ON UNIVERSALITY OF FINITE PRODUCTS OF POLISH SPACES

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Abstract. We introduce and study the n-Dimensional Perfect Homotopy Approximation Property (briefly n-PHAP) equivalent to the discrete n-cells property in the realm of LC^n-spaces. It is shown that the product $X \times Y$ of a space X with n-PHAP and a space Y with m-PHAP has $(n+m+1)$-PHAP. We derive from this that for a (nowhere locally compact) locally connected Polish space X without free arcs and for each $n \geq 0$ the power X^{n+1} contains a closed topological copy of each at most n-dimensional compact (resp. Polish) space.

A topological space X is called *$\mathcal{C}$-universal*, where $\mathcal{C}$ is a class of spaces, if X contains a closed topological copy of each space $C \in \mathcal{C}$. By $\mathcal{M}_0$ and $\mathcal{M}_1$ we denote the classes of metrizable compacta and Polish (= separable complete-metrizable) spaces, respectively. For a class $\mathcal{C}$ of spaces by $\mathcal{C}[n]$ we denote the subclass of $\mathcal{C}$ consisting of all spaces $C \in \mathcal{C}$ with $\dim C \leq n$. *All topological spaces considered in the paper are metrizable and separable, all maps are continuous.*

In terms of the universality, the classical Menger-Nöbeling-Pontrjagin-Lefschetz Theorem states that the cube $[0,1]^{2n+1}$ is $\mathcal{M}_0[n]$-universal for every $n \geq 0$. It is well known that the exponent $2n+1$ in this theorem is the best possible: the Menger universal compactum μ_n cannot be embedded into $[0,1]^{2n}$. Nonetheless, P.Bowers [Bo$_1$] has proved that the $(n+1)$-th power D^{n+1} of any dendrite D with dense set of end-points does be $\mathcal{M}_0[n]$-universal for every non-negative integer n. Moreover, any such a dendrite D contains a locally connected G_δ-subspace G whose $(n+1)$-th power G^{n+1} is $\mathcal{M}_1[n]$-universal for every n, see [Bo$_1$]. Generalizing this Bowers' result we shall prove that the power X^{n+1} of any locally connected Polish space X without free arcs is $\mathcal{M}_0[n]$-universal for all $n \geq 0$; moreover the power X^{n+1} is $\mathcal{M}_1[n]$-universal provided X is nowhere locally compact.

The standard way to prove the $\mathcal{M}_1[n]$-universality of a Polish space X with nice local structure is to verify the discrete n-cells property for X, see [Bo$_1$]. We remind that a space X has *the discrete n-cells property* if for any map $f : \mathbb{N} \times [0,1]^n \to X$ and any open cover $\mathcal{U}$ of X there is a map $g : \mathbb{N} \times [0,1]^n \to X$ such that g is $\mathcal{U}$-near to f and the collection $\{g(\{i\} \times [0,1]^n)\}_{i \in \mathbb{N}}$ is discrete in X.

Let us recall that two maps $f, g : Z \to X$ are called *$\mathcal{U}$-near* with respect to a cover $\mathcal{U}$ of X (this is denoted by $(f,g) \prec \mathcal{U}$) if for any point $z \in Z$ there is an element $U \in \mathcal{U}$ such that $\{f(z), g(z)\} \subset U$. Two maps $f, g : Z \to X$ are called *$\mathcal{U}$-homotopic* if they can be linked by a homotopy $\{h_t : Z \to X\}_{t \in [0,1]}$ such that $h_0 = f$, $h_1 = g$ and for any $z \in Z$ there is $U \in \mathcal{U}$ with $\{h_t(z) : t \in [0,1]\} \subset U$. It is clear that $\mathcal{U}$-homotopic maps are $\mathcal{U}$-near while the converse is not true in general.

Unfortunately, the discrete n-cells property is applicable only for spaces having nice local structure. To overcome this obstacle we introduce a stronger property, called n-PHAP, which is equivalent to the discrete n-cells property in the realm of LC^n-spaces. We remind that a space X is called an *LC^n-space*, $n \geq 0$, if for any point $x \in X$ and any neighborhood $U \subset X$ of x there is a neighborhood $V \subset X$ of x such that any map

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$f : \partial I^n \to V$ from the boundary of the n-dimensional cube $I^n = [0,1]^n$ can be extended to a map $\bar{f} : I^n \to U$ defined on the whole n-cube I^n.

All simplicial complexes considered in this paper are countable and locally finite. We shall identify simplicial complexes with their geometric realizations.

Definition 1. A space X is defined to have *the n-dimensional perfect homotopy approximation property* (briefly n-PHAP) if for any map $f : K \to X$ from a simplicial complex K with $\dim K \le n$ and any open cover $\mathcal{U}$ of X there is a perfect map $g : K \to X$, $\mathcal{U}$-homotopic to f.

We remind that a map $f : X \to Y$ is *perfect* if f is closed and the preimage $f^{-1}(y)$ of any point $y \in Y$ is compact. According to [En, 3.7.18], a map $f : X \to Y$ between metrizable spaces is perfect if and only if f is *proper* in the sense that the preimage $f^{-1}(K)$ of any compact subset $K \subset Y$ is compact.

A map $f : X \to Y$ is called *simplicially approximable* if for any open cover $\mathcal{U}$ of X there are a simplicial complex K and two maps $p : X \to K$ and $q : K \to Y$ such that the composition $q \circ p$ is $\mathcal{U}$-homotopic to f. It follows from Corollary 6.6 [BP, p.80] that each map into an absolute neighborhood retract is simplicially approximable.

Some basic properties of spaces with n-PHAP are described by the following theorem which is the main result of this paper.

Theorem 1. *Let n, m be non-negative integers.*

1. *If a space X has n-PHAP, then each open subspace of X has that property too.*
2. *A space X has n-PHAP provided X admits a cover by open subspaces with n-PHAP.*
3. *If a space X has n-PHAP, then X has the discrete n-cells property.*
4. *An LC^n-space X has n-PHAP if and only if X has the discrete n-cells property.*
5. *If X is a space with n-PHAP and Y is a space with m-PHAP, then their product $X \times Y$ has $(n+m+1)$-PHAP.*
6. *If a Polish space X has n-PHAP, then for any open cover $\mathcal{U}$ of X and any simplicially approximable map $f : P \to X$ from a Polish space P with $\dim P \le n$ there is a perfect map $g : P \to X$, $\mathcal{U}$-homotopic to f.*
7. *If a Polish space X has n-PHAP, then for any open cover $\mathcal{U}$ of X and any simplicially approximable map $f : P \to X$ from a Polish space P with $\dim P \le n$ there is a closed embedding $g : P \to X$, $\mathcal{U}$-near to f.*
8. *If a Polish space X has n-PHAP, then X is $\mathcal{M}_1[n]$-universal.*

Statements 4, 5, and 8 of Theorem 1 imply

Corollary 1. *If X is a Polish LC^n-space with the discrete n-cells property, then for every $k \ge 0$ the power X^{k+1} is $\mathcal{M}_1[nk+n+k]$-universal.*

In its turn, the last corollary implies another two corollaries generalizing the mentioned Bowers' results on the universality of finite powers of dendrites.

