(n) : n \in \omega\}$ and $\mathrm{pr}_\alpha : \omega^\kappa \to \omega$ stands for the coordinate projection. Centralizing the multicover μ_u, we obtain a centered multicover $\mu_p = \mathsf{cen}(\mu_u)$ on ω^κ, equivalent to the multicover induced by the product uniformity of ω^κ. Combining the universality property of μ_u with the universality property of centralization we obtain that the multicovered space (ω^κ, μ_p) is universal in the class of centered properly ω-bounded multicovered spaces with cofinality $\le \kappa$. The multicovered space (ω^κ, μ_p) is isomorphic to topological groups $\mathbb{Z}^\kappa$ and $\mathbb{R}^\kappa$ endowed with the multicovers generated by their topological group structures. The universal space (ω^ω, μ_p) is also isomorphic to its subspace $\omega^{\uparrow\omega} = \{f \in \omega^\omega : f \text{ is non-decreasing}\}$.

The existence of universal spaces allows us to obtains the following non-trivial estimates.

Theorem 4. $\mathrm{bc}_\omega(X, \mu) \le \mathfrak{d}$ *(resp.* $\mathrm{bc}_\omega(X, \mu) \in \{1, \aleph_0\} \cup [\mathfrak{b}, \mathfrak{d}]$*) for every (properly) ω-bounded multicovered space X.*

1.5. **Uniformizability and metrizability of multicovered spaces.** A multicovered space X is defined to be *uniformizable* (resp. *metrizable*) if X is isomorphic to the multicovered space $(Y, \mu_{\mathcal{U}})$ (resp. (Y, μ_ρ)) for some uniform (resp. metric) space $(Y, \mathcal{U})$ (resp. (Y, ρ)). For example, the universal space (ω^κ, μ_p) is uniformizable (metrizable) for every (at most countable) cardinal κ.

Theorem 5. *An ω-bounded multicovered space X is uniformizable (metrizable) iff it is is centered, properly ω-bounded (and has countable cofinality $\mathrm{cof}(X)$).*

Selection Principles on uniform spaces were considered by L. Kočinac in [45].

1.6. **Cardinal characteristics of some natural multicovered spaces.** This subsection is devoted to cardinal characteristics of multicovered spaces described at the beginning of this section, i.e. topological, metric, uniform spaces, and topological groups.

First we shall calculate the cardinal characteristics of multicovered spaces of the form $(X, \mathcal{O})$ where X is a topological space. Since a subset B of a regular topological space X is $\mathcal{O}$-bounded if and only if B has compact closure in X, we conclude that $\mathrm{bc}(X, \mathcal{O}) = \mathrm{kc}(X)$, where $\mathrm{kc}(X)$, the *compact covering number* of X, is the smallest size of a cover of X by compact subsets. Another important cardinal invariant of the family $\mathcal{K}(X)$ of compact subspaces of X is its cofinality $\mathrm{cof}(\mathcal{K}(X))$ with respect to the inclusion relation, see [27].

In a sense, the cardinal invariants $\mathrm{cof}(\mathcal{K}(X))$ and $\mathrm{kc}(X)$ are dual to the character $\chi(X; \beta X)$ and pseudo-character $\psi(X; \beta X)$ of a Tychonov space X in its Stone-Čech compactification βX (see [30] for their definitions). The cardinal $\psi(X; \beta X) = \mathrm{kc}(\beta X \setminus X)$ is often called *the Čech number* of X and is denoted by $\check{C}(X)$. For a given topological space X the Čech number $\check{C}(X)$ gives a cardinal measure of the non-compactness of X: X is compact iff $\check{C}(X) = 0$, X is locally compact iff $\check{C}(X) \le 1$, X is Čech-complete iff $\check{C}(X) \le \aleph_0$.

Theorem 6. *Let X be a Tychonoff space and Y be a compactification of X. Then*

(1) $\mathrm{cof}(X, \mathcal{O}) = \chi(X; Y) = \mathrm{cof}(\mathcal{K}(Y \setminus X))$;
(2) $b\chi(X, \mathcal{O}) \le \check{C}(X) = \psi(X; Y)$;
(3) $b\chi(X, \mathcal{O}) = \check{C}(X)$ if $b\chi(X, \mathcal{O}) \le \aleph_0$;
(4) $\mathrm{cof}(X, \mathcal{O}) \ge \mathfrak{d}$ if $Y \setminus X$ is first countable and not locally countably compact.

Next, we consider uniform spaces. Given a uniform space $(X,\mathcal{U})$, for an entourage $U \in \mathcal{U}$ and a subset A of X let $U(A) = \{y \in X : \exists x \in A \text{ with } (x, y) \in U\}$ be the U-ball around A. The uniform cover of X corresponding to U will be denoted by c_U, i.e. $c_U = \{U(x) : x \in X\}$, and the multicover of X consisting of all uniform covers will be denoted by $\mu_{\mathcal{U}}$. We recall from [30, § 8.1] that the *weight* $w(X,\mathcal{U})$ of a uniform space $(X,\mathcal{U})$ is the smallest size $|\mathcal{B}|$ of a subfamily $\mathcal{B}\subset\mathcal{U}$ such that for each $U\in\mathcal{U}$ there is $V\in\mathcal{B}$ with $V\subset U$. It is evident that $\mathrm{cof}(X,\mu_{\mathcal{U}})\le w(X,\mathcal{U})$ for any uniform space. On the other hand, we have the subsequent

Theorem 7. *For every uniform space $(X,\mathcal{U})$ with $b\chi(X,\mu_{\mathcal{U}})\le\aleph_0$ the completion $(\tilde{X},\tilde{\mathcal{U}})$ of $(X,\mathcal{U})$ is Čech-complete, i.e. so is its underlying topological space.*

Question 1. *Let $(X,\mathcal{U})$ be a dense uniform subspace of a uniform space $(Y,\mathcal{V})$. Is it true that $b\chi(X,\mu_{\mathcal{U}})=b\chi(Y,\mu_{\mathcal{V}})$?*

The answer onto the above problem is affirmative if one of these characters is countable.

And finally we shall consider cardinal invariants of topological groups endowed with multicovers corresponding to their natural uniformities. Let G be a topological group and $\mathcal{O}$ be the collection of all neighborhoods of the unit e of G. We shall be interested in the following four natural multicovers on G: $\mu_L=\{\{gU:g\in G\}:U\in\mathcal{O}\}$, $\mu_R=\{\{Ug:g\in G\}:U\in\mathcal{O}\}$, $\mu_{L\vee R}=\{\{gU\cap Ug:g\in G\}:U\in\mathcal{O}\}$, and $\mu_{L\wedge R}=\{\{UgU:g\in G\}:U\in\mathcal{O}\}$ corresponding to the left, right, two-sided and Rölke uniformities on G, respectively. It should be mentioned that the multicovered spaces (G,μ_L) and (G,μ_R) are isomorphic via the map $x\mapsto x^{-1}$, while $(G,\mu_{L\vee R})$ is isomorphic to the diagonal of the product $(G,\mu_L)\times(G,\mu_R)$. This implies that $\mathrm{bc}(G,\mu_{L\vee R})=\mathrm{bc}(G,\mu_L)=\mathrm{bc}(G,\mu_R)$ and $b\chi(G,\mu_{L\vee R})\le b\chi(G,\mu_L)=b\chi(G,\mu_R)$.

Question 2. *Can the latter inequality be strict? More precisely, is there a topological group G with $b\chi(G,\mu_{L\vee R})<b\chi(G,\mu_L)$?*

The following characterization shows that this cannot happen at the countable level.

Theorem 8. *For a topological group G the following conditions are equivalent: (1) $b\chi(G,\mu_L)\le\aleph_0$; (2) $b\chi(G,\mu_R)\le\aleph_0$; (3) $b\chi(G,\mu_{L\vee R})\le\aleph_0$; (4) G is a subgroup of a Čech-complete group. Moreover, the equivalent conditions (1)–(4) imply $b\chi(G,\mu_{L\wedge R})\le\aleph_0$.*

In light of the above result it is interesting to note that the cardinals $b\chi(G,\mu_{L\vee R})$ and $b\chi(G,\mu_{L\wedge R})$ can differ as much as we wish: there is no upper bound on $b\chi(G,\mu_{L\vee R})$ for groups with $b\chi(G,\mu_{L\wedge R})\le\aleph_0$.

2. Classical selection principles

2.1. **Basic Selection Principles.** Here we introduce and study various selection properties of multicovered spaces, intermediate between the properties of σ-boundedness and ω-boundedness.

The strongest and the weakest among these properties are the Menger and Hurewicz properties. They take their origin in topology and have appeared as cover counterparts of the σ-compactness. Both Menger and Hurewicz properties are expressed in terms of covers so have multicover nature and can be defined for any multicovered space. We shall go a bit further in our generalizing attempts and observe that both Menger and Hurewicz properties allow us to construct sequences (v_n) satisfying certain property. In such a way we arrive to our main selection principle $\cup_{\mathrm{fin}}(\mu, \mathcal{P})$ introduced by M. Scheepers in [64]. Let (X, μ) be a multicovered space and $\mathcal{P}$ be a property of sequences $(B_n)_{n\in\omega}$ of subsets of (X, μ). We shall say that the multicovered space satisfies the selection principle $\cup_{\mathrm{fin}}(\mu, \mathcal{P})$ if for any sequence of covers $(u_n)_{n\in\omega} \in \mu^\omega$ there is a sequence $(B_n)_{n\in\omega}$ having the property $\mathcal{P}$ and consisting of u_n-bounded subsets $B_n \subset X$. The above definition works properly only for *monotone* properties $\mathcal{P}$. The latter means that a sequence $(B_n)_{n\in\omega}$ has the property $\mathcal{P}$ provided there is a sequence $(C_n)_{n\in\omega}$ with the property $\mathcal{P}$ such that $C_n \subset B_n$ for all $n \in \omega$.

Let $\{U_i : i \in \omega\}$ be an indexed family of subsets of a set X. Given a point $x \in X$ let $\mathcal{I}_x = \{i \in \omega : x \in U_i\}$. The indexed family $\{U_i : i \in \omega\}$ is defined to be

- a *large cover* of X, if each point x lies in infinitely many sets U_i;
- a *γ-cover* of X, if each point $x \in X$ belongs to almost all U_i;
- a *sub-γ-cover* of X, if there exists an infinite subset $A \subset \omega$ such that $\{U_i : i \in A\}$ is a γ-cover of X;
- an *ω-cover* of X, if each finite subset $F \subset X$ lies in infinitely many U_i.

Now we arrive at our principal definition: A multicovered space (X, μ) is called *Menger* (resp. *Scheepers, sub-Hurewicz, Hurewicz*) if for any sequence $(u_n)_{n\in\omega} \subset \mu$ there exists an sequence $(B_n : n \in \omega)$ of subsets of X such that each set B_n is u_n-bounded and the indexed collection $\{B_n : n \in \omega\}$ is a cover (resp. ω-, sub-γ-, γ-)cover of X. Let us note that a topological space X has the Menger (resp. Scheepers, Hurewicz) property if and only if so does the multicovered space $(X, \mathcal{O})$. A topological group G is o-bounded iff the multicovered space (G, μ_L) is Menger. These properties are related as follows:

$$\sigma\text{-bounded} \Rightarrow \text{Hurewicz} \Rightarrow \text{sub-Hurewicz} \Rightarrow \text{Scheepers} \Rightarrow \text{Menger} \Rightarrow \omega\text{-bounded}.$$

The first implication of this chain can be reversed for spaces with countable boundedness character. This follows from the characterization of Hurewicz spaces as multicovered spaces X with $\mathrm{bc}_\omega(X) \leq \aleph_0$. The second implication can be reversed in the centered case: *Each centered sub-Hurewicz space is Hurewicz.*

On the other hand, in section 2.4 we shall present an example of a sub-Hurewicz space which fails to be Hurewicz as well as an example of a centered Menger space which is not Scheepers. In section 3.4 we shall meet many centered Scheepers spaces which are not sub-Hurewicz. Finally, a Hurewicz metrizable non-σ- compact space was constructed in [42].

The problem of constructing a Menger non-Hurewicz topological space was posed by Hurewicz in [40] and then posed again in [42], [22], and [80].

The first known example (added in a footnote of [40] by W. Sierpinski) distinguishing these properties was a Luzin set (i.e., an uncountable subset L of $\mathbb{R}$ having countable intersection with each meager subset $M \subset \mathbb{R}$). But the existence of a Lusin set is independent of ZFC. The reason why each Lusin set X is Menger lies in the fact that it is concentrated at some countable subset. We recall that a topological space X is said to be concentrated at a subset $A \subset X$ if for any open neighborhood $U \subset X$ of A, the complement $X \setminus U$ has size $|X \setminus U| < |X|$. Using the fact that each properly ω-bounded multicovered space (X, μ) with $\mathrm{bc}(X, \mu) < \mathfrak{d}$ is Menger, one can prove that Lindelöf topological space X is Menger provided it has size $|X| \leq \mathfrak{d}$ and is concentrated at some countable subset $A \subset X$.

