ON UNIVERSALITY OF FINITE PRODUCTS OF POLISH SPACES

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ABSTRACT. We introduce and study the $n$-Dimensional Perfect Homotopy Approximation Property (briefly $n$-PHAP) equivalent to the discrete $n$-cells property in the realm of LC$^n$-spaces. It is shown that the product $X \times Y$ of a space $X$ with $n$-PHAP and a space $Y$ with $m$-PHAP has $(n+m+1)$-PHAP. We derive from this that for a (nowhere locally compact) locally connected Polish space $X$ without free arcs and for each $n \geq 0$ the power $X^{n+1}$ contains a closed topological copy of each at most $n$-dimensional compact (resp. Polish) space.

A topological space $X$ is called $\mathcal{C}$-universal, where $\mathcal{C}$ is a class of spaces, if $X$ contains a closed topological copy of each space $C \in \mathcal{C}$. By $\mathcal{M}_0$ and $\mathcal{M}_1$ we denote the classes of metrizable compacta and Polish (= separable complete-metrizable) spaces, respectively. For a class $\mathcal{C}$ of spaces by $\mathcal{C}[n]$ we denote the subclass of $\mathcal{C}$ consisting of all spaces $C \in \mathcal{C}$ with $\dim C \leq n$. All topological spaces considered in the paper are metrizable and separable, all maps are continuous.

In terms of the universality, the classical Menger-Nöbeling-Pontrjagin-Lefschetz Theorem states that the cube $[0,1]^{2n+1}$ is $\mathcal{M}_0[n]$-universal for every $n \geq 0$. It is well known that the exponent $2n+1$ in this theorem is the best possible: the Menger universal compactum $\mu_n$ cannot be embedded into $[0,1]^{2n}$. Nonetheless, P. Bowers [Bo1] has proved that the $(n+1)$-th power $D^{n+1}$ of any dendrite $D$ with dense set of end-points does be $\mathcal{M}_0[n]$-universal for every non-negative integer $n$. Moreover, any such a dendrite $D$ contains a locally connected $G_{\delta}$-subspace $G$ whose $(n+1)$-th power $G^{n+1}$ is $\mathcal{M}_1[n]$-universal for every $n$, see [Bo1]. Generalizing this Bowers’ result we shall prove that the power $X^{n+1}$ of any locally connected Polish space $X$ without free arcs is $\mathcal{M}_0[n]$-universal for all $n \geq 0$; moreover the power $X^{n+1}$ is $\mathcal{M}_1[n]$-universal provided $X$ is nowhere locally compact.

The standard way to prove the $\mathcal{M}_1[n]$-universality of a Polish space $X$ with nice local structure is to verify the discrete $n$-cells property for $X$, see [Bo1]. We remind that a space $X$ has the discrete $n$-cells property if for any map $f : N \times [0,1]^n \to X$ and any open cover $U$ of $X$ there is a map $g : N \times [0,1]^n \to X$ such that $g$ is $U$-near to $f$ and the collection $\{g(\{i\} \times [0,1]^n)\}_{i \in \mathbb{N}}$ is discrete in $X$.

Let us recall that two maps $f, g : Z \to X$ are called $U$-near with respect to a cover $U$ of $X$ (this is denoted by $(f,g) < U$) if for any point $z \in Z$ there is an element $U \in U$ such that $\{f(z), g(z)\} \subseteq U$. Two maps $f, g : Z \to X$ are called $U$-homotopic if they can be linked by a homotopy $\{h_t : Z \to X\}_{t \in [0,1]}$ such that $h_0 = f$, $h_1 = g$ and for any $z \in Z$ there is $U \in U$ with $\{h_t(z) : t \in [0,1]\} \subseteq U$. It is clear that $U$-homotopic maps are $U$-near while the converse is not true in general.

Unfortunately, the discrete $n$-cells property is applicable only for spaces having nice local structure. To overcome this obstacle we introduce a stronger property, called $n$-PHAP, which is equivalent to the discrete $n$-cells property in the realm of LC$^n$-spaces. We remind that a space $X$ is called an LC$^n$-space, $n \geq 0$, if for any point $x \in X$ and any neighborhood $U \subset X$ of $x$ there is a neighborhood $V \subset X$ of $x$ such that any map

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f : ∂I^n → V from the boundary of the n-dimensional cube I^n = [0, 1]^n can be extended to a map f : I^n → U defined on the whole n-cube I^n.