Corollary 2. *If X is a locally connected Polish nowhere locally compact space, then for every $k \ge 0$ the power X^{k+1} is $\mathcal{M}_1[k]$-universal.*

Proof. The Polish space X, being locally connected, is locally path-connected and hence LC^0 according to the classical Mazurkiewicz-Moore-Menger Theorem, see [Ku]. It is well-known (and easy) that the discrete 0-cells property is equivalent to the nowhere local compactness. In this situation it is legal to apply Corollary 1 to conclude that the power X^{k+1} is $\mathcal{M}_1[k]$-universal for every $k \ge 0$. □

We say that a topological space X *has no free arcs* if no open subset of X is homeomorphic to the open interval $(0,1)$.

Corollary 3. *If X is a locally connected Polish space without free arcs, then for every $k \geq 0$ the power X^{k+1} is $\mathcal{M}_0[k]$-universal.*

Proof. Corollary 3 will follow from Corollary 2 as soon as we prove that each locally connected Polish space X without free arcs contains a locally connected nowhere locally compact Polish subspace Y.

Replacing X by any of its connected component, we can assume that X is connected. Then by [Wy, Ch.VIII,§9] the space X admits a compatible metric d such that any points $x, y \in X$ can be linked by an arc whose diameter does not exceed $2d(x,y)$. Fix a countable dense subset $D \subset X$ and for any points $x, y \in D$ fix an arc $J(x,y) \subset X$ with $\operatorname{diam} J(x,y) \leq 2d(x,y)$. It is easy to see that any subspace $Y \subset X$ containing the set $A = \bigcup_{x,y \in D} J(x,y)$ is locally path-connected. Since the Polish space X has no free arcs, the Baire Theorem implies that the complement $X \setminus A$ is dense in X. Let $C \subset X \setminus A$ be a countable dense set. Then $Y = X \setminus C$ is a locally connected nowhere locally compact Polish subspace of X. □

1. Proof of Theorem 1

Our notations are standard. In particular, by $\bar{A}$ or $\mathrm{cl}_X(A)$ we denote the closure of a set A in a topological space X; $\mathrm{cov}(X)$ stands for the family of all open covers of a space X. For a cover $\mathcal{U}$ of X and a subset $A \subset X$, let $\mathcal{S}t\,(A,\mathcal{U}) = \cup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$, $\mathcal{S}t^{\,1}(\mathcal{U}) = \mathcal{S}t\,(\mathcal{U}) = \{\mathcal{S}t\,(U,\mathcal{U}) : U \in \mathcal{U}\}$, and $\mathcal{S}t^{\,n+1}(\mathcal{U}) = \mathcal{S}t\,(\mathcal{S}t^{\,n}(\mathcal{U}))$ for $n \geq 1$. Given two families $\mathcal{U}, \mathcal{V}$ of subsets of a space X we write $\mathcal{U} \prec \mathcal{V}$ if any $U \in \mathcal{U}$ lies in some $V \in \mathcal{V}$. For a map $f : Z \to X$ and a family $\mathcal{U}$ of subsets of X we put $f^{-1}(\mathcal{U}) = \{f^{-1}(U) : U \in \mathcal{U}\}$.

For a metric space (X,d) and a point $x_0 \in X$ by $B(x_0,\varepsilon) = \{x \in X : d(x,x_0) < \varepsilon\}$ we denote the open ε-ball centered at x_0. Also we put $\operatorname{mesh}\mathcal{U} = \sup_{U \in \mathcal{U}} \operatorname{diam} U$ for a cover $\mathcal{U}$ of X. A homotopy $h : Z \times [0,1] \to X$ is called an *ε-homotopy* if $\operatorname{diam} h(\{z\} \times [0,1]) < \varepsilon$ for all $z \in Z$.

For a simplicial complex K, denote by $K^{(n)}$ the n-dimensional skeleton of K and let $\mathcal{S}t\,(K) = \{\mathcal{S}t\,(v,K) : v \in K^{(0)}\}$ where $\mathcal{S}t\,(v,K)$ stands for the open star of a vertex v in K. Several times we shall use the following homotopy extension property of simplicial pairs (see Corollary 5 of [Spa, p.112]): *If L is a subcomplex of a simplicial complex K, $f : K \to X$ is a continuous map into a space X, and $h : L \times [0,1] \to X$ is a homotopy with $h(z,0) = f(z)$ for all $z \in L$, then there is a homotopy $H : K \times [0,1] \to X$ such that $H|L \times [0,1] = h$ and $H(z,0) = f(z)$ for all $z \in K$. If h is a $\mathcal{U}$-homotopy for some open cover $\mathcal{U}$ of X, then H can be chosen to be a $\mathcal{U}$-homotopy. If $\operatorname{diam} h(\{x\} \times [0,1]) < \varepsilon \circ f(x)$, $x \in L$, for some continuous map $\varepsilon : X \to (0,\infty)$, then H can be chosen so that $\operatorname{diam} H(\{x\} \times [0,1]) < \varepsilon \circ f(x)$ for all $x \in K$.*

In the proof of Theorem 1 we shall exploit some known facts about proper maps.

Lemma 1. *For a perfect map $f : K \to X$ from a locally compact space K there is an open cover $\mathcal{U}$ of X such that each map $g : K \to X$ with $(f,g) \prec \mathcal{U}$ is perfect.*

Proof. Let $\overline{X}$ be any metrizable compactification of X. It follows from [En, 3.7.21] that the image $f(K)$ of the locally compact space K under the perfect map $f : K \to X$ is a closed locally compact subspace of X. Consequently, $f(K)$, being locally compact, is open in its closure $\mathrm{cl}_{\overline{X}}(f(K))$ in $\overline{X}$ and hence the complement $F = \mathrm{cl}_{\overline{X}}(f(K)) \setminus f(K)$ is closed in $\overline{X}$. It follows that $\widetilde{X} = \overline{X} \setminus F$ is a locally compact space containing X so that the map $f : K \to X \subset \widetilde{X}$ still is perfect. Now it is legal to apply Theorem 4.1 of [Ch] to find an open cover $\widetilde{\mathcal{U}}$ of $\widetilde{X}$ such that each map $g : K \to \widetilde{X}$ with $(f, g) \prec \widetilde{\mathcal{U}}$ is perfect. Then the open cover $\mathcal{U} = \{U \cap X : U \in \widetilde{U}\}$ satisfies our requirements. □

Lemma 2. *If $f : K \to X$ is a map from a locally compact space K and the restriction $f|L : L \to X$ of f onto a closed subset $L \subset K$ is perfect, then $f|\overline{W}$ is perfect for some closed neighborhood $\overline{W}$ of L in K.*

Proof. Fix any metric d generating the topology of X and write $K = \bigcup_{i\geq 0} K_i$ as the countable union of an increasing sequence $(K_i)_{i\geq 0}$ of compact subsets such that $K_0 = \emptyset$ and each K_n lies in the interior of K_{n+1}. For each $i \geq 1$ and $z \in K_i \setminus K_{i-1}$ find a neighborhood $O(z) \subset K$ such that $O(z) \subset K_{i+1} \setminus K_{i-1}$ and $f(O(z)) \subset B(f(z), \frac{1}{i}) = \{x \in X : d(x, f(z)) < \frac{1}{i}\}$. Let $\overline{W}$ be any closed neighborhood of L in K with $\overline{W} \subset \bigcup_{z\in L} O(z)$.