By a dichotomic argument (depending on the relation between the small cardinals $\mathfrak{b}$ and $\mathfrak{d}$) J. Chaber and R. Pol [24] constructed a subset $X \subset \omega^\omega$ of size $|X| = \mathfrak{b}$ that is concentrated at some countable set but cannot be included into a σ-compact subset of ω^ω. This construction resolves the original Hurewicz's problem giving a ZFC-example of a Menger non- Hurewicz metrizable space X of size $|X| = \mathfrak{b}$. Moreover, the non-Hurewicz space X constructed by Chaber and Pol has all finite powers X^n Menger. Consequently, X is Scheepers but not Hurewicz. Another space with the same properties was constructed by a non-dichotomic argument in [84].

A more refined version of the Hurewicz problem will also be considered in section 4.3.

2.2. **Preservation of selection properties by operations.** The selection properties defined above are preserved by many operations over multicovered spaces. In particular, they are preserved by uniformly bounded images.

Also *for a properly ω-bounded multicovered space X and a family $\mathcal{A}$ of Menger* (*resp. Hurewicz, sub-Hurewicz*) *subspaces of X with $|\mathcal{A}| < \mathfrak{b}$* (*resp. $|\mathcal{A}| < \mathfrak{b}$, $|\mathcal{A}| < \mathfrak{t}$*) *the union $\cup\mathcal{A}$ is Menger* (*resp. Hurewicz, sub-Hurewicz*), see [13, 2.3] for a corresponding topological result.

There is a deep connection between Scheepers and Menger properties via finite powers:

Theorem 9. *A (centered) multicovered space (X, μ) is Scheepers if (and only if) the power (X^n, μ^n) is Menger for every $n \in \omega$.*

In framework of topological groups this characterization was proved in [3]. Therefore the finite powers X^n, $n \in \omega$, of any centered Scheepers multicovered space X are Scheepers. The same assertion holds for Hurewicz spaces. Moreover, the product $X \times Y$ of a sub-Hurewicz multicovered space X and a Menger (resp. Scheepers) space Y is Menger (resp. Scheepers). In the framework of uniform spaces this result was proved by L.Kočinac [45, Th. 17, 18]. However under the negation of NCF there are two different centered Scheepers spaces with non-Scheepers product, see [8] or [28].

2.3. **Selection properties of subspaces of** $(\omega^{\uparrow\omega},\mu_p)$ **and** $(\mathbb{R}^\omega,\mu_G)$**.** The above selection properties of subspaces of the universal space $(\omega^{\uparrow\omega},\mu_p)$ have simple combinatorial characterizations. A subspace X of $\omega^{\uparrow\omega}$ is Menger (resp. Scheepers, Hurewicz) with respect to the multicover μ_p if and only if it is not dominating (resp. is not finitely dominating, is bounded) with respect to the preorder $\le^*$ of eventual dominance. For subspaces of $(\mathbb{R}^\omega,\mu_G)$ the characterization is similar.

Theorem 10. *A subspace A of $(\mathbb{R}^\omega,\mu_G)$ is Menger (resp. Scheepers, Hurewicz) if and only if the set $\|A\|=\{\|x\|:x\in A\}$ is not dominating (resp. not finitely dominating, bounded) in $(\mathbb{R}^\omega,\le^*)$, where $\|x\|_n=\max_{k\le n}|x_k|$.*

These characterizations combined with the countable nature of the selection properties and the universality of the space $(\omega^{\uparrow\omega},\mu_p)$ imply that *a uniformizable multicovered space (X,μ) is Menger (resp. Scheepers, Hurewicz) iff for every uniformly bounded morphism $\Phi:(X,\mu)\Rightarrow(\omega^{\uparrow\omega},\mu_p)$ the image $\Phi(X)$ is not dominating (resp. not finitely dominating, bounded) with respect to $\le^*$.* Consequently, *a properly ω-bounded multicovered space with $\mathrm{bc}_\omega(X)<\mathfrak{d}$ is Scheepers.* For topological spaces similar characterizations can be found in [40] (see also [42, Th. 4.3, 4.4]) and [81, Th. 2.1].

2.4. **Five natural multicovers of the Baire space** ω^ω**.** In this section we shall introduce five natural multicovers μ_m, μ_p, μ_u, μ_c, and μ_ℓ of the Baire space ω^ω. In fact, μ_u and μ_p have been introduced earlier.

We shall consider the infinite product ω^ω as $[\omega^{<\omega}]$, the set of branches of the tree $\omega^{<\omega}$. We set $\mu_m=\mathcal{O}(\omega^\omega)$, $\mu_\ell=\big\{\{\omega^\omega\setminus{\uparrow}s\}\cup\{{\uparrow}s^\wedge i:i\in\omega\}:s\in\omega^{<\omega}\big\}$, and $\mu_c=\mathsf{cen}(\mu_\ell)$, where $\uparrow s=\{x\in\omega^\omega: s \text{ is an initial segment of } x\}$.

The relations between these five multicovers are described by the following diagram in which the arrow from μ_i to μ_j means that the identity morphism $(\omega^\omega,\mu_i)\Rightarrow(\omega^\omega,\mu_j)$ is uniformly bounded.

$$\begin{tikzcd}
 & & \mu_u \arrow[dr] & \\
\mu_m \arrow[r] & \mu_p \arrow[ur] \arrow[dr] & & \mu_\ell \\
 & & \mu_c \arrow[ur] &
\end{tikzcd}$$

The multicovers μ_m, μ_p and μ_c are centered, while μ_ℓ and μ_u are not. $\cup\mu_i$ is a subbase of the usual product topology on ω^ω for every $i\in\{m,p,u,c,\ell\}$, and a subset of ω^ω is μ_i-bounded if and only if its closure in ω^ω is compact. Their cardinal characteristics also are very close: $b\chi(\omega^\omega,\mu_i)=\mathrm{cof}(\omega^\omega,\mu_i)=\aleph_0$, $\mathrm{bc}(\omega^\omega,\mu_i)=\mathfrak{d}$, for every $i\in\{p,u,c,\ell\}$ while $b\chi(\omega^\omega,\mu_m)=\aleph_0<\mathfrak{d}=\mathrm{cof}(\omega^\omega,\mu_m)=\mathrm{bc}(\omega^\omega,\mu_m)$. In spite of these similarities, selection properties of these multicovers differ substantially.

Theorem 11.
(1) The multicovered spaces (ω^ω, μ_u), (ω^ω, μ_p), and (ω^ω, μ_m) are not Menger.
(2) The multicovered space (ω^ω, μ_c) is Menger but its square $(\omega^\omega \times \omega^\omega, \mu_c \cdot \mu_c)$ is not Menger; consequently (ω^ω, μ_c) is not Scheepers.
(3) The multicovered space $(\omega^\omega, \mu_\ell)$ is sub-Hurewicz but not Hurewicz.

3. F-Menger multicovered spaces

In this section we shall look at selection principles from a bit more general point of view leading to an interesting and fruitful interplay between selection principles and semifilters.

3.1. **Semifilters.** First we give a very short survey of the theory of semifilters. For more detail information we refer the reader to the book [9] or the survey [11] of that book.

By a *semifilter* we understand a family $\mathcal{F}$ of nonempty subsets of ω, closed under taking almost supersets (which means that $\mathcal{F} \ni A \subset^* B \subset \omega \;\Rightarrow\; B \in \mathcal{F}$). The family of semifilters forms a lattice SF with respect to the operations of intersection and union. Besides these two operations there is an important operation of *transversal* assigning to each semifilter $\mathcal{F}$ its dual semifilter $\mathcal{F}^\perp = \{E \subset \omega : \forall F \in \mathcal{F} \;\; F \cap E \neq \emptyset\}$. The smallest element of the lattice SF is the Fréchet filter $\mathfrak{Fr}$ consisting of all cofinite subsets. Its dual $\mathfrak{Fr}^\perp$ is the largest semifilter consisting of all infinite subsets of ω. A semifilter $\mathcal{F}$ lies in its dual $\mathcal{F}^\perp$ if and only if $\mathcal{F}$ is *linked* which means that $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{F}$. Semifilters $\mathcal{F}$ with $\mathcal{F} = \mathcal{F}^\perp$ can be characterized as maximal linked semifilters. For example, any ultrafilter is a maximal linked semifilter.

Each family $\mathcal{B}$ of infinite subsets of ω induces a semifilter $\langle \mathcal{B} \rangle = \{F \subset \omega : \exists B \in \mathcal{B} \;\; B \subset^* F\}$. The smallest size of a family $\mathcal{B} \subset \mathfrak{Fr}^\perp$ with $\mathcal{F} = \langle \mathcal{B} \rangle$ (resp. $\mathcal{F} \subset \langle \mathcal{B} \rangle$) is called the *character* (resp. π- *character*) of a semifilter $\mathcal{F}$ and is denoted by $\chi(\mathcal{F})$ (resp. $\pi\chi(\mathcal{F})$).

For two semifilters $\mathcal{F}, \mathcal{U}$ we write $\mathcal{F} \Subset \mathcal{U}$ and say that $\mathcal{F}$ is *subcoherent* to $\mathcal{U}$ if $\Phi(\mathcal{F}) \subset \mathcal{U}$ for some finite-to-finite multifunction $\Phi : \omega \Rightarrow \omega$ (the latter means that for every $n \in \omega$ the sets $\Phi(n)$ and $\Phi^{-1}(n)$ are finite and non-empty). Two semifilters $\mathcal{F}, \mathcal{U}$ are called *coherent* if $\mathcal{F} \Subset \mathcal{U}$ and $\mathcal{U} \Subset \mathcal{F}$. In this case we write $\mathcal{F} \asymp \mathcal{U}$. The subset $[\mathcal{F}] = \{\mathcal{U} \in \mathsf{SF} : \mathcal{U} \asymp \mathcal{F}\}$ is called the *coherence class* of a semifilter $\mathcal{F}$. It follows from the Talagrand Theorem [69] that the coherence class $[\mathfrak{Fr}]$ (resp. $[\mathfrak{Fr}^\perp]$) consists of all meager (resp. comeager) semifilters, considered as subspaces of the power-set $\mathcal{P}(\omega)$ endowed with the natural compact metrizable topology. A semifilter $\mathcal{F}$ is *bi-Baire* if both $\mathcal{F}$ and $\mathcal{F}^\perp$ are Baire (equivalently non-meager) semifilters. This happens if and only if $\mathfrak{Fr} \not\asymp \mathcal{F} \not\asymp \mathfrak{Fr}^\perp$. For example, each maximal linked semifilter is bi- Baire. In the sequel by BS (resp. FF, UF, ML) we denote the subset of SF consisting of bi-Baire semifilters (resp. free filters, ultrafilters, maximal linked semifilters). A family $\mathsf{F} \subset \mathsf{SF}$ of semifilters is called $\asymp$*-invariant* if $[\mathcal{F}] \subset \mathsf{F}$ for every semifilter $\mathcal{F} \in \mathsf{F}$.

3.2. **Defining F-covers and F-Menger multicovered spaces.** Semifilters give a convenient tool for measuring "largeness" of subsets: in a sense elements of a semifilter can be thought as "large" subsets of ω. This allows us to assign to each family $\mathsf{F} \subset \mathsf{SF}$ of semifilter a selection property.

This will be done with help of the operation of so-called Marczewski numerization of an indexed cover u of X. Namely, given an indexed family $u = \{U_n : n \in \omega\}$ of subsets of X, let $u^\star_x = \{n \in \omega : x \in U_n\}$ be the *numerical star* of a point $x \in X$. The family $u^\star_X = \{u^\star_x : x \in X\} \subset \mathcal{P}(\omega)$ is called the *Marczewski numerization* of the indexed cover $u = \{U_n : n \in \omega\}$. If $u = \{U_n : n \in \omega\}$ is a large cover of X, then $u^\star_X$ consists of infinite subsets of ω and generates a semifilter $\langle u^\star_X\rangle = \{F \subset \omega : u^\star_x \subset^* F \text{ for some } x \in X\}$ called the *Marczewski semifilter* of u. If X is clear from the context, we shall omit the subscript and write $u^\star$ and $\langle u^\star\rangle$ instead of $u^\star_X$ and $\langle u^\star_X\rangle$.

In such a way we arrive at an important

Definition. Let $\mathsf{F} \subset \mathsf{SF}$ be a family of semifilters. An indexed sequence $v = \{B_n : n \in \omega\}$ of subsets of a set X is called an F-*cover* of X if its Marczewski numerization $v^\star_X$ lies in some semifilter $\mathcal{F} \in \mathsf{F}$.

A multicovered space (X, μ) is defined to be F-*Menger* (or else, satisfies the selection principle $\bigcup_{\mathrm{fin}}(\mu, \mathsf{F}^\star)$) if for any sequence of covers $(u_n)_{n\in\omega} \in \mu^\omega$ there is an F-cover $v = \{B_n : n \in \omega\}$ of X by u_n-bounded subsets $B_n \subset X$.

F-covers generalize large covers (=SF-covers), ω-covers (=UF-covers), γ-covers (=$\{\mathfrak{Fr}\}$-covers), and sub-γ-covers (=F_1-covers for the family $\mathsf{F}_1 = \{\mathcal{F} \in \mathsf{SF} : \chi(\mathcal{F}) = 1\}$ of semifilters generated by a single set).

Respectively, the selection properties corresponding to a particular type of cover are particular cases of the F-Menger property for suitable family F of semifilters.