All simplicial complexes considered in this paper are countable and locally finite. We shall identify simplicial complexes with their geometric realizations.

**Definition 1.** A space X is defined to have the n-dimensional perfect homotopy approximation property (briefly n-PHAP) if for any map f : K → X from a simplicial complex K with dim K ≤ n and any open cover U of X there is a perfect map g : K → X, U-homotopic to f.

We remind that a map f : X → Y is perfect if f is closed and the preimage f^{-1}(y) of any point y ∈ Y is compact. According to [En, 3.7.18], a map f : X → Y between metrizable spaces is perfect if and only if f is proper in the sense that the preimage f^{-1}(K) of any compact subset K ⊂ Y is compact.

A map f : X → Y is called simplicially approximable if for any open cover U of X there is a simplicial complex K and two maps p : X → K and q : K → Y such that the composition q ◦ p is U-homotopic to f. It follows from Corollary 6.6 [BP, p.80] that each map into an absolute neighborhood retract is simplicially approximable.

Some basic properties of spaces with n-PHAP are described by the following theorem which is the main result of this paper.

**Theorem 1.** Let n, m be non-negative integers.

1. If a space X has n-PHAP, then each open subspace of X has that property too.
2. A space X has n-PHAP provided X admits a cover by open subspaces with n-PHAP.
3. If a space X has n-PHAP, then X has the discrete n-cells property.
4. An LC^n-space X has n-PHAP if and only if X has the discrete n-cells property.
5. If X is a space with n-PHAP and Y is a space with m-PHAP, then their product X × Y has (n + m + 1)-PHAP.
6. If a Polish space X has n-PHAP, then for any open cover U of X and any simplicially approximable map f : P → X from a Polish space P with dim P ≤ n there is a perfect map g : P → X, U-homotopic to f.
7. If a Polish space X has n-PHAP, then for any open cover U of X and any simplicially approximable map f : P → X from a Polish space P with dim P ≤ n there is a closed embedding g : P → X, U-near to f.
8. If a Polish space X has n-PHAP, then X is M_1[n]-universal.

Statements 4, 5, and 8 of Theorem 1 imply

**Corollary 1.** If X is a Polish LC^n-space with the discrete n-cells property, then for every k ≥ 0 the power X^{k+1} is M_1[nk + n + k]-universal.

In its turn, the last corollary implies another two corollaries generalizing the mentioned Bowers’ results on the universality of finite powers of dendrites.

**Corollary 2.** If X is a locally connected Polish nowhere locally compact space, then for every k ≥ 0 the power X^{k+1} is M_1[k]-universal.

**Proof.** The Polish space X, being locally connected, is locally path-connected and hence LC^0 according to the classical Mazurkiewicz-Moore-Menger Theorem, see [Ku]. It is well-known (and easy) that the discrete 0-cells property is equivalent to the nowhere local compactness. In this situation it is legal to apply Corollary 1 to conclude that the power X^{k+1} is M_1[k]-universal for every k ≥ 0. □
We say that a topological space $X$ has no free arcs if no open subset of $X$ is homeomorphic to the open interval $(0, 1)$.

**Corollary 3.** If $X$ is a locally connected Polish space without free arcs, then for every $k \geq 0$ the power $X^{k+1}$ is $\mathcal{M}_0[k]$-universal.

**Proof.** Corollary 3 will follow from Corollary 2 as soon as we prove that each locally connected Polish space $X$ without free arcs contains a locally connected nowhere locally compact Polish subspace $Y$.

Replacing $X$ by any of its connected component, we can assume that $X$ is connected. Then by [Wy, Ch.VIII.9] the space $X$ admits a compatible metric $d$ such that any points $x, y \in X$ can be linked by an arc whose diameter does not exceed $2d(x, y)$. Fix a countable dense subset $D \subset X$ and for any points $x, y \in D$ fix an arc $J(x, y) \subset X$ with $\text{diam } J(x, y) \leq 2d(x, y)$. It is easy to see that any subspace $Y \subset X$ containing the set $A = \bigcup_{x,y \in D} J(x, y)$ is locally path-connected. Since the Polish space $X$ has no free arcs, the Baire Theorem implies that the complement $X \setminus A$ is dense in $X$. Let $C \subset X \setminus A$ be a countable dense set. Then $Y = X \setminus C$ is a locally connected nowhere locally compact Polish subspace of $X$. \hfill \Box