Let us show that the restriction $f|\overline{W}$ is perfect. Assuming the converse we could find a sequence $(x_i)_{i\geq 1} \subset \overline{W}$ that has no cluster point in $\overline{W}$ but $(f(x_i))_{i\geq 1}$ converges to some point a in X. Passing to a subsequence, if necessary, we can assume that $x_i \notin K_i$. For every $i \geq 1$ find a point $z_i \in L$ with $x_i \in O(z_i)$. Taking into account that $x_i \notin K_i$ and $O(z) \subset K_i$ for all $z \in K_{i-1}$, we conclude that $z_i \notin K_{i-1}$ for all $i \geq 1$. Then $d(f(x_i), f(z_i)) < \frac{1}{i}$ for $i \geq 1$ and thus the sequence $(f(z_i))$ converges to $a = \lim f(x_i)$ which is not possible since $f|L$ is perfect and the sequence (z_i) has no cluster point in L. □

Applying n-PHAP it will be convenient to work with its stronger version.

Lemma 3. *If a space X has n-PHAP, then for any open cover $\mathcal{U}$ of X, any simplicial complex K with $\dim K \leq n$, any closed subspace $F \subset K$, and any map $f : K \to X$ whose restriction $f|F : F \to X$ is perfect, there is a perfect map $g : K \to X$, $\mathcal{U}$-homotopic to f via a $\mathcal{U}$-homotopy $h : K \times [0, 1] \to X$ such that $h(x, 1) = g(x)$ for all $x \in K$ and $h(x, t) = f(x)$ for all $(x, t) \in K \times \{0\} \cup F \times [0, 1]$.*

Proof. By Lemma 2, the restriction $f|\overline{W}$ is perfect for some closed neighborhood $\overline{W}$ of F in K. By Lemma 1, there is a cover $\mathcal{V} \in \mathrm{cov}(X)$, $\mathcal{V} \prec \mathcal{U}$, such that a map $g : \overline{W} \to X$ is perfect, whenever it is $\mathcal{V}$-near to $f|\overline{W}$. Using n-PHAP of X, find a perfect map $\tilde{f} : K \to X$, $\mathcal{V}$-homotopic to f via a homotopy $\tilde{h} : K \times [0, 1] \to X$ such that $\tilde{h}(x, 0) = f(x)$ and $\tilde{h}(x, 1) = \tilde{f}(x)$ for all $x \in K$. Fix any continuous map $\lambda : K \to [0, 1]$ with $\lambda(F) \subset \{0\}$ and $\lambda(K \setminus W) \subset \{1\}$ and consider the homotopy $h : K \times [0, 1] \to X$ defined by $h(x, t) = \tilde{h}(x, \lambda(x)t)$ for $(x, t) \in K \times [0, 1]$. It is easy to see that the map $g : K \to X$, $g : x \mapsto h(x, 1)$, and the $\mathcal{U}$-homotopy h satisfy the requirements of the lemma. □

The following lemma gives a proof of Theorem 1(1).

Lemma 4. *If X is a space with n-PHAP, then each open subspace of X has n-PHAP.*

Proof. Let U be an open subspace of X, $\mathcal{U}$ be an open cover of U and $f_0 : K \to U$ be a map of a simplicial complex K with $\dim K \leq n$. We have to construct a perfect map $f_\infty : K \to U$ which is $\mathcal{U}$-homotopic to f_0.

Fix any metric $\rho < 1$ generating the topology of X. For every $n \geq 0$ let $K_n = \{x \in K : \rho(f_0(x), X \setminus U) \geq 2^{-n}\}$. It is clear that each set K_n is closed in K and lies in the interior of K_{n+1}. Since $\rho < 1$, $K_0 = \emptyset$.

Let $(\mathcal{U}_n)_{n\geq 0}$ be a sequence of open covers of X such that $\mathrm{mesh}\,\mathcal{U}_n < 2^{-(n+1)}$ and $\mathcal{S}t\,\mathcal{U}_{n+1} \prec \mathcal{U}_n$ for any $n \geq 0$. We can additionally assume that the covers $\mathcal{U}_n$ are so fine that $\{\mathcal{S}t\,(x, \mathcal{U}_n) : \rho(x, X \setminus U) \geq 2^{-n}\} \prec \mathcal{U}$ for every $n \geq 0$.

By induction, we shall construct a function sequence $\{f_n : K \to X\}_{n\in\omega}$ satisfying the following conditions for every $n \in \mathbb{N}$:

(1_n) $f_n(x) = f_{n-1}(x)$ for any $x \in K_{n-1} \cup (K \setminus K_{n+1})$;
(2_n) the map $f_n|K_n : K_n \to X$ is perfect;
(3_n) the map f_n is $\mathcal{U}_{n+2}$-homotopic to f_{n-1} via a $\mathcal{U}_{n+2}$-homotopy $h_n : K \times [0,1] \to X$ such that $h_n(x,t) = f_n(x)$ for $(x,t) \in K \times \{1\}$ and $h_n(x,t) = f_{n-1}(x)$ for all $(x,t) \in K \times \{0\} \cup (K_{n-1} \cup (K \setminus K_{n+1})) \times [0,1]$.

Assume that for some $n \in \mathbb{N}$ the function f_{n-1} has been constructed. Using Lemma 3 find a perfect map $g : K \to X$ and a $\mathcal{U}_{n+2}$-homotopy $h : K \times [0,1] \to X$ such that $h(x,1) = g(x)$ for any $x \in K$ and $h(x,t) = f_{n-1}(x)$ for any $(x,t) \in K \times \{0\} \cup K_{n-1} \times [0,1]$. Let $\lambda : K \to [0,1]$ be a continuous function such that $\lambda^{-1}(0) \supset K \setminus K_{n+1}$ and $\lambda^{-1}(1) \supset K_n$. Finally, consider the function $f_n : K \to X$ defined by $f_n(x) = h(x, \lambda(x))$ for $x \in K$ and the homotopy $h_n : K \times [0,1] \to X$ defined by $h_n(x,t) = h(x, \lambda(x)\cdot t)$ for $(x,t) \in K \times [0,1]$. The construction of f_n and h_n imply that the conditions (1_n)–(3_n) are satisfied. The conditions (1_n) imply that for each $x \in K$ the sequence $(f_n(x))$ eventually stabilizes and thus the limit map $f_\infty = \lim_{n\to\infty} f_n : K \to X$ is well-defined. Observe that f_∞ is homotopic to the map f_0 via the homotopy $h_\infty : K \times [0,\infty] \to X$ defined by $h_\infty(x,\infty) = f_\infty(x)$ for $x \in K$ and $h_\infty(x,t) = h_n(x, t-n+1)$ for $x \in K$ and $t \in [n-1, n]$, $n \geq 1$.

Since $\rho(f_0(X), X \setminus U) \geq 2^{-n}$, for $x \in K_n \setminus K_{n-1}$, we get

$$h_\infty(\{x\} \times [0,\infty]) = \bigcup_{i=-1}^{1} h_{n+i}(\{x\} \times [0,1]) \subset \mathcal{S}t\,(f_0(x), \mathcal{U}_n) \subset \mathcal{S}t\,(f_0(x), \mathcal{U}). \tag{1}$$

This means that h_∞ is a $\mathcal{U}$-homotopy, which yields $h_\infty(K \times [0,\infty]) \subset U$ and $f_\infty(K) \subset U$. Also (1) implies that $\rho(f_\infty(x), f_0(x)) \leq \operatorname{mesh} \mathcal{U}_n < 2^{-(n+1)}$ for any $x \in K_n \setminus K_{n-1}$.