Proposition 12. *A multicovered space (X, μ) is (1) Scheepers iff it is UF-Menger; (2) Menger iff it is SF-Menger iff it is $\{\mathfrak{Fr}^\perp\}$-Menger; (3) Hurewicz iff it is $\{\mathfrak{Fr}\}$-Menger; (4) sub-Hurewicz iff it is F_1-Menger for the family $\mathsf{F}_1 = \{\mathcal{F} \in \mathsf{SF} : \chi(\mathcal{F}) = 1\}$.*

Like the basic selection principles the F-Menger property is preserved by uniformly bounded images, and has countable nature in the sense that a multicovered space (X, μ) is F-Menger if and only if for each countable subcollection $\delta \subset \mu$ so is the multicovered space (X, δ). This observation allows us to prove that *a uniformizable multicovered space X is F-Menger if and only if for any uniformly bounded function $f : X \to (\omega^{\uparrow\omega}, \mu_p)$ the image $f(X)$ is an F-Menger subspace of $(\omega^{\uparrow\omega}, \mu_p)$.*

For families $\mathsf{F} \subset \mathsf{SF}$ closed under countable unions of semifilters, the F-Menger property is closed under countable unions as well.

3.3. **F-Menger property for an $\asymp$-invariant family F.** To obtain interesting results on the F-Menger property, one need take a relatively large family F of semifilters, containing together with each semifilter $\mathcal{F} \in \mathsf{F}$ all its finite-to-one images. In fact, each family $\mathsf{F} \subset \mathsf{SF}$ can be included into two larger families: $\mathsf{F}_\downarrow = \{\varphi(\mathcal{F}) : \mathcal{F} \in \mathsf{F},\ \varphi : \omega \to \omega$ is a finite-to-one map$\}$ and $\mathsf{F}_\asymp = \bigcup_{\mathcal{F}\in\mathsf{F}}[\mathcal{F}] = \{\mathcal{E} \in \mathsf{SF} : \exists \mathcal{F} \in \mathsf{F}$ with $\mathcal{E} \asymp \mathcal{F}\}$ possessing this property.

It turns out that *the* $\mathsf{F}_\downarrow$*-Menger and* $\mathsf{F}_\asymp$*- Menger properties coincide for uniformizable multicovered spaces.* Moreover, $\mathsf{F}_\downarrow$-Menger subspaces of the universal space $(\omega^{\uparrow\omega}, \mu_p)$ admit a simple description: they lie in sets of the form

$$M(b, \mathcal{F}) = \{(x_n) \in \omega^{\uparrow\omega} : \exists F \in \mathcal{F}\ \forall n \in F\ \max_{i\le n} x_i \le b(n)\}$$

for a suitable semifilter $\mathcal{F} \in \mathsf{F}_\downarrow$ and an increasing function $b : \omega \to \omega$. Therefore the study of the $\mathsf{F}_\asymp$-Menger property can be reduced to the studying the $[\mathcal{F}]$-Menger property for a suitable semifilter $\mathcal{F} \in \mathsf{F}$. The following statement collects some elementary properties of $[\mathcal{F}]$-Menger spaces.

Proposition 13. *(1) If a semifilter $\mathcal{F}$ is subcoherent to a semifilter $\mathcal{U}$, then each $[\mathcal{F}]$-Menger multicovered space is $[\mathcal{U}]$-Menger; (2) A centered multicovered space X (with $\operatorname{cof}(X,\mu) \le \aleph_0$) is Scheepers if (and only if) it is $[\mathcal{F}]$-Menger for some filter $\mathcal{F}$; (3) A centered multicovered space is Hurewicz iff it is sub-Hurewicz iff it is $[\mathfrak{Fr}]$-Menger.*

3.4. **Universal $[\mathcal{F}]$-Menger spaces.** In section 1.4 we presented examples of universal spaces in the class of properly ω-bounded metrizable multicovered spaces. The class of metrizable $[\mathcal{F}]$-Menger multicovered spaces has a universal element as well. Namely, for every semifilter $\mathcal{F}$ the subspace $\mathbb{M}(\mathcal{F}) = M(\mathrm{id}_\omega, \mathcal{F})$ of $(\omega^{\uparrow\omega}, \mu_p)$ endowed with the induced multicover $\mu_p|\mathbb{M}(\mathcal{F})$ is universal in the class of metrizable $[\mathcal{F}]$-Menger multicovered spaces.

Theorem 14. *A semifilter $\mathcal{F}$ is (sub)coherent to a semifilter $\mathcal{U}$ iff $\mathbb{M}(\mathcal{F})$ is isomorphic to (a subspace of) $\mathbb{M}(\mathcal{U})$.*

For example, for the Fréchet filter $\mathfrak{Fr}$ we get that the space $\mathbb{M}(\mathfrak{Fr})$ is universal in the class of metrizable σ-bounded multicovered spaces. The other extreme is the multicovered space $\mathbb{M} = \mathbb{M}(\mathfrak{Fr}^\perp)$, which is universal in the class of metrizable Menger multicovered spaces. $\mathbb{M}$ is a dense G_δ-subset of $\omega^{\uparrow\omega}$.

For a family $\mathsf{F} = \mathsf{F}_\downarrow \subset \mathsf{SF}$ there exists a universal multicovered space in the class of metrizable F-Menger spaces if and only if there exists a semifilter $\mathcal{F} \in \mathsf{F}$ such that $\mathsf{F} \subset [\mathcal{F}]$. Since the classes of Scheepers and UF-Menger spaces coincide, we conclude that there is a universal space in the class of Scheepers metrizable multicovered spaces if and only if the principle NCF holds (i.e. any two ultrafilters are coherent).

3.5. **Menger π-character of a Menger multicovered space.** As we already know each large cover $v = \{V_n : n \in \omega\}$ generates a corresponding Marczewski semifilter $v^\star_X$. This naturally leads us to the idea of applying $\asymp$-invariant cardinal characteristics of semifilters for studying multicovered spaces. Probably the most important among such cardinal characteristics is the π- character $\pi\chi[\mathcal{F}] = \min\{\pi\chi(\mathcal{U}) : \mathcal{U} \in [\mathcal{F}]\}$ of the coherence class of $\mathcal{F}$.

Definition. The *Menger π-character* $\pi\chi[X]$ of a Menger multicovered space (X, μ) is the smallest cardinal κ such that for each sequence of covers $(u_n)_{n\in\omega} \in \mu^\omega$ there is a large cover $v = \{B_n : n \in \omega\}$ of X by u_n-bounded subsets $B_n \subset X$ with $\pi\chi(\langle v_X^\star\rangle) \le \kappa$.

Obviously, a multicovered space (X, μ) is $[\mathcal{F}]$-Menger for any semifilter $\mathcal{F}$ with $\mathrm{non}[\mathcal{F}] > \pi\chi[X]$. Consequently, a centered multicovered space X is Hurewicz if $\pi\chi[X] < \mathfrak{b}$. We recall that $\mathrm{non}[\mathcal{F}]$ is the smallest cardinal κ such that any semifilter $\mathcal{U}$ with $\pi\chi(\mathcal{U}) < \kappa$ is subcoherent to $\mathcal{F}$.

There is a non-trivial equality $\mathrm{bc}_\omega(X) = \min\{\mathfrak{d}, \pi\chi[X]\}$ linking the Menger character of a Menger uniformizable multicovered space X with its countably-bounded covering number. This equality can be applied to show that each semifilter $\mathcal{F}$ with $\pi\chi[\mathcal{F}] < \mathfrak{d}$ is subcoherent to a filter.

Since each semifilter $\mathcal{F}$ with $\pi\chi[\mathcal{F}] < \mathfrak{b}$ (resp. $\pi\chi[\mathcal{F}] < \mathfrak{d}$, $\pi\chi[\mathcal{F}] < \mathfrak{c}$) is meager (resp. subcoherent to a filter, not comeager), we obtain the subsequent

Proposition 15. *A Menger multicovered space X is $[\mathfrak{Fr}]$-Menger (resp.* UF-*Menger,* BS-*Menger) provided $\pi\chi[X] < \mathfrak{b}$ (resp. $\pi\chi[X] < \mathfrak{d}$, $\pi\chi[X] < \mathfrak{c}$).*

3.6. **$\forall\mathsf{F}$-Menger multicovered spaces.** As we saw in the previous subsection, a multicovered space X with $\pi\chi[X] < \mathfrak{b}$, being $[\mathfrak{Fr}]$-Menger, is $[\mathcal{F}]$-Menger for **all** semifilters $\mathcal{F}$; if $\pi\chi[X] < \mathrm{non}_{\mathsf{BS}} = \min\{\mathrm{non}[\mathcal{F}] : \mathcal{F} \in \mathsf{BS}\}$, then X is $[\mathcal{F}]$-Menger for **all non-meager** semifilters, i.e., semifilters which are not minimal with respect to the subcoherence preorder.

Definition. Let $\mathsf{F} \subset \mathsf{SF}$ be a family of semifilters. A multicovered space X is called $\forall\mathsf{F}$-*Menger* if it is $[\mathcal{F}]$-Menger for each semifilter $\mathcal{F} \in \mathsf{F}$.

For example, a multicovered space X is $[\mathfrak{Fr}]$-Menger iff it is $\forall\mathsf{SF}$-Menger. For any family of semifilters $\mathsf{F} \subset \mathsf{SF}$ the class of $\forall\mathsf{F}$-Menger multicovered spaces is closed with respect to the operations of taking a subspace, the image under a uniformly bounded morphism, countable union of multicovered spaces. Also the $\forall\mathsf{F}$-Menger property has countable nature.

For a Menger multicovered space X and a family $\mathsf{F} \subset \mathsf{SF}$ we get

$$\pi\chi[X] < \mathrm{non}_{\mathsf{F}} \Rightarrow (X \text{ is } \forall\mathsf{F}\text{-Menger}) \Rightarrow \pi\chi[X] \le \pi\chi_{\mathsf{F}}$$

where $\mathrm{non}_{\mathsf{F}} = \min\{\mathrm{non}[\mathcal{F}] : \mathcal{F} \in \mathsf{F}\}$ and $\pi\chi_{\mathsf{F}} = \min\{\pi\chi[\mathcal{F}] : \mathcal{F} \in \mathsf{F}\}$.

Next we consider two particular cases of this concept: $\forall\mathsf{BS}$-Menger and $\forall\mathsf{UF}$- Menger multicovered spaces, where BS and UF stand for the families of bi-Baire semifilters and ultrafilters, respectively. Taking into account that $\pi\chi_{\mathsf{BS}} = \mathfrak{b}$, $\pi\chi_{\mathsf{UF}} = \mathfrak{r}$, and $\mathrm{non}_{\mathsf{SF}} = \mathfrak{b}$ we get the following statement.

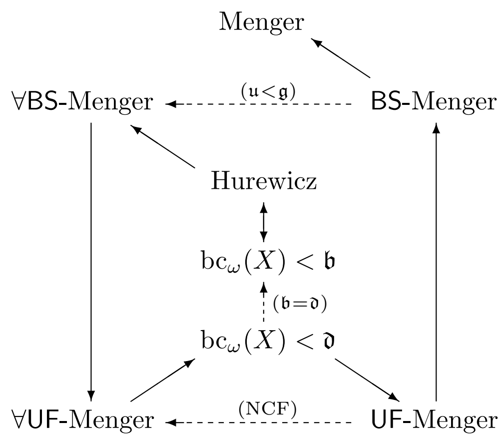
Theorem 16. *For every Menger multicovered space X the subsequent implication hold:*

- $\pi\chi[X] < \mathfrak{b} \quad \Rightarrow (X$ *is* $\forall\mathsf{SF}$- *Menger*$) \Leftrightarrow (X$ *is* $[\mathfrak{Fr}]$-*Menger*$)$*;*
- $\pi\chi[X] < \mathrm{non}_{\mathsf{BS}} \Rightarrow (X$ *is* $\forall\mathsf{BS}$- *Menger*$)$*;*

• $\pi\chi[X] \le \mathfrak{b} \quad\quad \Leftarrow$ *(X is $\forall\mathsf{BS}$- Menger);*
• $\pi\chi[X] < \mathrm{non}_{\mathsf{UF}} \Rightarrow$ *(X is $\forall\mathsf{UF}$- Menger);*
• $\pi\chi[X] \le \mathfrak{r} \quad\quad \Leftarrow$ *(X is $\forall\mathsf{UF}$- Menger).*

Moreover, if X is a $\forall\mathsf{UF}$-Menger uniformizable multicovered space, then
• $\mathrm{bc}_\omega(X) = \pi\chi[X] < \mathfrak{d}$*;*
• *X is $\forall\mathsf{SF}_{\mathrm{non}=\mathfrak{d}}$-Menger for the family $\mathsf{SF}_{\mathrm{non}=\mathfrak{d}} = \{\mathcal{F} \in \mathsf{SF} : \mathrm{non}[\mathcal{F}] = \mathfrak{d}\}$;*
• *X is $\mathsf{FF}_{\chi<\mathfrak{d}}$-Menger for the family $\mathsf{FF}_{\chi<\mathfrak{d}} = \{\mathcal{F} \in \mathsf{SF} : \mathcal{F}$ is a filter with $\chi(\mathcal{F}) < \mathfrak{d}\}$.*

The preceding results imply the following "spiral" of Menger properties holding for any uniformizable multicovered space X. The dashed arrows indicate implications holding under a suitable set-theoretic assumption.