1. **Proof of Theorem 1**

Our notations are standard. In particular, by $\bar{A}$ or $\text{cl}_X(A)$ we denote the closure of a set $A$ in a topological space $X$; $\text{cov}(X)$ stands for the family of all open covers of a space $X$. For a cover $U$ of $X$ and a subset $A \subset X$, let $\text{St}(A, U) = \{U \in U : U \cap A \neq \emptyset\}$, $\text{St}^1(U) = \text{St}(U) = \{\text{St}(U, U) : U \in U\}$, and $\text{St}^n(U) = \text{St}(\text{St}^{n-1}(U))$ for $n \geq 1$. Given two families $U, V$ of subsets of a space $X$ we write $U \prec V$ if any $U \in U$ lies in some $V \in V$.

For a map $f : Z \to X$ and a family $U$ of subsets of $X$ we put $f^{-1}(U) = \{f^{-1}(U) : U \in U\}$.

For a metric space $(X, d)$ and a point $x_0 \in X$ by $B(x_0, \varepsilon) = \{x \in X : d(x, x_0) < \varepsilon\}$ we denote the open $\varepsilon$-ball centered at $x_0$. Also we put $\text{mesh } U = \sup_{U \in U} \text{diam } U$ for a cover $U$ of $X$. A homotopy $h : Z \times [0, 1] \to X$ is called an $\varepsilon$-homotopy if $\text{diam } h(\{z\} \times [0, 1]) < \varepsilon$ for all $z \in Z$.

For a simplicial complex $K$, denote by $K^{(n)}$ the $n$-dimensional skeleton of $K$ and let $\text{St}(K) = \{\text{St}(v, K) : v \in K^{(0)}\}$ where $\text{St}(v, K)$ stands for the open star of a vertex $v$ in $K$. Several times we shall use the following homotopy extension property of simplicial pairs (see Corollary 5 of [Spa, p.112]): If $L$ is a subcomplex of a simplicial complex $K$, $f : K \to X$ is a continuous map into a space $X$, and $h : L \times [0, 1] \to X$ is a homotopy with $h(z, 0) = f(z)$ for all $z \in L$, then there is a homotopy $H : K \times [0, 1] \to X$ such that $H|L \times [0, 1] = h$ and $H(z, 0) = f(z)$ for all $z \in K$. If $h$ is a $U$-homotopy for some open cover $U$ of $X$, then $H$ can be chosen to be a $U$-homotopy. If $\text{diam } h(\{x\} \times [0, 1]) < \varepsilon \circ f(x)$, $x \in L$, for some continuous map $\varepsilon : X \to (0, \infty)$, then $H$ can be chosen so that $\text{diam } H(\{x\} \times [0, 1]) < \varepsilon \circ f(x)$ for all $x \in K$.

In the proof of Theorem 1 we shall exploit some known facts about proper maps.

**Lemma 1.** For a perfect map $f : K \to X$ from a locally compact space $K$ there is an open cover $U$ of $X$ such that each map $g : K \to X$ with $(f, g) \prec U$ is perfect.

**Proof.** Let $\bar{X}$ be any metrizable compactification of $X$. It follows from [En, 3.7.21] that the image $f(K)$ of the locally compact space $K$ under the perfect map $f : K \to X$ is a closed locally compact subspace of $X$. Consequently, $f(K)$, being locally compact, is open in its closure $\text{cl}_{\bar{X}}(f(K))$ in $\bar{X}$ and hence the complement $F = \text{cl}_{\bar{X}}(f(K)) \setminus f(K)$ is closed in $\bar{X}$. It follows that $\tilde{X} = \bar{X} \setminus F$ is a locally compact space containing $X$ so that
the map $f : K \to X \subseteq \widetilde{X}$ still is perfect. Now it is legal to apply Theorem 4.1 of [Ch] to find an open cover $\mathcal{U}$ of $X$ such that each map $g : K \to \widetilde{X}$ with $\langle f, g \rangle \prec \mathcal{U}$ is perfect. Then the open cover $\mathcal{U} = \{ U \cap X : U \in \mathcal{U} \}$ satisfies our requirements.

Lemma 2. If $f : K \to X$ is a map from a locally compact space $K$ and the restriction $f|L : L \to X$ of $f$ onto a closed subset $L \subseteq K$ is perfect, then $f|\overline{W}$ is perfect for some closed neighborhood $\overline{W}$ of $L$ in $K$.