Let us show finally that the map $f_\infty : K \to U$ is perfect. Take any compact subset $C \subset U$ and find $n \geq 0$ such that $\rho(C, X \setminus U) > 2^{-n}$. We claim that $f_\infty^{-1}(C) \subset K_{n+1}$. Fix any $x \in K \setminus K_{n+1}$ and find a unique number m such that $x \in K_m \setminus K_{m-1}$. It follows that $m \geq n+2$ and $\rho(f_\infty(x), f_0(x)) < 2^{-(m+1)} \leq 2^{-(n+3)}$. By the definition of the set K_{m-1}, we get $\rho(f_0(x), X \setminus U) < 2^{-(m-1)} \leq 2^{-(n+1)}$ and thus

$$\rho(f_0(x), C) \geq \rho(C, X \setminus U) - \rho(f_0(x), X \setminus U) > 2^{-n} - 2^{-(n+1)} = 2^{-(n+1)}.$$

Then $\rho(f_\infty(x), C) \geq \rho(f_0(x), C) - \rho(f_\infty(x), f_0(x)) > 2^{-(n+1)} - 2^{-(n+3)} > 0$ and thus $f_\infty(x) \notin C$. Therefore $f_\infty^{-1}(C) \subset K_{n+1}$. Since the map $f_\infty|K_{n+1} = f_{n+2}|K_{n+1}$ is perfect we conclude that the preimage $f_\infty^{-1}(C) = (f_\infty|K_{n+1})^{-1}(C)$ is compact. This means that the map $f_\infty : K \to U$ is perfect. □

Lemma 5. *A space X has n-PHAP provided X is a union of two open subspaces with n-PHAP.*

Proof. Suppose $X = U_0 \cup U_1$ where U_0, U_1 are open subspaces of X having n-PHAP. Find two open subsets $V_0, V_1 \subset X$ such that $V_0 \cup V_1 = X$ and $\overline{V}_i \subset U_i$ for $i = 0,1$.

To show that X has n-PHAP, fix an open cover $\mathcal{U}$ of X and a map $f : K \to X$ of a simplicial complex K with $\dim K \leq n$. Pick an open cover $\mathcal{V}$ of X such that $\mathcal{S}t\,\mathcal{V} \prec \mathcal{U}$ and $\mathrm{cl}_X(\mathcal{S}t\,(\overline{V}_i, \mathcal{S}t\,\mathcal{V})) \subset U_i$ for $i = 0,1$.

Let $W_i = f^{-1}(V_i)$ and $W_i' = f^{-1}(U_i)$ for $i = 0,1$. Taking a sufficiently fine triangulation of K, we can assume that each simplex of K lies in W_0 or W_1. Then the union K_i of simplexes lying in W_i is a subcomplex of K and $K_0 \cup K_1 = K$.

Since the space $W_0' \subset K$ is triangulable, the n-PHAP of U_0 allows us to find a proper map $f_0 : W_0' \to U_0$ which is $\mathcal{V}$-homotopic to $f|W_0'$ via a $\mathcal{V}$-homotopy $h_0 : W_0' \times [0,1] \to$

U_0 such that $h_0(x,0)=f(x)$ and $h_0(x,1)=f_0(x)$ for $x\in W_0'$. Note that $f_0(K_0)\subset \mathcal{S}t\,(f(K_0),\mathcal{V})\subset \mathcal{S}t\,(\overline{V}_0,\mathcal{V})\subset \mathrm{cl}_X(\mathcal{S}t\,(\overline{V}_0,\mathcal{V}))\subset U_0$ which implies that the map $f_0|K_0:K_0\to X$ is perfect.

Let $\lambda:K\to[0,1]$ be a continuous map such that $\lambda^{-1}(1)\supset K_0$ and $\lambda^{-1}(0)\supset K\setminus W_0$. Since $\overline{W}_0\subset W_0'$, we can define a homotopy $\tilde h_0:K\times[0,1]\to X$ letting $\tilde h_0(x,t)=h_0(x,\lambda(x)\cdot t)$ for $(x,t)\in W_0'\times[0,1]$ and $\tilde h_0(x,t)=f(x)$ for $x\notin W_0$ and $t\in[0,1]$. Let $\tilde f_0(x)=\tilde h_0(x,1)$. Since $\tilde f_0|K_0=f_0|K_0$ the map $\tilde f_0|K_0:K_0\to X$ is perfect.

Observe that $\tilde f_0(K_1)\subset \mathcal{S}t\,(f(K_1),\mathcal{V})\subset \mathcal{S}t\,(\overline{V}_1,\mathcal{V})\subset U_1$ and applying Lemma 3, find a perfect map $f_1:K_1\to U_1$ which is $\mathcal{V}$-homotopic to the restriction $\tilde f_0|K_1$ via a $\mathcal{V}$-homotopy $h_1:K_1\times[0,1]\to U_1$ such that $h_1(x,1)=f_1(x)$ and $h_1(x,t)=\tilde f_0(x)$ for $(x,t)\in K_1\times\{0\}\cup(K_0\cap K_1)\times[0,1]$. Then $f_1(K_1)\subset \mathcal{S}t\,(\tilde f_0(K_1),\mathcal{V})\subset \mathcal{S}t\,(\mathcal{S}t\,(f(K_1),\mathcal{V}),\mathcal{V})\subset \mathrm{cl}_X\mathcal{S}t\,(\overline{V}_1,\mathcal{S}t\,\mathcal{V})\subset U_1$ and hence the map $f_1|K_1:K_1\to X$ is perfect.

Finally, consider the map $g:K\to X$ defined by $g|K_0=\tilde f_0|K_0$ and $g|K_1=f_1$. The map g is perfect because so are its restrictions onto the closed sets K_0 and K_1. It is easy to show that g is $\mathcal{V}$-homotopic to $\tilde f_0$ and hence is $\mathcal{S}t\,\mathcal{V}$-homotopic to f. □

Now we can prove the second item of Theorem 1. We shall exploit the classical Michael result [Mi] on local properties. Following E. Michael we call a property $\mathcal{P}$ of topological spaces to be *local* if a space X has $\mathcal{P}$ if and only if each point of X has an open neighborhood with the property $\mathcal{P}$. According to [Mi] (see also Proposition 4.1 of [BP, Ch.II]) a property $\mathcal{P}$ is local if and only if $\mathcal{P}$ is *open-hereditary* (open subspaces of a space with the property $\mathcal{P}$ have that property), *open-additive* (a space has the property $\mathcal{P}$ if it is a union of two open subspaces with that property), and *discrete additive* (a space has $\mathcal{P}$ provided it is the union of a discrete family of open subspaces with the property $\mathcal{P}$).

Lemmas 4 and 5 imply that the n-PHAP is an open-hereditary and open-additive property. It is trivial to check that the discrete union of spaces with n-PHAP has n-PHAP. Applying the Michael Theorem, we conclude that n-PHAP is a local property. In other words the following lemma implying Theorem 1(2) is true.

Lemma 6. *A space X has n-PHAP provided X admits an open cover by subspaces with n-PHAP.*

The third statement of Theorem 1 follows from

Lemma 7. *If a space X has n-PHAP, then X has the discrete n-cells property.*

Proof. This lemma trivially follows from a result of [Cu] asserting that a space X has the discrete n-cells property if and only if each map $f:I^n\times\omega\to X$ can be approximated by a map g sending $\{I^n\times\{i\}\}_{i\in\omega}$ onto a locally finite collection in X. □

To reverse the preceding lemma we will need one classical result concerning LC^n-spaces.