3.7. **Products of $[\mathcal{F}]$-Menger multicovered spaces.** Since the class of centered Scheepers spaces with countable cofinality consists of centered $[\mathcal{F}]$-Menger spaces where $\mathcal{F}$ runs over filters, the problem of preservation of the Scheepers property by products can be reduced to studying products of $[\mathcal{F}]$-Menger spaces for different (semi)filters $\mathcal{F}$.

We shall say that semifilters $\mathcal{F}_1, \dots, \mathcal{F}_n$ are *jointly subcoherent to a semifilter* $\mathcal{F}$ on ω if there is a finite-to-finite multifunction $\Phi : \omega \Rightarrow \omega$ such that $\bigcap_{i=1}^n \Phi(F_i) \in \mathcal{F}$ for any elements $F_i \in \mathcal{F}_i$, $i \in [1, n]$. For example, for every semifilter $\mathcal{F}$ the semifilters $\mathcal{F}, \mathcal{F}^\perp$ are jointly subcoherent to the largest semifilter $\mathfrak{Fr}^\perp$. Also for every filter $\mathcal{F}$ the semifilters $\mathcal{F}, \dots, \mathcal{F}$ are jointly subcoherent to $\mathcal{F}$.

The joint subcoherence appears quite naturally in questions about products and unions of $[\mathcal{F}]$-Menger spaces.

Theorem 17. *For semifilters $\mathcal{F}, \mathcal{F}_1, \dots, \mathcal{F}_n$ the following conditions are equivalent:*
(1) The product $X_1 \times \cdots \times X_n$ of centered $[\mathcal{F}_i]$-Menger multicovered spaces X_i is $[\mathcal{F}]$-Menger;
(2) The semifilters $\mathcal{F}_1, \dots, \mathcal{F}_n$ are jointly subcoherent to $\mathcal{F}$.

Moreover, if $\mathcal{F}$ is a filter, then the conditions (1),(2) *are equivalent to*

(3) The semifilters $\mathcal{F}_1, \dots, \mathcal{F}_n$ are subcoherent to $\mathcal{F}$;
(4) The union $\mathbb{M}(\mathcal{F}_1) \cup \dots \cup \mathbb{M}(\mathcal{F}_n)$ is $[\mathcal{F}]$-Menger in $(\omega^{\uparrow\omega}, \mu_p)$.

Consequently, for any filter $\mathcal{F}$, the $[\mathcal{F}]$-Menger as well as $\forall\mathsf{UF}$-Menger properties are preserved by finite products of uniformizable multicovered spaces. The situation with products of $\forall\mathsf{BS}$-Menger spaces is not so clear.

Question 3. *Is the $\forall\mathsf{BS}$-Menger property preserved by finite products of metrizable multicovered spaces?*

The answer is affirmative provided $\mathfrak{b} < \mathrm{non}_{\mathsf{BS}}$ or $\mathfrak{b}^{\perp}([\mathsf{SF}]^{\circ}) > 1$.

The above results permit us to characterize the F-Menger and $\forall\mathsf{F}$- Menger spaces via their product properties.

Corollary 18. *A uniformizable multicovered space (X, μ) is $[\mathcal{F}]$-Menger for a semifilter $\mathcal{F}$ if and only if the product $X \times \mathbb{M}(\mathcal{F}^{\perp})$ is Menger.*

Consequently, a uniformizable multicovered space X is $\forall\mathsf{F}$-Menger for a family $\mathsf{F} = \mathsf{F}_{\downarrow} \subset \mathsf{SF}$ of semifilters if and only if for every semifilter $\mathcal{F} \in \mathsf{F}$ the product $X \times \mathbb{M}(\mathcal{F}^{\perp})$ is Menger.

This yields a characterization of $\forall\mathsf{UF}$-Menger spaces as the spaces X whose product $X \times Y$ with any Scheepers space Y is Scheepers.

And finally, the above results give us an approach to studying multicovered spaces whose finite powers are Menger. Namely, *for a centered multicovered space X (of countable cofinality) an n-th power X^n of X is Menger if and only if X is L_n-Menger for the family L_n of n-linked semifilters (if and only if X is $[\mathcal{L}]$-Menger for some n-linked semifilter $\mathcal{L}$).*

It is consistent that all centered multicovered spaces with Menger square are Scheepers. This is equivalent to the set-theoretic assumption that *each maximal linked semifilter is coherent to an ultrafilter,* which implies $\mathrm{NCF}_{<\omega}$ and follows from $\mathfrak{u} < \max\{\mathfrak{g}, \mathfrak{s}\}$, see [10]. We recall that the principle $\mathrm{NCF}_{<\omega}$ asserts that there are only finitely many non- coherent ultrafilters. This is one of two possibilities allowed by the Finite-$2^{\mathfrak{c}}$ Dichotomy [7] asserting that the number of distinct coherence classes of ultrafilters is either finite or $2^{\mathfrak{c}}$. Under the negation of $\mathrm{NCF}_{<\omega}$, for every $n \geq 2$ there is a maximal n-linked semifilter $\mathcal{F}$ such that $\mathcal{F} \Subset \mathcal{L}$ for no $(n+1)$-linked semifilter $\mathcal{L}$. For such a semifilter $\mathcal{F}$ the space $\mathbb{M}(\mathcal{F})^n$ is Menger, while $\mathbb{M}(\mathcal{F})^{n+1}$ is not.

3.8. **Implications of NCF, CML, and $(\mathfrak{u} < \mathfrak{g})$.** In this section we collect some results on selection properties of multicovered spaces, which can be proved only under certain additional set-theoretic assumptions. We shall consider effects of the following three assumptions: NCF, CML, and $(\mathfrak{u} < \mathfrak{g})$, which are independent of ZFC and relate as follows:

$$(\mathfrak{u} < \mathfrak{g}) \Rightarrow (\mathrm{CML}) \Rightarrow (\mathrm{NCF}) \Rightarrow (\mathrm{NCF}_{<\omega}) \Rightarrow (\mathfrak{u} < \mathfrak{d}) \Rightarrow (\neg\mathrm{MA})$$

3.8.1. *Implications of NCF.* Since NCF is responsible for coherence of (ultra)filters, many natural questions concerning universal spaces, products or unions of Scheepers spaces can be resolved affirmatively if and only if the principle NCF holds. We recall that NCF asserts that all ultrafilters are coherent.

Theorem 19. *The following conditions are equivalent: (1) the principle NCF holds; (2) for any ultrafilter $\mathcal{U}$ on ω the space $\mathbb{M}(\mathcal{U})$ is universal in the class of metrizable Scheepers multicovered spaces; (3) any* UF*-Menger multicovered space is* $\forall\mathsf{UF}$*-Menger; (4)* $\mathrm{bc}_\omega(X) < \mathfrak{d}$ *for any uniformizable Scheepers space X; (5) the union $\cup\mathcal{A}$ of any family $\mathcal{A}$ of Scheepers subspaces of an ω-bounded uniformizable multicovered space X of size $|\mathcal{A}| < \mathfrak{d}$ is Scheepers; (6) the union of two Scheepers metrizable multicovered spaces is Scheepers; (7) the product of finitely many centered Scheepers multicovered spaces is Scheepers; (8) the product of two Scheepers metrizable multicovered spaces is Menger.*

3.8.2. *Implications of CML.* The principle CML asserts that any two maximal linked semifilters on ω are coherent. Formally, it is stronger than NCF and is responsible for the behavior of ML-Menger spaces. Let us recall that a centered multicovered space is ML-Menger iff it has Menger square X^2.

Theorem 20. *The following conditions are equivalent: (1) the principle CML holds; (2) there is a universal space in the class of metrizable multicovered spaces with Menger square; (3) any centered multicovered space with Menger square is* $\forall\mathsf{UF}$*-Menger.*

3.8.3. *Implications of* $(\mathfrak{u} < \mathfrak{g})$. Finally, we establish what happens under $(\mathfrak{u} < \mathfrak{g})$, the strongest among the considered assumptions contradicting Martin's Axiom. The consistency of $(\mathfrak{u} < \mathfrak{g})$ was proved by Blass and Shelah [17] or [18].

The following theorem characterizing the assumption $(\mathfrak{u} < \mathfrak{g})$ shows that it has as same effect on the class of BS-Menger spaces as NCF has on the class of Scheepers (= UF-Menger) spaces.

Theorem 21. *The following conditions are equivalent: (1)* $\mathfrak{u} < \mathfrak{g}$*; (2) any two bi-Baire semifilters are coherent; (3) each (metrizable) uniformizable* BS*-Menger multicovered space has Menger square; (4) any* BS*-Menger multicovered space is* $\forall\mathsf{BS}$*-Menger; (5) the product of two centered* BS*-Menger multicovered spaces is* BS*-Menger.*

3.9. **Cardinal characteristics of the family of F-Menger spaces.** In this section we calculate cardinal characteristics of families of multicovered spaces satisfying Selection Principles considered in preceding sections. We shall be interested in the following four classical cardinal characteristics defined for any family $\mathcal{I} \subset \mathcal{P}(X)$ with $\cup\mathcal{I} = X \notin \mathcal{I}$:

$\mathrm{add}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{I}$ and $\bigcup\mathcal{A} \notin \mathcal{I}\}$,
$\mathrm{non}(\mathcal{I}) = \min\{|A| : A \subset X$ and $A \notin \mathcal{I}\}$,
$\mathrm{cov}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{I}$ and $\cup\mathcal{A} = X\}$,
$\mathrm{cof}(\mathcal{I}) = \min\{|\mathcal{B}| : \mathcal{B} \subset \mathcal{I}$ and $\forall A \in \mathcal{I}\ \exists B \in \mathcal{B}$ with $A \subset B\}$.

These cardinal characteristics will be studied for the collection $\mathcal{I} = \mathsf{M}_\mathsf{F}(X)$ of all F-Menger subspaces of a given non-Menger multicovered space X, where F is a suitable family of semifilters. instead of $\mathsf{M}_{\mathsf{SF}}(X)$. Observe that the family $\mathsf{M}(X)$ consists of all Menger subspaces of the multicovered space X. If $X = (\omega^{\uparrow\omega}, \mu_p)$, then we shall write M_{F} instead of $\mathsf{M}_{\mathsf{F}}(X) = \mathsf{M}_{\mathsf{F}}(\omega^{\uparrow\omega}, \mu_p)$. The space $X = (\omega^{\uparrow\omega}, \mu_p)$ deserves a special attention because it is universal in the class of ω-bounded metrizable multicovered spaces.

We shall compare these characteristics with cardinal characteristics of the family F, see [11] for their definitions. The relationships between them are described by the diagram in Figure 1. In this diagram the family $\mathrm{Supp}(\mathsf{F})$ consists of semifilters $\mathrm{Supp}(\mathcal{F})$ for $\mathcal{F} \in \mathsf{F}$ where $\mathrm{Supp}(\mathcal{F}) = \{S \in \mathcal{F}^\perp : S \wedge \mathcal{F} \Subset \mathcal{F}\}$ with $S \wedge \mathcal{F} = \langle S \cap F : F \in \mathcal{F}\rangle$.

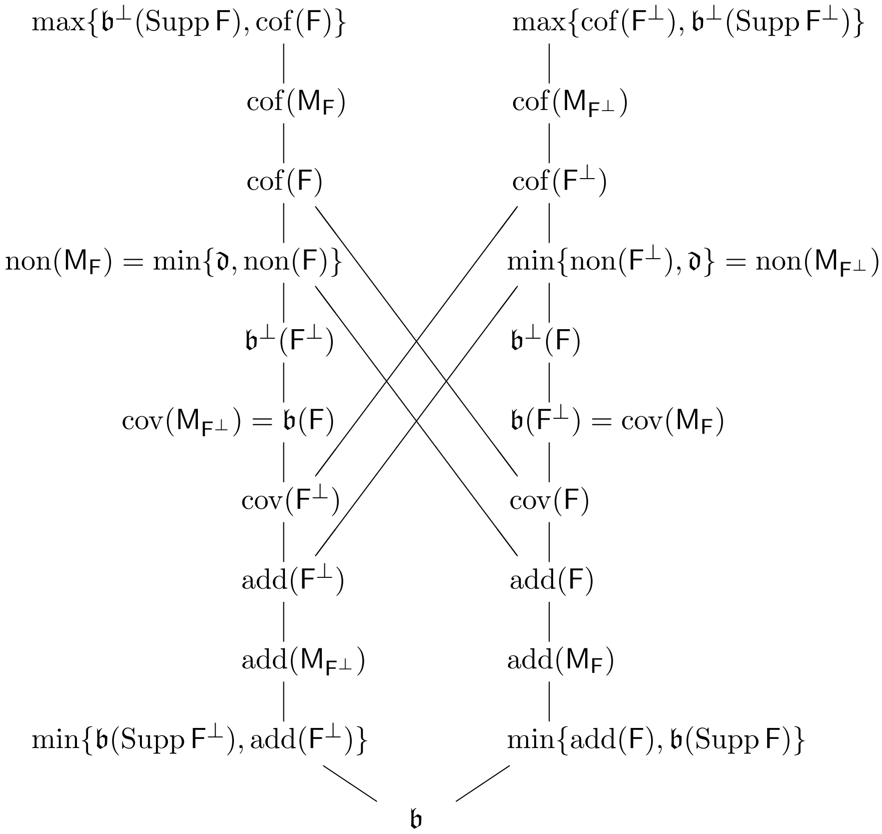


FIGURE 1. Inequalities between cardinal characteristics of the family M_{F} of F-Menger subspaces of $(\omega^{\uparrow\omega}, \mu_p)$ for an $\asymp$-invariant family F of bi-Baire semifilters.