Proof. Fix any metric $d$ generating the topology of $X$ and write $K = \bigcup_{i \geq 0} K_i$ as the countable union of an increasing sequence $(K_i)_{i \geq 0}$ of compact subsets such that $K_0 = \emptyset$ and each $K_i$ lies in the interior of $K_{i+1}$. For each $i \geq 1$ and $z \in K_i \setminus K_{i-1}$ find a neighborhood $O(z) \subset K$ such that $O(z) \subset K_{i+1} \setminus K_{i-1}$ and $f(O(z)) \subset B(f(z), \frac{1}{i}) = \{ x \in X : d(x, f(z)) < \frac{1}{i} \}$. Let $\overline{W}$ be any closed neighborhood of $L$ in $K$ with $\overline{W} \subset \bigcup_{z \in L} O(z)$.

Let us show that the restriction $f|\overline{W}$ is perfect. Assuming the converse we could find a sequence $(x_i)_{i \geq 1} \subset \overline{W}$ that has no cluster point in $\overline{W}$ but $(f(x_i))_{i \geq 1}$ converges to some point $a$ in $X$. Passing to a subsequence, if necessary, we can assume that $x_i \notin K_i$. For every $i \geq 1$ find a point $z_i \in L$ with $x_i \in O(z_i)$. Taking into account that $x_i \notin K_i$ and $O(z) \subset K_i$ for all $z \in K_{i-1}$, we conclude that $z_i \notin K_{i-1}$ for all $i \geq 1$. Then $d(f(x_i), f(z_i)) < \frac{1}{i}$ for $i \geq 1$ and thus the sequence $(f(z_i))$ converges to $a = \lim f(x_i)$ which is not possible since $f|L$ is perfect and the sequence $(z_i)$ has no cluster point in $L$.

Applying $n$-PHAP it will be convenient to work with its stronger version.

Lemma 3. If a space $X$ has $n$-PHAP, then for any open cover $\mathcal{U}$ of $X$, any simplicial complex $K$ with $\dim K \leq n$, any closed subspace $F \subset K$, and any map $f : K \to X$ whose restriction $f|F : F \to X$ is perfect, there is a perfect map $g : K \to X$, $\mathcal{U}$-homotopic to $f$ via a $\mathcal{U}$-homotopy $h : K \times [0, 1] \to X$ such that $h(x, 1) = g(x)$ for all $x \in K$ and $h(x, t) = f(x)$ for all $(x, t) \in K \times \{ 0 \} \cup F \times [0, 1]$.

Proof. By Lemma 2, the restriction $f|\overline{W}$ is perfect for some closed neighborhood $\overline{W}$ of $F$ in $K$. By Lemma 1, there is a cover $\mathcal{V} \in \text{cov}(X)$, $\mathcal{V} \prec \mathcal{U}$, such that a map $g : \overline{W} \to X$ is perfect, whenever it is $\mathcal{V}$-near to $f|\overline{W}$. Using $n$-PHAP of $X$, find a perfect map $\hat{f} : K \to X$, $\mathcal{V}$-homotopic to $f$ via a homotopy $\hat{h} : K \times [0, 1] \to X$ such that $\hat{h}(x, 0) = f(x)$ and $\hat{h}(x, 1) = \hat{f}(x)$ for all $x \in K$. Fix any continuous map $\lambda : K \to [0, 1]$ with $\lambda(F) \subset \{ 0 \}$ and $\lambda(K \setminus \overline{W}) \subset \{ 1 \}$ and consider the homotopy $h : K \times [0, 1] \to X$ defined by $h(x, t) = \hat{h}(x, \lambda(x)t)$ for $(x, t) \in K \times [0, 1]$. It is easy to see that the map $g : K \to X$, $g : x \mapsto h(x, 1)$, and the $\mathcal{U}$-homotopy $h$ satisfy the requirements of the lemma.

The following lemma gives a proof of Theorem 1(1).

Lemma 4. If $X$ is a space with $n$-PHAP, then each open subspace of $X$ has $n$-PHAP.

Proof. Let $U$ be an open subspace of $X$, $U$ be an open cover of $U$ and $f_0 : K \to U$ be a map of a simplicial complex $K$ with $\dim K \leq n$. We have to construct a perfect map $f_\infty : K \to U$ which is $\mathcal{U}$-homotopic to $f_0$.