Lemma 8. ([Hu, V.5.1]) *For any cover $\mathcal{U}\in\mathrm{cov}(X)$ of an LC^n-space X there is a cover $\mathcal{V}\in\mathrm{cov}(X)$ such that any two $\mathcal{V}$-near maps $f,g:K\to X$ from a space K with $\dim K\le n$ are $\mathcal{U}$-homotopic.*

Now we are able to prove the item 4 of Theorem 1.

Lemma 9. *An LC^n-space has n-PHAP if and only if it has the discrete n-cells property.*

Proof. The "only if" part follows from Lemma 7. The "if" part will be proven by induction. Fix any finite $n\ge 0$ and assume that Lemma 9 has been proved for all $k<n$. To show that an LC^n-space X with the discrete n-cells property has n-PHAP, fix a cover $\mathcal{U}\in\mathrm{cov}(X)$ and a map $f:K\to X$ from an n-dimensional simplicial complex K.

Let $\mathcal{U}_1 \in \mathrm{cov}(X)$ be an open cover with $\mathcal{S}t\,\mathcal{U}_1 \prec \mathcal{U}$. Let $K^{(n-1)}$ denote the $(n-1)$-dimensional skeleton of K. By the inductive hypothesis, the space X has $(n-1)$-PHAP which allows us to find a perfect map $g : K^{(n-1)} \to X$ which is $\mathcal{U}_1$-homotopic to $f|K^{(n-1)}$. Since the pair $(K, K^{(n-1)})$ has the homotopy extension property, the map g admits a continuous extension $\bar{g} : K \to X$, $\mathcal{U}_1$-homotopic to f.

By Lemma 2, the restriction $\bar{g}|\overline{W}$ is perfect for some closed neighborhood $\overline{W}$ of $K^{(n-1)}$ in K. By Lemma 1, there is a cover $\mathcal{U}_2 \in \mathrm{cov}(X)$ such that $\mathcal{U}_2 \prec \mathcal{U}_1$ and any map $p : \overline{W} \to X$, $\mathcal{U}_2$-near to $\bar{g}|\overline{W}$ is perfect. By Lemma 8 there is a cover $\mathcal{U}_3 \in \mathrm{cov}(X)$ such that any two $\mathcal{U}_3$-near maps from a space D with $\dim D \le n$ into X are $\mathcal{U}_2$-homotopic.

Write the complement $K \setminus K^{(n-1)} = \bigcup_{i\in I} \sigma_i$ as the disjoint union of open n-dimensional simplexes of K and consider the discrete topological sum $D = \bigsqcup_{i\in I} \bar{\sigma}_i$ of their closures in K. Denote by $i : K \setminus K^{(n-1)} \to D$ the natural embedding. There is a natural surjective perfect map $\pi : D \to K$ such that $\pi(\bigcup_{i\in I} \partial\bar{\sigma}_i) = K^{(n-1)}$.

Since X has the discrete n-cells property, there is a perfect map $q : D \to X$ such that $(q, \bar{g}\circ\pi) \prec \mathcal{U}_3$. By the choice of the cover $\mathcal{U}_3$, there is a $\mathcal{U}_2$-homotopy $h : D \times [0,1] \to X$ connecting the maps $\bar{g}\circ\pi$ and q in the sense that $h(x,0) = \bar{g}\circ\pi(x)$ and $h(x,1) = q(x)$ for $x \in D$. Let $\lambda : K \to [0,1]$ be a continuous map such that $\lambda^{-1}(0)$ is a neighborhood of $K^{(n-1)}$ and $K \setminus W \subset \lambda^{-1}(1)$. Finally, consider the map $p : K \to X$ defined by

$$p(x) = \begin{cases} g(x) & \text{if } x \in K^{(n-1)}, \\ h(i(x), \lambda(x)) & \text{otherwise.} \end{cases}$$

It is easy to see that the map p is continuous and $\mathcal{U}_2$-homotopic to $\bar{g}$. Taking into account that $\mathcal{U}_2 \prec \mathcal{U}_1$, $\mathcal{S}t\,\mathcal{U}_1 \prec \mathcal{U}$, and $\bar{g}$ is $\mathcal{U}_1$-homotopic to f, we conclude that the map p is $\mathcal{U}$-homotopic to f.

Finally, let us show that the map p is perfect. For this observe that the restriction $p|\overline{W}$, being $\mathcal{U}_2$-homotopic to $\bar{g}$, is perfect while the restriction $p|K \setminus W$, being equal to $q \circ i|K \setminus W$ is perfect too. □

For the proof of Theorem 1(5) we shall need

Lemma 10. *Let K be a simplicial complex and $\emptyset = L_0 \subset L_1 \subset \cdots$ be a tower of subcomplexes of K such that $K = \bigcup_{i\in\omega} L_i$ and each L_i lies in the interior of L_{i+1}. Then for any map $f : K \to X$ into a metric space (X,d) with n-PHAP and any sequence $(\varepsilon_i)_{i\in\omega}$ in $(0,1]$ there exists a map $\tilde{f} : K \to X$ and a homotopy $H : K \times [0,1] \to X$ satisfying the following conditions:*

(a) *$H(z,0) = f(z)$, $H(z,1) = \tilde{f}(z)$ for all $z \in K$;*
(b) *$\mathrm{diam}\, H(\{z\} \times [0,1]) < \varepsilon_k$ for all $z \in L_k \setminus L_{k-1}$ and $k \in \omega$;*
(c) *$\tilde{f}|L_k^{(n)}$ is perfect for every $k \in \omega$.*

Proof. Without loss of generality, $\varepsilon_{k+1} < \varepsilon_k/2$ for all $k \in \omega$. Put $f_0 = f$. By induction, for every $k \in \mathbb{N}$ we shall construct a map $f_k : K \to X$ and a homotopy $H_k : K \times [0,1] \to X$ satisfying the following conditions:

(1_k) $H_k(z,0) = f_{k-1}(z)$ and $H_k(z,1) = f_k(z)$ for all $z \in K$;
(2_k) $H_k(z,t) = f_{k-1}(z)$ for all $z \in L_{k-1} \cup \overline{K \setminus L_{k+1}}$ and $t \in [0,1]$;
(3_k) $\mathrm{diam}\, H_k(\{z\} \times [0,1]) < \varepsilon_{k+1}$ for all $z \in K$;
(4_k) $f_k|L_k^{(n)}$ is perfect.

Suppose that functions f_i and homotopies H_i have been constructed for $i \le k$. Take any open cover $\mathcal{U}$ of X with $\mathrm{mesh}\,\mathcal{U} < \varepsilon_{k+2}$. Using Lemma 3, find a perfect map $g : K^{(n)} \to X$, $\mathcal{U}$-homotopic to f_k via a homotopy $h : K^{(n)} \times [0,1] \to X$ such that $h(z,1) = g(z)$

for $z \in K^{(n)}$ and $h(z,t) = f_k(z)$ for $(z,t) \in K^{(n)} \times \{0\} \cup L_k^{(n)} \times [0,1]$. Then $M = L_k \cup L_{k+1}^{(n)} \cup \overline{K \setminus L_{k+2}}$ is a simplicial subcomplex of K and the homotopy extension property of the simplicial pair (K, M) allows us to find a $\mathcal{U}$-homotopy $H_{k+1} : K \times [0,1] \to X$ such that $H_{k+1}(z,t) = f_k(z)$ if $(z,t) \in K \times \{0\} \cup (L_k \cup \overline{K \setminus L_{k+2}}) \times [0,1]$ and $H_{k+1}(z,t) = h(z,t)$ if $(z,t) \in L_{k+1}^{(n)} \times [0,1]$. Letting $f_{k+1}(z) = H_{k+1}(z,1)$ for $z \in K$ we finish the inductive step.