This diagram implies that *for a non-Menger ω-bounded metrizable multicovered space X and a semifilter $\mathcal{F}$ whose dual $\mathcal{F}^\perp$ is a filter, we get*

- $\mathrm{add}(\mathsf{M}_{[\mathcal{F}]}(X)) = \mathrm{add}[\mathcal{F}] = \min\{\mathfrak{b}^\perp(\mathcal{F}), \mathfrak{q}^\perp(\mathcal{F})\} = \mathrm{cov}[\mathcal{F}] = \mathrm{cov}(\mathsf{M}_{[\mathcal{F}]}(X))$*;*
- $\mathrm{non}(\mathsf{M}_{[\mathcal{F}]}(X)) = \mathrm{non}[\mathcal{F}] = \max\{\mathfrak{b}(\mathcal{F}), \mathfrak{q}(\mathcal{F})\} = \mathrm{cof}[\mathcal{F}] \geq \mathrm{cof}(\mathsf{M}_{[\mathcal{F}]}(X))$*.*

If X is isomorphic to (ω^ω, μ_p), then $\mathrm{cof}(\mathsf{M}_{[\mathcal{F}]}(X)) = \mathrm{cof}[\mathcal{F}]$.

Also we can estimate the cardinal characteristics of the family M_{F} for families $\mathsf{F} \in \{\mathsf{BS}, \mathsf{UF}, \mathsf{ML}\}$:

- $\mathfrak{g} = \mathrm{add}(\mathsf{M}_{\mathsf{BS}}) \le \max\{\mathfrak{b}, \mathfrak{g}\} = \mathrm{cov}(\mathsf{M}_{\mathsf{BS}}) \le \mathfrak{d} = \mathrm{non}(\mathsf{M}_{\mathsf{BS}}) \le \mathrm{cof}(\mathsf{M}_{\mathsf{BS}})$;
- $\{2, \mathfrak{d}\} \ni \mathrm{add}(\mathsf{M}_{\mathsf{UF}}) \le \mathrm{cov}(\mathsf{M}_{\mathsf{UF}}) \le \mathrm{non}(\mathsf{M}_{\mathsf{UF}}) = \mathfrak{d} \le \mathrm{cof}(\mathsf{M}_{\mathsf{UF}}) \in \{\mathfrak{d}, 2^{\mathfrak{c}}\}$;
- $\{2, \mathfrak{d}\} \ni \mathrm{add}(\mathsf{M}_{\mathsf{ML}}) \le \mathrm{cov}(\mathsf{M}_{\mathsf{ML}}) \le \mathrm{non}(\mathsf{M}_{\mathsf{ML}}) = \mathfrak{d} \le \mathrm{cof}(\mathsf{M}_{\mathsf{ML}}(X)) \le 2^{\mathfrak{c}}$.

4. Selection Principles in topological spaces

This section is devoted to studying Selection Principles in topological spaces. This is the most elaborated part in Selection Principles Theory and first problems and concepts of the theory appeared just in the topological context. The great number of recent survey papers (see, e.g., [46], [47], [65], [80], [79], [83], and [66]) devoted to this subject is an evidence of its intensive development.

Of course, many results discussed in the preceding sections apply also to topological spaces X (identified with the multicovered spaces $(X, \mathcal{O})$ carrying the multicover $\mathcal{O}$ consisting of all open covers of X). However, multicovered spaces appearing in the topological setting have some distinctive features.

Firstly, they rarely have countable cofinality. Because of that feature, it is not so easy to construct topological examples distinguishing between various $[\mathcal{F}]$-Menger properties. In fact, even the problem of constructing Menger topological space which is not Hurewicz turned out to be non-trivial.

The second distinctive feature of multicovered spaces appearing from topological spaces is their odd behavior with respect to products. Namely, for two topological spaces $(X, \mathcal{O}(X))$ and $(Y, \mathcal{O}(X))$ the product multicover $\mathcal{O}(X)\cdot\mathcal{O}(Y)$ on $X\times Y$ generally is not equivalent to the multicover $O(X\times Y)$, which makes impossible application of general result on products of multicovered spaces to studying selection principles in products of topological spaces. As an illustration of this pathology one can look at [42, 2.11] describing a Hurewicz subspace X of reals whose square X^2 admits a (uniformly) continuous map onto the irrationals and thus is not Menger. So, studying selection principles in products of topological spaces usually requires a careful separate treatment.

The third distinctive feature of topological multicovers is rather of positive character and is connected with the fact that such a multicover $\mathcal{O}$ is closed under forming countable covers by elements of the union $\cup\mathcal{O}$ (which is the topology). This feature allows us to obtain some specific (and very helpful) characterizations of the F-Menger property in topological spaces, not valid for general multicovered spaces.

Throughout the section, saying that a topological space X has some selection property we shall understand that the multicovered space $(X, \mathcal{O})$ has that property.

4.1. **F-Menger property in (strongly) diagonalizable multicovered spaces.** We define a multicovered space (X,μ) to be *(strongly) diagonalizable*, if the multicover μ contains any countable cover of X by elements of $\cup\mu$ (of the smallest family containing $\cup\mu$ and closed under finite intersections). The class of strongly diagonalizable multicovered spaces includes multicovered spaces of the form $(X,\mathcal{O})$, $(X,\mathcal{O}_\omega)$, and $(X,\mathcal{B}_\omega)$, where X is a topological space and $\mathcal{B}_\omega$ ($\mathcal{O}_\omega$) denotes the family of all countable Borel (open) covers of X, respectively.

There is a tight connection between selection properties of strongly diagonalizable multicovered spaces X and topological properties of Marczewski semifilters of their μ-open large covers, where a subset $U\subset X$ is *μ-open* with respect to the multicover μ if it can be written in the form $U=\cup v$ for some countable subfamily v of a cover $u\in\mu$. In the sequel Marczewski semifilters are considered as topological spaces endowed with the (metrizable separable) topology inherited from $\mathcal{P}(\omega)$. This permits us to speak about topological properties of semifilters, for example the (F-)Menger property. A crucial observation is that for every strongly diagonalizable multicovered space (X,μ) and any μ-open large cover $v=\{V_n:n\in\omega\}$ of X the multifunction $\uparrow v^\star : x\mapsto \uparrow v^\star_x=\{A\subset\omega : v^\star_x\subset A\}$ is uniformly bounded as a morphism $\uparrow v^\star : (X,\mu)\Rightarrow(\langle v^\star_X\rangle,\mathcal{O})$.

A topological version of the above result [88] states that the set-valued map $\uparrow v^\star$ is compact-valued upper semicontinuous, i.e. for every open $W\subset\wp(\omega)$ the preimage $(\uparrow v^\star)^{-1}_{\subset}(W)=\{x\in X:\uparrow v^\star(x)\subset W\}$ is open in X. This implies that the Marczewski semifilter $\langle v^\star\rangle$ of v is F- Menger as a topological space. The topological properties of semifilters appear to be much stronger than corresponding combinatorial ones: every F-Menger semifilter $\mathcal{U}$ lies in some semifilter $\mathcal{F}\in\mathsf{F}$, while there exists a meager semifilter without the Menger property.

The above observation implies the following characterization.

Theorem 22. *A Menger strongly diagonalizable multicovered space X is F-Menger for an $\asymp$-invariant family of semifilters F iff each large cover $v=\{V_n:n\in\omega\}\in\mu$ of X is an F-cover iff $\langle v^\star_X\rangle$ is F-Menger for each large μ-open cover $v=\{V_n:n\in\omega\}$ of X iff X is F_{top}-Menger.*

(Here F_{top} denotes the family of F-Menger semifilters.)

The preceding result motivates the study of F-Menger semifilters. Such semifilters are relatively small. More precisely, they cannot be coherent to rapid semifilters. A semifilter $\mathcal{F}$ is defined to be *rapid*, if for each function $f:\omega\to\omega$ there is a set $F\in\mathcal{F}$ whose enumerating function $e_F\ge f$. If F contains no rapid semifilter and the multicovered space (X,μ) is strongly paracompact, then the "Menger" assumption can be dropped in the above characterization of the F-Menger property. (We define a multicovered space X to be *strongly paracompact* if for each cover $u\in\mu$ there is a star-finite cover $v\in\mu$ such that $v\succ u$. A cover v of X is *star-finite* if for any $V\in v$ the set $\{V'\in v : V'\cap V\ne\emptyset\}$ is finite.) Examples of strongly paracompact spaces include regular Lindelöf spaces endowed with the multicover of open covers. Since no meager semifilter is rapid, we get the following characterization of the Hurewicz property obtained in [88] and [78] independently.

Theorem 23. *A regular Lindelöf topological space (X, μ) is Hurewicz iff the Marczewski semifilter $\langle v^\star_X \rangle$ of each large cover v of X is meager.*

One can introduce some new selection principles using stars of covers. For example, a multicovereds space (X, μ) is *star-Menger*, if so the space $(X, \{\mathcal{S}t(u) : u \in \mu\})$. It is easy to see that these selection principles coincide with the classical ones for strongly paracompact spaces. Such the properties are intensively studied in the realm of topological spaces, see [19], [20], and [44].

4.2. **Characterizing the F-Menger property in topological spaces.** The F-Menger property in topological spaces can be characterized in many different ways.

4.2.1. *Characterizing F-Menger metrizable spaces.* For a metrizable topological space X, the multicovered space $(X, \mathcal{O})$ is F-Menger if and only if for every metric ρ generating the topology of X the multicovered space (X, μ_ρ) is F-Menger. The latter happens if and only if for every $\varepsilon > 0$ there is an F-cover $v = \{V_n : n \in \omega\}$ of X with $\sup_{n\in\omega} \mathrm{diam}_\rho(V_n) < \varepsilon$ and $\lim\limits_{n\to\infty} \mathrm{diam}_\rho(V_n) = 0$.

There are also equivalents of the F-Menger property in terms of properties of bases of its topology with respect to some (all) admissible metrics, namely that the family v as above can be chosen from an arbitrary base $\mathcal{B}$ of the topology of X.

Another condition characterizing the F-Menger property does not appeal to any admissible metrics.

Theorem 24. *A metrizable topological space X is F-Menger iff every open base for X contains an F-cover u of X which is almost point (or locally) finite in the sense that for every open subset W of X the family $\{U \in u : u\backslash W \neq \emptyset\}$ is point (locally) finite at each point $x \in W$.*

Eliminating “F-” from the above characterizations we get a characterization of the Menger property due to Lelek [55].

4.2.2. *Characterizing by continuous images.* This kind of characterization traces its history back to Hurewicz [40] who showed that the negation of the Menger property is equivalent to the existence of a dominating image with respect to $\leq^*$. Similar characterization of some other selection properties were obtained in [81], [82], [89].

The most general form of such a characterization involves compact-valued upper semicontinuous maps.

Theorem 25.
(1) A Lindelöf topological space X is F-Menger iff for any compact-valued

upper semicontinuous multifunction $\Phi : X \Rightarrow \omega^\omega$ *the image* $\Phi(X)$ *is an* F*-Menger subspace of* (ω^ω, μ_p).
(2) A (zero-dimensional) Lindelöf topological space X *is* F*-Menger iff for any compact-valued upper semicontinuous multifunction* $\Phi : X \Rightarrow \mathbb{R}^\omega$ *the image* $\Phi(X)$ *is an* F*-Menger subspace of* $(\mathbb{R}^\omega, \mu_G)$ *(iff for any continuous map* $f : X \to \omega^\omega$ *the image* $f(X)$ *is an* F*-Menger subspace of* (ω^ω, μ_p) *).*

Some related results (including partial cases $\mathsf{F} = \{\mathfrak{Fr}\}$ and $\mathsf{F} = \{\mathfrak{Fr}^\perp\}$) may be found in [48].

The Lindelöfness of Menger topological spaces makes it impossible to extend the above characterization beyond the class of Lindelöf spaces in its present form. But we can replace the multicover $\mathcal{O}$ of a topological space X by a smaller one. One of the natural choices for such a smaller multicover seems to be $\mathcal{O}_\omega = \{u \in \mathcal{O} : |u| = \omega\}$. But what can we really characterize via continuous images are the selection properties of the multicover $\mathcal{O}^f_\omega$ of all countable covers by functionally open subsets of X. We recall that a subset $U \subset X$ is *functionally open* if $U = f^{-1}(V)$ for some continuous function $f : X \to \mathbb{R}$ and some open set $V \subset \mathbb{R}$.

Proposition 26. *For a topological space* X *the multicovered space* $(X, \mathcal{O}^f_\omega)$ *is* F*-Menger iff for any continuous function* $f : X \to \mathbb{R}^\omega$ *the image* $f(X)$ *is an* F*-Menger subspace of* $(\mathbb{R}^\omega, \mu_G)$.