The conditions (1_k)–(3_k) imply that the limit map $\tilde{f} = \lim_{k\to\infty} f_k$ is well-defined and continuous. Using the homotopies H_k it is easy to compose a homotopy H connecting the maps f and $\tilde{f}$ and satisfying the conditions (a)–(c) of the lemma. □

With Lemma 10 in disposition we can prove the fifth item of Theorem 1. It should be mentioned that a particular case of Lemma 11 was proven by P.Bowers in [Bo$_2$, 4.6].

Lemma 11. *If X_1 is a space with n_1-PHAP and X_2 is a space with n_2-PHAP, then the product $X_1 \times X_2$ has $(n_1 + n_2 + 1)$-PHAP.*

Proof. Let $n = n_1 + n_2 + 1$, K be a simplicial complex with $\dim K \le n, \mathcal{U} \in \operatorname{cov}(X_1 \times X_2)$, and $f = (f_1, f_2) : K \to X_1 \times X_2$ be a map. For every $i \in \{1,2\}$ fix an admissible metric $d_i < 1$ on X_i. On the product $X_1 \times X_2$ consider the metric $d((x_1,x_2),(x_1',x_2')) = \max\{d_1(x_1,x_1'), d_2(x_2,x_2')\}$. Find a continuous map $\varepsilon : X_1 \times X_2 \to (0,1]$ such that $\{B(x, 6\varepsilon(x)) : x \in X_1 \times X_2\} \prec \mathcal{U}$. Replacing K by its sufficiently fine subdivision, we can assume that for any simplex σ of K we have

(1) $\min\{\varepsilon \circ f(z) : z \in \sigma\} > \frac{1}{2} \max\{\varepsilon \circ f(z) : z \in \sigma\}$ and

(2) $\operatorname{diam} f(\sigma) < \min\{\varepsilon \circ f(z) : z \in \sigma\}$.

For every $k \in \omega$ let $F_k = (\varepsilon \circ f)^{-1}([2^{-k}, 1])$. It follows from (1) that any simplex of K meeting F_k lies in the interior of F_{k+1}. Consequently, the simplicial subcomplex L_k of K, composed by simplexes meeting F_k lies in the interior of the subcomplex L_{k+1}. Evidently, the subcomplexes L_k, $k \in \omega$, cover the complex K.

Denote by K_1 the n_1-dimensional skeleton of K and let K_2 be the full subcomplex of the barycentric subdivision of K, generated by the barycenters of simplexes of dimension $> n_1$. Then K_2 is a subcomplex of dimension $\dim K - (n_1 + 1) \le n_2$ of the barycentric subdivision of K. Applying Lemma 10 with $\varepsilon_k = 2^{-(k+1)}$, for every $i \in \{1,2\}$ we can find a map $\bar{f}_i : K \to X_i$ and a homotopy $H_i^1 : K \times [0,1] \to X_i$ such that the following conditions hold

(3) $H_i^1(z,0) = f_i(z)$ and $H_i^1(z,1) = \bar{f}_i(z)$ for $z \in K$;

(4) $\operatorname{diam} H_i(\{z\} \times [0,1]) < \varepsilon \circ f(z)$ for $z \in K$;

(5) $\bar{f}_i | K_i \cap L_k$ is perfect for all $k \in \omega$.

Observe that for points z, z' of a simplex σ of K, the conditions (1), (2) and (4) imply

$$d_i(\bar{f}_i(z), \bar{f}_i(z')) \le d_i(\bar{f}_i(z), f_i(z)) + \operatorname{diam} f_i(\sigma) + d_i(f_i(z'), \bar{f}_i(z')) < \varepsilon \circ f(z) + \operatorname{diam} f_i(\sigma) + \varepsilon \circ f(z') < 5 \min \varepsilon \circ f_i(\sigma),$$

which yields $\operatorname{diam} \bar{f}_i(\sigma) < 5 \min \varepsilon \circ f(\sigma)$.

Each point $z \in K$ can be written as $z = sz_1 + (1-s)z_2$ with $z_i \in K_i$ and $s \in [0,1]$ and such a representation is unique if $z \notin K_1 \cup K_2$. The set C_1 (resp. C_2) of points z for which $s \ge \frac{1}{2}$ (resp. $s \le \frac{1}{2}$) is closed in K and $K = C_1 \cup C_2$. For every $i \in \{1,2\}$ there is a homotopy $\Phi_i : K \times [0,1] \to K$ such that $\Phi_i(z,0) = z$, $\Phi_i(C_i \times \{1\}) \subset K_i$ and $\Phi_i(\sigma \times [0,1]) \subset \sigma$ for each simplex σ of K (such a homotopy Φ_i can be defined by $\Phi_i(z,t) = \alpha_i(s,t)z_1 + (1-\alpha_i(s,t))z_2$ for $z = sz_1 + (1-s)z_2$, where $\alpha_1(s,t) = \min\{1, (1+t)s\}$ and $\alpha_2(s,t) = \max\{0, s + t(s-1)\}$).

For $i \in \{1,2\}$, define a homotopy $H_i^2 : K \times [0,1] \to X_i$ by $H_i^2(z,t) = \bar{f}_i \circ \Phi_i(z,t)$ and let $g_i(z) = H_i^2(z,1)$. Let $z \in K$ and σ be a simplex of K, containing the point z. Since $\Phi_i(\sigma \times [0,1]) \subset \sigma$ we get $\operatorname{diam} H_i^2(\{z\} \times [0,1]) \le \operatorname{diam} \bar{f}_i(\sigma) < 5\varepsilon \circ f(z)$. Since $H_i^1(z,1) = \bar{f}_i(z) = H_i^2(z,0)$, we can glue H_i^1 and H_i^2 together and define a homotopy H_i linking f_i and g_i and such that $\operatorname{diam} H_i(\{z\} \times [0,1]) < 6\varepsilon \circ f(z)$ for all $z \in K$. Then $H = (H_1, H_2)$ is a homotopy between f and $g = (g_1, g_2)$ such that $\operatorname{diam} h(\{z\} \times [0,1]) < 6\varepsilon \circ f(z)$ for all $z \in K$. The choice of ε guarantees that H is a $\mathcal{U}$-homotopy.

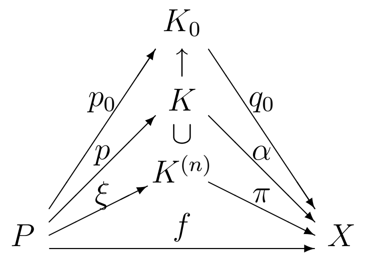
Let us show that the map g is perfect. Assuming the converse we would find a sequence $\{z_r\}$ without limit points in K and such that the sequence $\{g(z_r)\}$ converges to some point $x = (x_1, x_2) \in X$. Since $C_1 \cup C_2 = K$, we can suppose that $\{z_r\} \subset C_i$ for some $i \in \{1,2\}$. The inclusion $\Phi_i(\sigma \times [0,1]) \subset \sigma$ for any simplex σ of K implies that the homotopy Φ_i is proper and $\Phi_i(L_k \times [0,1]) \subset L_k$ for all k. In particular, $\Phi_i((C_i \cap L_k) \times \{1\}) \subset K_i \cap L_k$ and since the restriction $\bar{f}_i | K_i \cap L_k$ is proper, we get that the restriction of g_i onto the closed subset $C_i \cap L_k$ is proper. Then $C_i \cap L_k$ contains only finitely many points z_r which yields $\varepsilon \circ f(z_r) < 2^{-k}$ for all sufficiently large r and thus $\lim_{r\to\infty} \varepsilon \circ f(z_r) = 0$. Since $d(f(z_r), g(z_r)) < 6\varepsilon \circ f(z_r)$, we get that the sequence $\{f(z_r)\}$ converges to x and thus $\varepsilon(x) = \lim_{r\to\infty} \varepsilon \circ f(z_r) = 0$, which is impossible. ☐

Let X be a topological space and $\mathcal{U} \in \operatorname{cov}(X)$. We define a subset $B \subset X$ to be *$\mathcal{U}$-bounded*, if $B \subset \cup\mathcal{F}$ for some finite subcollection $\mathcal{F}$ of $\mathcal{U}$.

Lemma 12. *Let X be a space with n-PHAP and $\mathcal{U} \in \operatorname{cov}(X)$. Then for any simplicially approximable map $f : P \to X$ from a space P with $\dim P \le n$ and any open cover $\mathcal{V}$ of P there exists an open cover $\mathcal{W}$ of X and a map $g : P \to X$, $\mathcal{U}$-homotopic to f and such that $g^{-1}(A)$ is $\mathcal{V}$-bounded in P for any $\mathcal{W}$-bounded subset $A \subset X$.*

Proof. Given a cover $\mathcal{U} \in \operatorname{cov}(X)$ let $\mathcal{U}' \in \operatorname{cov}(X)$ be any cover with $\mathcal{S}t^2\mathcal{U}' \prec \mathcal{U}$. Since f is simplicially approximable, there are a simplicial complex K_0 and two maps $p_0 : P \to K_0$ and $q_0 : K_0 \to X$ such that the map $q_0 \circ p_0$ is $\mathcal{U}'$-homotopic to f. Replacing the triangulation of K_0 by a sufficiently fine subdivision, if necessary, we can assume that $\mathcal{S}t\,(K_0) \prec q_0^{-1}(\mathcal{U}')$.

Let $\mathcal{V}_1 \prec \mathcal{V}$ be an open star-finite cover of P, K_1 be the nerve of $\mathcal{V}_1$ and $p_1 : P \to K_1$ be a canonical map such that $p_1^{-1}(\mathcal{S}t\,(K_1)) \prec \mathcal{V}$. Let $K = K_0 \times K_1$, $p = (p_0, p_1) : P \to K$ and $\alpha = q_0 \circ \mathrm{pr}_{K_0} : K \to X$. Endow K with a triangulation such that the projections of K onto K_0 and K_1 are simplicial maps. Then $\mathcal{S}t\,(K) \prec (\mathrm{pr}_{K_0})^{-1}(\mathcal{S}t\,(K_0)) \prec \alpha^{-1}(\mathcal{U}')$ while $p^{-1}(\mathcal{S}t\,(K)) \prec p_1^{-1}(\mathcal{S}t\,(K_1)) \prec \mathcal{V}$.



Since $\dim P \le n$, there is a continuous function $\xi : P \to K^{(n)}$ such that for any $x \in P$ the point $\xi(x)$ belongs to the minimal simplex containing $p(x)$. Then ξ is $\mathcal{S}t\,(K)$-homotopic to p and hence $\alpha \circ \xi$ is $\mathcal{U}'$-homotopic to $\alpha \circ p = q_0 \circ p_0$. On the other hand, for every vertex v of K, $\xi^{-1}(\mathcal{S}t\,(v,K)) \subset p^{-1}(\mathcal{S}t\,(v,K))$ and thus $\xi^{-1}(\mathcal{S}t\,(K))$ refines $\mathcal{V}$.

Using the n-PHAP of X, we can find a perfect map $\pi : K^{(n)} \to X$, $\mathcal{U}'$-homotopic to $\alpha | K^{(n)}$. Then $g = \pi \circ \xi$ is $\mathcal{U}'$-homotopic to $\alpha \circ \xi$ and consequently, $\mathcal{S}t^2(\mathcal{U}')$-homotopic to f.

Since π is perfect and $\mathcal{S}t\,(K)$ is locally finite, each point $x \in X$ has an open neighborhood $O(x)$ such that $\pi^{-1}(O(x))$ is $\mathcal{S}t\,(K)$-bounded. Then $g^{-1}(O(x))$ is $\xi^{-1}(\mathcal{S}t\,(K))$-bounded and hence $\mathcal{V}$-bounded. Consequently, the cover $\mathcal{W} = \{O(x) : x \in X\}$ has the desired properties. □

Next, we prove the sixth item of Theorem 1.

Lemma 13. *For any simplicially approximable map $f : P \to X$ from a Polish space P with $\dim P \le n$ into a Polish space X with n-PHAP and any open cover $\mathcal{U} \in \mathrm{cov}(X)$ there is a perfect map $g : P \to X$, $\mathcal{U}$-homotopic to f.*

Proof. We assume that the Polish spaces P and X are endowed with some complete metrics generating their topology.

Let $f_{-1} = f$ and $\mathcal{U}_{-1} = \mathcal{U}$. Using Lemma 12 we can construct by induction two sequences of star-finite open covers $(\mathcal{V}_n)_{n\in\omega} \subset \mathrm{cov}(P)$ and $(\mathcal{U}_n)_{n\in\omega} \subset \mathrm{cov}(X)$ and a sequence $(f_n)_{n\in\omega}$ of continuous maps from P into X satisfying the following conditions:

(a) $\lim\limits_{n\to\infty} \mathrm{mesh}(\mathcal{V}_n) = 0$;
(b) $\mathrm{mesh}(\mathcal{U}_n) < \frac{1}{n^2}$ for every $n \in \omega$;
(c) $\mathcal{S}t\,(\mathcal{U}_{n+1}) \prec \mathcal{U}_n$ for every $n \in \omega$;
(d) $f_n^{-1}(B)$ is $\mathcal{V}_n$-bounded in P for any $\mathcal{U}_n$-bounded subset $B \subset X$;
(e) f_n and f_{n-1} are $\mathcal{U}_{n-1}$-homotopic for all $n \in \omega$.

It follows from (b), (c) and (e) that the limit map $g = \lim_{n\to\infty} f_n : P \to X$ is a well-defined continuous function, $\mathcal{S}t\,(\mathcal{U}_n)$-homotopic to each f_n.

We claim that the map g is proper. Indeed, let C be a compact subset of X. We have to show that $g^{-1}(C)$ is compact. Since $g^{-1}(C)$ is closed in the complete metric space P, we may prove the total boundedness of $g^{-1}(C)$. Due to (a) it suffices to verify that for every $n \in \omega$ the set $g^{-1}(C)$ is $\mathcal{V}_n$-bounded. Since $(g, f_n) \prec \mathcal{S}t\,(\mathcal{U}_n)$, we get $g^{-1}(C) \subset f_n^{-1}\big(\mathcal{S}t\,(C, \mathcal{S}t\,(\mathcal{U}_n))\big)$. Taking into account that the cover $\mathcal{U}_n$ is star-finite and the set C is compact, we conclude that the set $\mathcal{S}t\,(C, \mathcal{S}t\,(\mathcal{U}_n))$ is $\mathcal{U}_n$-bounded. Then (d) implies that $f_n^{-1}(\mathcal{S}t\,(C, \mathcal{S}t\,(\mathcal{U}_n))) \supset g^{-1}(C)$ is $\mathcal{V}_n$-bounded. □

For the proof of two last items of Theorem 1 we need to recall some definitions from [BRZ]. Given two spaces X, Y denote by $C(X,Y)$ the space of all continuous functions from X to Y, endowed with the limitation topology whose neighborhood base at an $f \in C(X,Y)$ consists of the sets $B(f,\mathcal{U}) = \{g \in C(X,Y) : (g,f) \prec \mathcal{U}\}$, where $\mathcal{U}$ runs over all open covers of Y, see [Bo$_3$]. If the space Y is Polish, then the space $C(X,Y)$ is Baire, see [To] or [BRZ, 3.2.1].