In light of this it is important to detect topological spaces X for which the multicovers $\mathcal{O}_\omega$ and $\mathcal{O}^f_\omega$ are equivalent. The class of such spaces includes regular Lindelöf, countably compact, and perfectly normal spaces, and is closed under taking closed subspaces.

4.2.3. *Characterizing* F*-Menger property in terms of covers by compacta.* It was shown in [42] that a subspace $X \subset \mathbb{R}$ is Hurewicz iff for every G_δ-subset G of $\mathbb{R}$ containing X there exists a σ-compact set C with $X \subset C \subset G$. The same approach still works in a more general situation.

Theorem 27. *A Lindelöf regular space* X *is Hurewicz iff for every compactification* cX *of* X *and every* G_δ*-subset* G *of* cX *containing* X *there exists a family* $\mathcal{C}$ *of compact subsets of* cX *such that* $X \subset \cup\mathcal{C} \subset G$ *and* $|\mathcal{C}| < \mathfrak{b}$.

Under some additional set-theoretic assumptions this characterization may be extended to Scheepers topological spaces. Namely the Scheepers property can be characterized in the same way via the families $\mathcal{C}$ of compact subsets of size $|\mathcal{C}| < \mathfrak{d}$ provided the NCF principle holds. In addition, under $\mathfrak{u} < \mathfrak{g}$ the Scheepers and Menger properties coincide for topological spaces, and hence the same characterization holds for the Menger property under $\mathfrak{u} < \mathfrak{g}$. This means that the condition

> For every Čech-complete space $G \supset X$ there exists a family $\mathcal{C}$ of compact subsets of G such that $X \subset \cup\mathcal{C} \subset G$ and $|\mathcal{C}| < \mathfrak{d}$

is universal in the sense that under $\mathfrak{b} = \mathfrak{d}$, NCF, and $\mathfrak{u} < \mathfrak{g}$ it is equivalent to the Hurewicz, Scheepers, and Menger properties of a Lindelöf regular topological space X, respectively.

It also implies that under NCF (under $\mathfrak{u} < \mathfrak{g}$) the Scheepers (Menger) property is preserved by unions of families of subspaces of regular hereditarily Lindelöf spaces of size $< \mathfrak{d}$.

4.3. **Constructing examples of $[\mathcal{F}]$-Menger topological spaces.** As we saw there are ZFC-examples of Menger topological spaces which are not Hurewicz. In this section we address a more precise question: *For which pairs of semifilters $\mathcal{F}, \mathcal{U}$ is there an $[\mathcal{F}]$-Menger topological space which is not $[\mathcal{U}]$-Menger?*

To construct examples of topological $[\mathcal{F}]$-Menger spaces we develop the machinery of concentrated sets already exploited in section 2.1, see also [24] and [84]. Let $\mathsf{F} \subset \mathsf{SF}$ be a family of semifilters and $\omega_*^{\uparrow\omega}$ denote the set of bounded non-decreasing functions in $\omega^{\uparrow\omega}$. A subset $X \subset \omega^\omega$ is called F-*concentrated* at $\omega_*^{\uparrow\omega}$, if $\omega_*^{\uparrow\omega} \subset X \subset \omega^{\uparrow\omega}$ and for any unbounded function $f \in \omega^{\uparrow\omega}$ and any semifilter $\mathcal{F} \in \mathsf{F}$ the complement $X \setminus M(f, \mathcal{F})$ has size $< |X|$. For example, a subset $X \subset \omega_*^{\uparrow\omega}$ is concentrated at $\omega_*^{\uparrow\omega}$ in the sense of the definition used in section 2.1 if (and only if) it is $\{\mathfrak{Fr}^\perp\}$- concentrated at $\omega_*^{\uparrow\omega}$ (and the size $|X|$ of X has uncountable cofinality).

Let $\mathcal{F}$ be a semifilter. As expected, every $[\mathcal{F}]$-concentrated at $\omega_*^{\uparrow\omega}$ set $X \subset \omega^{\uparrow\omega}$ of size $|X| \le \mathrm{non}(\mathsf{M}_{[\mathcal{F}]})$ is $[\mathcal{F}]$-Menger as a topological space. The above statement has also a product version. Given semifilters $\mathcal{F}_1, \dots, \mathcal{F}_n$ let $\mathcal{F}_1 \wedge \dots \wedge \mathcal{F}_n = \{F_1 \cap \dots \cap F_n : F_i \in \mathcal{F}_i \text{ for } i \le n\}$. The family $\mathcal{F}_1 \wedge \dots \wedge \mathcal{F}_n$ is a semifilter if it does not contain the empty set. In such a situation the product $X = \prod_{i=1}^n X_i$ is $\mathcal{F}_1 \wedge \dots \wedge \mathcal{F}_n$-Menger provided $X_i \subset \omega^{\uparrow\omega}$ is an $[\mathcal{F}_i]$-concentrated subset at $\omega_*^{\uparrow\omega}$ of size $|X_i| \le \mathrm{add}(\mathsf{M}_{[\mathcal{F}_1 \wedge \dots \wedge \mathcal{F}_n]})$ for all $i \le n$. In particular, if $\mathcal{F}$ is a filter and $|X| \le \mathrm{add}(\mathsf{M}_{[\mathcal{F}]})$, then all finite powers of X are $[\mathcal{F}]$- Menger.

Our next aim is to construct an $[\mathcal{F}]$-Menger space X failing to be $[\mathcal{U}]$-Menger for a semifilter $\mathcal{U}$. Of course, such a semifilter $\mathcal{U}$ cannot be arbitrary: it must lie in the set $\mathsf{SF} \setminus \{\mathcal{U} \in \mathsf{SF} : \mathcal{F} \not\in \mathcal{U}\}$.

Theorem 28. *Let $\mathcal{F}$ be a semifilter with $\kappa = \mathrm{cof}(\mathsf{M}_{[\mathcal{F}^\perp]}) = \mathrm{cov}(\mathsf{M}_{[\mathcal{F}^\perp]}) = \mathrm{add}(\mathsf{M}_{[\mathcal{F}]})$ for some cardinal κ. Then for any semifilter $\mathcal{U}$ with $\mathcal{F} \not\in \mathcal{U}$ and $\mathrm{cof}(\mathsf{M}_{[\mathcal{U}]}) \le \kappa$ there is an $[\mathcal{F}]$-concentrated subset $X \subset \omega^\omega$ at $\omega_*^{\uparrow\omega}$ of size $|X| = \kappa$ such that (1) $(X, \mathcal{O})$ is $[\mathcal{F}]$-Menger; (2) (X, μ_p) is not $[\mathcal{U}]$-Menger; (3) if $\mathcal{F}$ is a filter, then all finite powers of X are $[\mathcal{F}]$-Menger.*

Since the cardinal characteristics of families $\mathsf{M}_{[\mathcal{F}]}$ lie in the interval $[\mathfrak{b}, \mathfrak{d}]$ all of them coincide under $\mathfrak{b} = \mathfrak{d}$. Thus under ($\mathfrak{b} = \mathfrak{d}$) for all semifilters $\mathcal{F} \not\in \mathcal{U}$ there exists an $[\mathcal{F}]$-Menger zero-dimensional metrizable space X that fails to be $[\mathcal{U}]$-Menger. This is the base of a dichotomic proof of the existence of a non-Hurewicz subspace $X \subset \omega^\omega$ of size $|X| = \mathfrak{b}$ whose all finite powers of X are Scheepers (it suffices to consider the cases $\mathfrak{b} = \mathfrak{d}$ and $\mathfrak{b} < \mathfrak{d}$). However we do not know the answer to the following

Question 4. *Let $\mathcal{F}$ be an ultrafilter. Is there an $[\mathcal{F}]$-Menger topological space which is not Hurewicz?*

S. Garćia-Ferreira and A. Tamariz-Mascarúa deeply considered the presence of an ultrafilter parameter in the Frechét-Urysohn property, sequentiality, γ-property of Gerlits and Nagy, see [33] and references therein. In spirit Question 4 seems to be close to [33, Problem 3.14].

4.4. **A selection principle varying between Hurewicz and Menger.** In this section we discuss the selection principle $\bigcup_{fin}(\mathcal{O}, T^*)$ introduced by Tsaban [82]. In our terminology this selection principle coincides with the SPF-Menger property corresponding to the family SPF of all simple P-filters. Recall from [10] that a filter $\mathcal{F}$ on ω is a *simple P-filter*, if it is generated by a tower. Following [86] by a *tower* we understand a $\subset^*$-decreasing transfinite sequence of infinite subsets of ω, i.e. a sequence $(T_\alpha)_{\alpha<\lambda}$ such that $T_\alpha \subset^* T_\beta$ for all $\alpha \geq \beta$. The cardinality λ is called the *length* of this tower. We denote by $\mathrm{Depth}^+([\omega]^{\aleph_0})$ the smallest cardinality κ such that there is no tower of length κ. (Thus $\mathfrak{t} < \mathrm{Depth}^+([\omega]^{\aleph_0})$. A model with $\mathfrak{b} \geq \mathrm{Depth}^+([\omega]^{\aleph_0})$ was constructed in [26]. Some other applications of $\mathrm{Depth}^+([\omega]^{\aleph_0})$ in Selection Principles may be found in [68].)

The following result result was proven in [90].

Theorem 29. *Under* $\mathrm{Depth}^+([\omega]^{\aleph_0}) \leq \mathfrak{b}$ *the* SPF*-Menger property is equivalent to the Hurewicz property, while under* $\mathfrak{u} < \mathfrak{g}$ *it is equivalent to the Menger property.*

The statement above can be compared with the fact that the Hurewicz and Menger properties differ in ZFC, see, e.g., [24].

4.5. **Applications to function spaces $C_p(X)$.** This section is devoted to characterizations of the F-Menger property of a topological space X via properties of the function space $C_p(X)$. We recall that $C_p(X)$ is the space of continuous real-valued functions on X, endowed with the topology of pointwise convergence (inherited from the Tychonoff product $\mathbb{R}^X$).

4.5.1. *Characterizing the* F*-Menger spaces via convergence properties of function sequences.* Here we characterize F-Menger topological spaces X via convergence properties of sequences in $C_p(X)$. Our results develop ideas of Bukovsky, Reclaw, Repický [23] who investigated the interplay between the pointwise and so-called quasi-normal convergence in $C_p(X)$.

A function sequence $(f_n : X \to \mathbb{R})_{n\in\omega}$ is defined to converge *quasinormally* to a function $f : X \to \mathbb{R}$ if there exists a vanishing sequence $(\varepsilon_n)_{n\in\omega}$ of positive reals such that for every $x \in X$ the equation $|f_n(x) - f(x)| \leq \varepsilon_n$ holds for all but finitely many $n \in \omega$. Introducing a semifilter parameter $\mathsf{F} \subset \mathsf{SF}$ in this definition we obtain a notion of F-normal convergence so that the quasinormal convergence will correspond to the $\{\mathfrak{F}r\}$-normal convergence: a function sequence $(f_n : X \to \mathbb{R})_{n\in\omega}$ is defined to converge F-*normally* to a function $f : X \to \mathbb{R}$ if there exists a vanishing sequence $(\varepsilon_n)_{n\in\omega}$ of positive reals such that the family $\big\{\{n \in \omega : |f_n(x) - f(x)| < \varepsilon_n\} : x \in X\big\}$ lies in some semifilter $\mathcal{F} \in \mathsf{F}$.

Our general strategy is to find conditions on a topological space X implying the F-normal convergence of any pointwise convergent function sequence in $C_p(X)$. Spaces X with that property are referred to as QN_{F}*-spaces*.

We shall write $\inf A = 0$ for a subset $A \subset \mathbb{R}^X$ if $\inf\{f(x) : f \in A\} = 0$ for each $x \in X$. By $C_p^+(X) \subset C_p(X)$ we denote the set of all strictly positive functions on X.

Theorem 30. *For an $\asymp$-invariant family $\mathsf{F} \subset \mathsf{SF}$ of semifilters and a topological space X the following conditions are equivalent: (1) X is an QN_{F}-space; (2) The multicovered space $(X, \mathcal{O}_\omega^f)$ is F-Menger; (3) Any function sequence $(f_n)_{n\in\omega} \subset C_p^+(X)$ with $\inf\{f_n\}_{n\in\omega} = 0$ is F-normally convergent.*

In particular, a Lindelöf regular topological space X is F-Menger iff it is a QN_{F}-space.

For the family $\mathsf{F} = [\mathfrak{F}r]$ of meager semifilters this yields a characterization [22] of Hurewicz regular topological spaces as regular Lindelöf spaces with the property that every monotone pointwise convergent function sequence $(f_n) \subset C_p(X)$ converges quasinormally.

The above statements yield a characterization of the F-Menger property of X via convergence properties of $C_p(X)$. However, this characterization involves the partial order of $C_p(X)$ and thus is not purely topological. So, Arkhangelski's question [2] on the t-invariance of the Menger property still remains open. We extend it to the subsequent

Question 5. *Is the F-Menger property t-invariant? What about $\asymp$-invariant families F?*

4.5.2. *The F-Menger property and fan tightness.* In this section we discuss another duality between selection properties of a Tychonoff space X and local properties of the function space $C_p(X)$.

Let F be a family of semifilters. Generalizing the definition of countable fan tightness introduced in [1] we define a topological space X to have F-*fan tightness*, if for every point $x \in X$ and every sequence $(A_n)_{n\in\omega}$ of subsets of X with $x \in \bigcap_{n\in\omega} \overline{A}_n$ there exists a sequence $(B_n)_{n\in\omega}$ of finite subsets of X such that $A_n \supset B_n$ and the collection $\{\{n \in \omega : U \cap B_n \neq \emptyset\} : U$ is a neighborhood of $x\}$ lies in some semifilter $\mathcal{F} \in \mathsf{F}$. Thus a topological space X has countable fan tightness if and only if X has $\{\mathfrak{F}r^\perp\}$-fan tightness if and only if it has SF-fan tightness.

Theorem 31. *The function space $C_p(X)$ has F-fan tightness for an $\asymp$-invariant family $\mathsf{F} \subset \mathsf{SF}$ if and only if all finite powers of X are F-Menger.*

In two extreme cases, namely $\mathsf{F} = [\mathfrak{Fr}^{\perp}]$ and $\mathsf{F} = [\mathfrak{Fr}]$, the above result gives known characterizations of the Menger and Hurewicz properties of all finite powers proved in [1] and [50].

5. Games on Multicovered spaces

5.1. **Definitions and elementary properties.** In this section we shall consider classes of multicovered spaces situated between σ-bounded and ω-bounded spaces, and defined with help of infinite games. All multicovered spaces are assumed to be centered.

Given a family $\mathsf{F} \subset \mathsf{SF}$ of semifilters, consider the following F-Menger game on a multicovered space (X, μ) played by two players I and II: player I chooses a cover $u_0 \in \mu$ and player II responds choosing a u_0-bounded subset B_0 of X; in the second turn, player I again chooses some $u_1 \in \mu$ and player II responds choosing a u_1-bounded subset B_1 of X, and so on. At the end of this game they will obtain infinite sequences $(u_n)_{n\in\omega} \in \mu^\omega$ and $(B_n)_{n\in\omega}$, where $B_n \subset X$ is u_n-bounded for all $n \in \omega$. We say, that player II (=the second player) wins this game, if the indexed family $v = \{B_n : n \in \omega\}$ is an F-cover of X. Otherwise player I(=the first player) wins.

For families $\{\mathfrak{Fr}\}$, UF, and $\{\mathfrak{Fr}^{\perp}\}$ we obtain the games corresponding to the Hurewicz, Scheepers, and Menger properties respectively. The last one was introduced by R. Telgarsky in [72], who proved that a hereditarily Lindelöf topological space X is σ-compact provided the second player has a winning strategy in the Menger game on X, see [71] and references therein.

¿From now on we shall call a multicovered space (X, μ) *winning* F*-Menger* (resp. *non-loosing* F*-Menger,* F*-Menger determined,* F*-Menger undetermined*) if the second player has a winning strategy (resp. the first has no winning strategy, one of the players has a winning strategy, none of the players has a winning strategy) in the F-Menger game on (X, μ).

All "winning" and "non-loosing" properties are preserved by uniformly bounded surjective morphisms. One of the most important results is that all "winning" properties coincide, i.e. every winning $\{\mathfrak{Fr}^{\perp}\}$-Menger multicovered space is winning $\{\mathfrak{Fr}\}$-Menger. Therefore it is natural to call a multicovered space (X, μ) *winning*, if it is winning F-Menger for some (equivalently any) family F of semifilters.

Proposition 32. *Every winning multicovered space X satisfies the inequality* $\mathrm{bc}(X) \le \aleph_0 \cdot b\chi(X)$.

5.2. **The interplay between the F-Menger and non-loosing F-Menger properties.** It is clear that every non-loosing F-Menger multicovered space is F- Menger. The inverse implication depends on the properties of F. Given a family $\mathsf{F} = \mathsf{F}_\downarrow \subset \mathsf{SF}$ of semifilters and a map $\theta : \mathsf{F} \to \mathsf{SF}$ such that $\theta(\mathcal{F}) \subset \mathcal{F}^\perp$ and $\theta(\mathcal{F}) \not\asymp \mathfrak{Fr}$ for all $\mathcal{F} \in \mathsf{F}$, denote by F_θ the family $\{\mathcal{F} \wedge \theta(\mathcal{F}) : \mathcal{F} \in \mathsf{F}\}$. Then every F-Menger uniformizable multicovered space is non-loosing F_θ-Menger for all θ as above. In particular, every F-Menger uniformizable multicovered space is non-loosing F-Menger for every family F of filters (in this case $\theta = \mathrm{id}_{\mathsf{F}}$). This implies that every Scheepers (Hurewicz) uniformizable space is non-loosing Scheepers (Hurewicz). In case of topological spaces this implies the subsequent result essentially proven by Hurewicz [39], see also [64, Th. 13].

Theorem 33. *Each Menger topological space is* BS*-Menger, and hence non-loosing Menger.*

On the other hand, the multicovered space (ω^ω, μ_c) is Menger but not non-loosing Menger. Therefore it fails to be BS-Menger. As we shall see later, the multicovered space (ω^ω, μ_c) has nontrivial applications to K-analytic spaces.

It is worth to mention here that in our proofs we use the technique based on (semi)filter games deeply considered in [53], [52] and [12]. Surprisingly, but the results about games on multicovered spaces from the previous section enable us to find simple alternative proofs of some results from [52]. Therefore, semifilter games is a place where multicovered spaces can be effectively applied as well as the tool for considering games on multicovered spaces.

5.3. **Unions and products of multicovered spaces.** It is easy to check that the winning property is preserved by countable unions of subspaces of a multicovered space. Moreover, for an ω-bounded multicovered space X with $b\chi(X) \leq \aleph_0$ each Hurewicz subspace of X is winning. This observation implies that the union of less than $\mathfrak{b}$ winning subspaces of X is winning.

For multicovered spaces (X, μ) with uncountable $b\chi(X, \mu)$ the situation is rather different. Let X be a Bernstein subset of the real line and $\mu = \mathcal{O}(X)$ be the family of all open covers of X. (Recall, that a subset X of $\mathbb{R}$ is *Bernstein* if it splits every uncountable compact $C \subset \mathbb{R}$ in sense $C \cap X \neq \emptyset$ and $C \setminus X \neq \emptyset$.) Then every winning subspace of (X, μ) is at most countable, consequently the winning property is not preserved by uncountable unions of subspaces of (X, μ).

Concerning the "non-loosing" properties, for a family $\mathsf{F} = \mathsf{F}_\downarrow \subset \mathsf{SF}$ the non-loosing F-Menger property is preserved by unions of less than $\mathfrak{t}$ subspaces of an ω-bounded uniformizable multicovered space. If, moreover, F is a family of filters, then the F-Menger and non-loosing F-Menger properties coincide. So we can exploit the results on preservation of F-Menger property by operations over multicovered spaces, discussed in sections 2.2, 3.7 and 3.9.

Theorem 34. *The product* $(X \times Y, \mu_X \cdot \mu_Y)$ *of a non-loosing* F*-Menger (winning) multicovered space* (X, μ_X) *and a winning space* (Y, μ_Y) *is non-loosing* F*-Menger (winning).*

5.4. **A transfinite extension of the Menger game.** Let us fix any ordinal α and consider the following Menger(α) game: players I and II at each step $\beta < \alpha$ choose a cover $u_\beta \in \mu$ and a u_β-bounded subset B_β of X. Player

II is declared the winner in the Menger(α) game, if $\bigcup_{\beta<\alpha} B_\beta = X$, otherwise player I wins. Thus we arrive to classes of *winning*(α) and *non-loosing Menger*(α) multicovered spaces.

Observe, that the Menger game is a particular case of a Menger(α) game with $\alpha = \omega$. It is clear, that every multicovered space X is winning Menger(α) for a sufficiently large α (for example $\alpha = |X|$). This naturally leads us to the concept of the *ordinal winning index* $owi(X)$ of a multicovered space X, equal to the least ordinal α such that X is winning(α).

It is clear, that the multicovered space X is Menger provided $owi(X) < \omega_1$. On the other hand, every topological space X has $owi(X, \mathcal{O}) \le hl(X)^+$, where $hl(X)$ is the *hereditary Lindelöf number* of X. The ordinal winning index $owi(X)$ can not be arbitrary: it equals either to a limit ordinal or to the successor of a limit ordinal. The latter case occurs for the (non-centered) multicovered space $(\omega^\omega, \mu_\ell)$: $owi(\mathbb{N}^\omega, \mu_\ell) = \omega + 1$.

Concerning centered multicovered spaces with intermediate ordinal winning index, we have no naive examples.

Question 6. *Is there a ZFC-example of a centered multicovered space (X, μ) such that $\omega < owi(X, \mu) < \omega_1$? More generally, for which $\alpha \in (\omega, \omega_1)$ there is a consistent (resp. ZFC-)example of a centered multicovered space (X, μ) with $owi(X, \mu) = \alpha$?*

Some partial answer onto the second part of the above question is already known.

Theorem 35. *(1) Under* $(\mathrm{add}(\mathcal{M}) = \mathfrak{d})$ *there exists a divisible subgroup H of $\mathbb{R}^\omega$ with $owi(H) = \omega{\cdot}2$. (2) Under* $(\mathrm{add}(\mathcal{M}) = \mathfrak{c})$ *there exists a decreasing sequence $(H_n)_{n\in\mathbb{N}}$ of divisible nonmeager subgroups of $\mathbb{R}^\omega$ with $owi(H_n^n) = \omega \cdot (n+1)$ for every $n \in \mathbb{N}$. (3) Under* $(\mathfrak{b} = \mathfrak{d})$ *there exists a subgroup G of $\mathbb{Z}^\omega$ such that $owi(G^n) = \omega^2$ for all $n \in \omega$; (4) Under* $(\mathfrak{b} = \mathfrak{d})$ *there exists a zero-dimensional metrizable space X such that $owi(X^n, \mathcal{O}) = \omega \cdot (n+1)$ for every $n \in \mathbb{N}$.*

Similar results can be proven for the Rothberger version of the Menger(α) game (the difference is that at the β-th inning the second player select an element $B_\beta \in u_\beta$ in the cover u_β chosen by the first player).

5.5. **Applications to topological groups.**

5.5.1. *Straightforward applications.* In this section we survey some applications of the results from sections 4 and 5 to topological groups. This will allow us to answer some questions posed in [38] and [74]. Some of them were independently answered in [4]. The crucial observation is that the game OF on a topological group G coincides with the Menger game on the multicovered space (G, μ_L), where $\mu_L = \{\{gU : g \in G\} : U$ is open subset of $G\}$ is the multicover of G corresponding to its left uniform structure. Thus a topological group G is: (strictly) o-bounded, if the multicovered space (G, μ_L) is (winning) Menger; OF-(un)determined, if (G, μ_L) is Menger (un)determined.

Theorem 36. *(1) The classes of (strictly) o-bounded, OF-determined and OF-undetermined topological groups are closed under multiplication of such groups by strictly o-bounded groups.*

(2) A strictly o-bounded topological group G is σ-bounded provided the Raikov completion of G is Čech-complete.

(3) Assume ($\mathfrak{u} < \mathfrak{g}$) and let $\{G_i : i \le n\}$ be a finite family of topological groups such that the topological space G_i is Menger for all $i \le n$. Then the product $G_0 \times \cdots \times G_n$ is either strictly o-bounded or OF-undetermined.

(4) Let G be a topological group which fails to be strictly o-bounded. If the square G^2 is o-bounded, then G is OF-undetermined.

(5) The union $\bigcup_{\alpha<\tau} G_\alpha$ of an increasing family $\{G_\alpha : \alpha < \tau\}$ of $\tau < \mathfrak{t}$ many OF-undetermined subgroups of an ω-bounded topological group is OF-undetermined.

5.5.2. *Selection properties of free groups over a Tychonoff space.* In this section we characterize Tychonoff spaces X whose free (Abelian) topological group $F(X)$ ($A(X)$) is [strictly] o-bounded, thus answering a question from [38].

In what follows topological groups are considered with the multicover corresponding to their left uniformity provided converse is not stated.

Theorem 37. *For a Tychonoff space X and an $\asymp$-invariant family of semifilters $\mathsf{F} \subset \mathsf{SF}$ the following conditions are equivalent: (1) the free Abelian topological group $A(X)$ of X is F-Menger; (2) $A(X)^n$ is F-Menger for all $n \in \omega$; (3) the free topological group $F(X)$ of X is F-Menger; (4) $F(X)^n$ is F-Menger for all $n \in \omega$; (5) All finite powers of the multicovered space $(X, \mu_{\mathcal{U}(X)})$ are F-Menger.*

Here $\mathcal{U}(X)$ is the maximal uniformity generating the topology of X. For $\mathsf{F} = [\mathfrak{Fr}^{\perp}]$ this gives a characterization of o-boundedness of free (Abelian) topological groups. The characterization of the strict o-boundedness is analogical.