By a multivalued map $\mathcal{F} : Z \Rightarrow Y$ we understand a function assigning to each point $z \in Z$ a (possibly empty) subset $\mathcal{F}(z) \subset Y$. Such a multivalued map $\mathcal{F} : Z \Rightarrow Y$ is called *perfect* if for any compact subsets $A \subset Z$, $B \subset Y$ the sets $\mathcal{F}(A) = \bigcup_{z\in A} \mathcal{F}(z)$ and $\mathcal{F}^{-1}(B) = \{z \in Z : \mathcal{F}(z) \cap B \ne \emptyset\}$ are compact.

Following [BRZ, p.124] we define a map $f : X \to Y$ to be $\mathcal{F}$*-injective* if $|f^{-1}(\mathcal{F}(z))| \le 1$ for all $z \in Z$. A map $f : X \to Y$ is called a $(\mathcal{U},\mathcal{F})$*-map*, where $\mathcal{U}$ is an open cover of X, if there is an open cover $\mathcal{V}$ of Y such that $\big\{f^{-1}\big(\mathcal{S}t\,(\mathcal{F}(z), \mathcal{V})\big)\big\}_{z\in Z} \prec \mathcal{U}$.

Lemma 14. *Let $U \subset \mathbb{R}^\omega$ be an open subspace of the countable product of lines and $\mathcal{F} : Z \Rightarrow U$ be a perfect multivalued map. For any Polish space P the set of all perfect $\mathcal{F}$-injective maps is dense in the function space $C(P,U)$.*

Proof. Fix a complete metric on the Polish space P and let $(\mathcal{U}_n)_{n\in\omega}$ be a sequence of open covers of P with $\mathrm{mesh}\,\mathcal{U}_n < 2^{-n}$ for all $n \in \omega$.

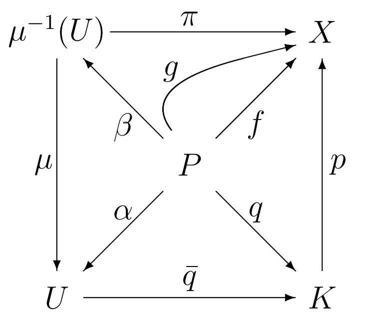
By [To] the set $\mathcal{E}$ of closed embeddings is dense G_δ in $C(P,U)$. By Lemma 3.2.14 of [BRZ] for every $n \in \omega$ the set $\mathcal{H}_n$ of $(\mathcal{U}_n, \mathcal{F})$-maps is open and dense in $C(P,U)$. Since the function space $C(P,U)$ is Baire (see [To, 1.1]), the intersection $\mathcal{I} = \mathcal{E} \cap \bigcap_{n\in\omega} \mathcal{H}_n$ is dense in $C(P,U)$. It is clear that each function $f \in \mathcal{I}$ is perfect and $\mathcal{F}$-injective. □

Our final lemma proves the item (7) of Theorem 1 and (8) follows from (7) applied to a constant map.

Lemma 15. *If a Polish space X has n-PHAP, then for any open cover $\mathcal{U}$ of X and any simplicially approximable map $f : P \to X$ from a Polish space P with $\dim P \le n$ there is a closed embedding $g : P \to X$, $\mathcal{U}$-near to f.*

Proof. Let $\mathcal{V} \in \mathrm{cov}(X)$ be any cover with $\mathcal{S}t\,(\mathcal{V}) \prec \mathcal{U}$. The map $f : P \to X$, being simplicially approximable, is $\mathcal{V}$-homotopic to the composition $p \circ q$ of maps $q : P \to K$, $p : K \to X$, where K is a simplicial complex. Identify the Polish space P with a closed subset of $s = (-1,1)^\omega$, the pseudo-interior of the Hilbert cube $Q = [-1,1]^\omega$. Since K is an ANR, the map q admits a continuous extension $\bar{q} : U \to K$ onto some open neighborhood U of P in s.

According to a result of Dranishnikov [Dr] (see also [BRZ, 2.3.5]), there is an map $\mu : N \to Q$ from an n-dimensional compactum N onto Q, which is n-invertible in the sense that for any map $\alpha : A \to Q$ from a space A with $\dim A \le n$ there is a map $\beta : A \to N$ such that $\alpha = \mu \circ \beta$. It follows that $\mu^{-1}(U)$ is a Polish space with $\dim \mu^{-1}(U) \le \dim N \le n$.



Consider the simplicially approximable map $p \circ \bar{q} \circ \mu : \mu^{-1}(U) \to X$. By Lemma 13, it is $\mathcal{V}$-near to a perfect map $\pi : \mu^{-1}(U) \to X$. It is easy to see that for any $t \in U$ we get $\pi(\mu^{-1}(t)) \subset \mathcal{S}t\,(p \circ \bar{q}(t), \mathcal{V})$. Since the map $\mu|\mu^{-1}(U)$ is perfect, we can find an open cover $\mathcal{W}$ of U such that $\pi\big(\mu^{-1}(\mathcal{S}t\,(t,\mathcal{W}))\big) \subset \mathcal{S}t\,(p \circ \bar{q}(t), \mathcal{V})$ for all $t \in U$.

Now consider the multivalued map $\mathcal{F} : U \Rightarrow U$ defined by $\mathcal{F}(x) = \mu \circ \pi^{-1} \circ \pi \circ \mu^{-1}(x)$ for $x \in U$ and observe that it is perfect (in the sense that for any compact set $C \subset U$ the sets $\mathcal{F}(C)$ and $\mathcal{F}^{-1}(C)$ are compact in U). By Lemma 14, there is a perfect $\mathcal{F}$-injective map $\alpha : P \to U$ which is $\mathcal{W}$-near to the inclusion $P \subset U$. By the choice of the map μ, there is a map $\beta : P \to \mu^{-1}(U)$ such that $\alpha = \mu \circ \beta$. The perfectness of the maps α and π implies the perfectness of the maps β and $g = \pi \circ \beta : P \to X$. Moreover, the $\mathcal{F}$-injectivity of the map α implies the injectivity of the map g. Thus g, being injective and perfect, is a closed embedding.

Observe that for each $t \in P$ we get

$$g(t) = \pi \circ \beta(t) \in \pi\big(\mu^{-1}(\alpha(t))\big) \subset \pi\big(\mu^{-1}(\mathcal{S}t\,(t,\mathcal{W}))\big) \subset \mathcal{S}t\,(p \circ q(t), \mathcal{V}),$$

which means that the maps g and $p \circ q$ are $\mathcal{V}$-near. Since f and $p \circ q$ are $\mathcal{V}$-near and $\mathcal{S}t\,\mathcal{V} \prec \mathcal{U}$ we get that f and g are $\mathcal{U}$-near. □

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