Theorem 38. *For a Tychonoff space X the following conditions are equivalent: (1) $A(X)$ is (strictly) o-bounded; (2) $A(X)^n$ is (strictly) o-bounded for all $n \in \omega$; (3) $F(X)$ is (strictly) o-bounded; (4) $F(X)^n$ is (strictly) o-bounded for all $n \in \omega$; (5) The multicovered space $(X, \mu_{\mathcal{U}(X)})$ is Scheepers (winning).*

For a Lindelöf topological space X the multicovers $\lambda_{\mathcal{U}(X)}$ and $\mathcal{O}(X)$ are equivalent. Therefore a Tychonoff space X is winning (resp. F-Menger) if and only if so is $(X, \lambda_{\mathcal{U}(X)})$ and X is Lindelöf. In combination with the A-invariance of the Lindelöf property [87] this implies that the winning property as well as the F-Menger property for a family $\mathsf{F} = \mathsf{F}_{\downarrow}$ of filters are A- and hence M-invariant. (A topological property is (A-) M-invariant, if a topological space X has this property whenever so does any topological space Y such that $F(X)$ and $F(Y)$ (resp. $A(X)$ and $A(Y)$) are topologically isomorphic).

The above characterizations enable us to resolve the problem of construction of OF-undetermined groups posed in [74] and solved in [51] and [6] (and, probably, somewhere else) independently. Namely, let X be a non-σ-compact metrizable space such that all finite powers of X are Menger (Hurewicz). Then all finite powers of $F(X)$ and $A(X)$ are OF-undetermined being non-strictly o-bounded groups whose underlying topological space are Menger (Hurewicz). Spaces X with properties specified above were constructed in [14], [24], and [84].

In all nontrivial cases free groups are not metrizable. In the next section we shall also present metrizable examples of OF-undetermined groups.

6. Determinacy of Games related to Selection Principles

This section is devoted to the determinacy of the F-Menger game. We introduce a class of so-called absolutely F-Menger-determined topological spaces and prove that it contains spaces with nice descriptive properties, namely all countably K-analytic or more generally quasi-analytic spaces. In section 6.3 we give an exposition of Menger-undetermined multicovered spaces possessing an additional algebraic structure.

6.1. K-analytic multicovered spaces.

We recall that a topological space X is (*countably-*) K*-analytic* if X is the image of ω^ω under an upper semi-continuous set-valued map $\Phi : \omega^\omega \Rightarrow X$ with (countably) compact values $\Phi(z)$, $z \in \omega^\omega$. If $|\Phi(z)| \le 1$ for any $z \in \omega^\omega$, then the space X is called *Souslin*. Metrizable Souslin spaces are frequently called *analytic*.

The (countable) K-analyticity can be characterized in terms of an infinite game, called "Cover-Subset$-K$" (resp. "Cover-Subset$-K_\omega$"). A decreasing sequence $(F_n)_{n\in\omega}$ of subsets of a topological space X is called a K*-sequence* (resp. K_ω*-sequence*) if the intersection $F_\omega = \bigcap_{n\in\omega} F_n$ is (countably) compact and the sequence $(F_n)_{n\in\omega}$ converges to F_ω in the sense that for any neighborhood $U \subset X$ of F_ω there is a number $m \in \omega$ with $F_m \subset U$. A sequence $(F_n)_{n\in\omega}$ of subsets of a topological space X is called a $\cap_{\ne\emptyset}$*-sequence* if $\bigcap_{n\in\omega} F_n \ne \emptyset$.

Now we are able to describe the infinite games "Cover-Subset$-K$", "Cover-Subset$-K_\omega$", and "Cover-Subset$-\cap_{\ne\emptyset}$" played by two players I and II on a topological space X as follows. The player I starts the game selecting a countable cover u_0 of X and II responds with an element $U_0 \in u_0$ and a non-empty closed subset F_0 of U_0. At the n-th inning the player I selects a countable cover u_n of the set F_{n-1} and II responds with an element $U_n \in u_n$ and a non-empty closed subset $F_n \subset U_n$. At the end of the game "Cover-Subset$-K$" (resp. "Cover-Subset$-K_\omega$", "Cover-Subset$-\cap_{\ne\emptyset}$") the player I is declared the winner if the constructed sequence $(F_n)_{n\in\omega}$ is a K-sequence (resp. K_ω-sequence, $\cap_{\ne\emptyset}$-sequence). These games give us the following useful characterization of (countably-)K-analytic spaces.

Theorem 39. *A topological space X is (countably-)K-analytic if and only if the first player has a winning strategy in the "Cover-Subset–K" (resp. "Cover-Subset–K_ω") game on X.*

We define a topological space X to be *quasi-analytic*, if the first player has a winning strategy in the game "Cover-Subset-$\cap_{\neq\emptyset}$". The preceding characterization means that quasi-analytic topological spaces are natural generalizations of (countably-)K-analytic spaces.

Question 7. *Is there a quasi-analytic topological space which is not countably K-analytic?*

The class of quasi-analytic spaces shares many useful properties of K-analytic spaces: it is closed under taking upper semicontinuous (countably-) compact-valued images and F_σ-subspaces of its elements. In addition, every metrizable quasi-analytic topological space X is analytic.

Generalizing the classical result of Hurewicz asserting that an analytic subset Z of a Polish space P is either σ-compact or else contains a closed in P copy of the Baire space, we prove the following dichotomy:

Theorem 40. *Let μ be a properly ω-bounded open multicover on a quasi-analytic topological space X. Then either (X, μ) is winning or else the multicovered space $(\omega^\omega, \mu_\ell)$ is the image of X under a surjective uniformly bounded morphism $\Phi : (X, \mu) \Rightarrow (\omega^\omega, \mu_\ell)$.*

Applying this dichotomy to quasi-analytic regular hereditarily Lindelöf spaces, we obtain that such a space X either is σ-compact or else contains a closed subspace that maps continuously onto ω^ω. A similar result may be found in the survey paper [61].

These dichotomic results have many corollaries. First of all, it implies that every quasi-analytic topological space X is *absolutely* F-*Menger* determined in the sense that every properly ω-bounded open multicover μ of X the multicovered space (X, μ) is F-Menger determined for every family F of semifilters.

Another corollary asserts that *the product of two centered properly ω-bounded quasi-analytic multicovered spaces is Menger if and only if one of these spaces is winning and the other is Menger.* In its turn this corollary helps us to prove that *an Abelian quasi-analytic topological group is o-bounded if and only if it is strictly o-bounded*, which shows that the group "constructed" in [37, 6.1] and "improved" in [6, Theorem 4] cannot exist in principle.

In connection with our games 'Cover-Subset–K", "Cover-Subset–K_ω" and "Cover-Subset-$\cap_{\neq\emptyset}$" let us mention Telgarsky who introduced and studied very similar games in [70] and [71] characterizing the analyticity.

6.2. **Other examples of Menger-determined multicovered spaces.** In this section we establish the Menger-determinacy of multicovered spaces that live on nice topological spaces.

A topological space X is called a *Choquet space* if the second player has a winning strategy in the following Choquet game G_X. Two players, I and II, at every step $k \in \omega$ choose non-empty open subsets U_k and V_k of X, respectively, so that $U_k \subset V_{k-1}$ and $V_k \subset U_k$. At the end of the game, the player II is declared the winner if the intersection $\bigcap_{k\in\omega} V_k \neq \emptyset$.

Theorem 41. *Let μ be an open regular multicover on a topological space X admitting a continuous surjective map $f : Z \to X$ from a hereditarily Lindelöf hereditarily Choquet topological space Z. The multicovered space (X, μ) is non-loosing Menger if and only if it is σ-bounded.*

Next, we reveal the interplay between the Menger-determinacy of multicovered spaces and the determinacy of the game $G(A)$ defined for any subset $A \subset \omega^\omega$ as follows. Two players, I and II, at every step $k \in \omega$ choose numbers n_{2k} and n_{2k+1}, respectively. At the end of the game, I is declared the winner if the constructed sequence $(n_k)_{k\in\omega}$ belongs to the set A. Otherwise, II wins. The game $G(A)$ is *determined* if one of the players has a winning strategy in the game $G(A)$.

According to the fundamental Martin Theorem (see [57] or [43]), the game $G(A)$ is determined for any Borel subset A of ω^ω. The question whether $G(A)$ can be undetermined for an analytic or projective subset $A \subset \omega^\omega$ depends on axioms of Set Theory. We recall that the algebra of projective subsets of a Polish space P is the smallest algebra of subsets of P closed under taking continuous images of its elements, see [43, 37.1].

Let *Det* denote the family of all subspaces $A \subset \omega^\omega$ such that for any closed subset $F \subset \omega^\omega$ and any continuous map $p : \omega^\omega \to \omega^\omega$ the game $G(p^{-1}(A) \cap F)$ is determined. It follows from the Martin Theorem that the family *Det* includes all Borel subsets of ω^ω. Under a suitable large cardinal assumption the collection *Det* includes all projective subsets of ω^ω.

Theorem 42. *Suppose a topological space X is the image of a space $Z \in Det$ under an upper semicontinuous countably-compact-valued map $\Phi : Z \Rightarrow X$ and μ is an open multicover of X such that the multicovered space (X, μ) is properly ω-bounded and $b\chi(X, \mu) \leq \aleph_0$. Then the multicovered space (X, μ) is Menger-determined. More precisely, either (X, μ) is σ-bounded or else Z contains a closed subset P of ω^ω whose image $\Phi(P)$ fails to be non-loosing Menger in (X, μ).*

6.3. **Menger- and Scheepers-undetermined multicovered spaces.** Generalizing the notion of the group hull of a subset in a topological group, we arrive at the concept of an *algebraic hull operator* by which we understand a function $H : \mathcal{P}(X) \to \mathcal{P}(X)$ such that for any subsets $A \subset B \subset X$ we get

$$A \subset H(A) \subset H(B) = \cup\{H(F) : F \text{ is a finite subset of } B\}.$$

An algebraic hull operator H on a topological space X is defined to be *σ-compact* if the set $H(K)$ is σ-compact for any compact subset $K \subset X$; H is called *G_δ-measurable* if for any compact subset $K \subset X$ the set-valued function $H_K : X \Rightarrow X$, $H_K : x \mapsto H(\{x\} \cup K)$, is G_δ-measurable in the sense that for any open subset U of X the set $\{x \in X : H_K(x) \subset U\}$ is a G_δ-subset of X.

σ-Compact G_δ-measurable algebraic hull operators often arise in topological algebra. Let us recall some related definitions, see [25]. By a *continuous signature* we understand a sequence $\mathcal{E} = \{E_n\}_{n\in\omega}$ of topological spaces. A continuous signature $\mathcal{E} = \{E_n\}_{n\in\omega}$ is *σ-compact* if all the spaces E_n are σ-compact. A *universal topological algebra of the continuous signature* $\mathcal{E}$ (briefly, a *topological $\mathcal{E}$-algebra*) is a topological space X endowed with a sequence of continuous maps $\{e_n : E_n \times X^n \to X\}_{n\in\omega}$. A subset A of a topological $\mathcal{E}$-algebra $(X, \{e_n\}_{n\in\omega})$ is called a *subalgebra* of X if $e_n(E_n \times Y^n) \subset Y$ for all $n \in \omega$. By the *algebraic hull* $H(A)$ of a subset A of a topological $\mathcal{E}$-algebra we understand the intersection of all subalgebras of X containing the subset A. The operator $H : \mathcal{P}(X) \to \mathcal{P}(X)$ assigning to each subset $A \subset X$ its algebraic hull $H(A)$ is an algebraic hull operator. Moreover, if the signature $\mathcal{E}$ is σ-compact, then the operator H is σ-compact and G_δ-measurable. Observe, that the notion of a topological $\mathcal{E}$-algebra generalizes concepts of a topological (semi)group, topological ring, linear topological space, and many other.

Theorem 43. *Let $H : \mathcal{P}(X) \Rightarrow \mathcal{P}(X)$ be a σ-compact G_δ-measurable algebraic hull operator on a Polish space X and μ be any properly ω-bounded open multicover on X. Then either (X, μ) is winning or else it contains a Menger-undetermined subspace A with Menger-undetermined hull $H(A)$.*

In case when algebraic hull operator H is generated by some structure of topological $\mathcal{E}$-algebra with σ-compact signature $\mathcal{E}$ on X, using the ideas of [24] (see also [84]) we can prove a bit more.

Theorem 44. *Let $H : \mathcal{P}(X) \to \mathcal{P}(X)$ be the algebraic hull operator generated by the structure of topological $\mathcal{E}$-algebra with σ-compact signature $\mathcal{E}$ on a K-analytic regular topological space X and let μ be an open centered multicover of X such that the multicovered space (X, μ) is properly ω-bounded but not winning. Then X contains not winning subspace Z such that all finite powers of $H(Z)$ have the Menger property. Consequently, $H(Z)$ is non-loosing Scheepers.*

If the space X is Polish and nowhere locally-compact, then under CH we can construct the subalgebra $H(Z)$ to be hereditarily Baire. In case of topological groups this gives a positive solution to Problem 5 posed in [38].

6.4. **Determinacy of transfinite games on quasi-analytic multicovered spaces.** For a centered open multicover μ on a quasi-analytic topological space X the Menger(α) game on (X, μ) is determined for every countable ordinal α. More precisely, either X is winning or else for every countable ordinal α the first player has a winning strategy in the Menger(α) game on X. Consequently, $owi(X, \mu) \in \{1, \omega, \omega_1\}$ for any open centered multicover μ on a regular Lindelöf quasi-analytic space X.

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