Fix any metric $\rho < 1$ generating the topology of $X$. For every $n \geq 0$ let $K_n = \{ x \in K : \rho(f_0(x), X \setminus U) \geq 2^{-n} \}$. It is clear that each set $K_n$ is closed in $K$ and lies in the interior of $K_{n+1}$. Since $\rho < 1$, $K_0 = \emptyset$.

Let $(U_n)_{n \geq 0}$ be a sequence of open covers of $X$ such that $\text{mesh} U_n < 2^{-(n+1)}$ and $St U_{n+1} \prec U_n$ for any $n \geq 0$. We can additionally assume that the covers $U_n$ are so fine that $\{ St(x, U_n) : \rho(x, X \setminus U) \geq 2^{-n} \} \prec U$ for every $n \geq 0$. 

By induction, we shall construct a function sequence \( \{ f_n : K \to X \}_{n \in \omega} \) satisfying the following conditions for every \( n \in \mathbb{N} \):

1. \( f_n(x) = f_{n-1}(x) \) for any \( x \in K_{n-1} \cup (K \setminus K_{n+1}) \);
2. the map \( f_n|K_n : K_n \to X \) is perfect;
3. the map \( f_n \) is \( \mathcal{U}_{n+2} \)-homotopic to \( f_{n-1} \) via a \( \mathcal{U}_{n+2} \)-homotopy \( h_n : K \times [0, 1] \to X \) such that \( h_n(x, t) = f_n(x) \) for \( (x, t) \in K \times \{1\} \) and \( h_n(x, t) = f_{n-1}(x) \) for all \( (x, t) \in K \times \{0\} \cup (K_{n-1} \cup (K \setminus K_{n+1})) \times [0, 1] \).

Assume that for some \( n \in \mathbb{N} \) the function \( f_{n-1} \) has been constructed. Using Lemma 3 find a perfect map \( g : K \to X \) and a \( \mathcal{U}_{n+2} \)-homotopy \( h : K \times [0, 1] \to X \) such that \( h(x, 1) = g(x) \) for any \( x \in K \) and \( h(x, t) = f_{n-1}(x) \) for any \( (x, t) \in K \times \{0\} \cup K_{n-1} \times [0, 1] \).

Let \( \lambda : K \to [0, 1] \) be a continuous function such that \( \lambda^{-1}(0) \supset K \setminus K_{n+1} \) and \( \lambda^{-1}(1) \supset K_n \).

Finally, consider the function \( f_n : K \to X \) defined by \( f_n(x) = h(x, \lambda(x)) \) for \( x \in K \) and the homotopy \( h_n : K \times [0, 1] \to X \) defined by \( h_n(x, t) = h(x, \lambda(x) \cdot t) \) for \( (x, t) \in K \times [0, 1] \). The construction of \( f_n \) and \( h_n \) imply that the conditions (1)–(3) are satisfied. The conditions (1) imply that for each \( x \in K \) the sequence \( \{f_n(x)\} \) eventually stabilizes and thus the limit map \( f_\infty = \lim_{n \to \infty} f_n : K \to X \) is well-defined. Observe that \( f_\infty \) is homotopic to the map \( f_0 \) via the homotopy \( h_\infty : K \times [0, \infty) \to X \) defined by \( h_\infty(x, \infty) = f_\infty(x) \) for \( x \in K \) and \( h_\infty(x, t) = h_n(x, t - n + 1) \) for \( x \in K \) and \( t \in [n - 1, n] \), \( n \geq 1 \).

Since \( \rho(f_0(X), X \setminus U) \geq 2^{-n} \), for \( x \in K \setminus K_{n-1} \), we get

\[
\tag{1} h_\infty(\{x\} \times [0, \infty)) = \bigcup_{i=-1}^{1} h_{n+i}(\{x\} \times [0, 1]) \subset St(f_0(x), U_n) \subset St(f_0(x), U).
\]

This means that \( h_\infty \) is a \( \mathcal{U} \)-homotopy, which yields \( h_\infty(K \times [0, \infty)) \subset U \) and \( f_\infty(K) \subset U \).

Also (1) implies that \( \rho(f_\infty(x), f_0(x)) \leq \text{mesh} U_n < 2^{-(n+1)} \) for any \( x \in K \setminus K_{n-1} \).

Let us show finally that the map \( f_\infty : K \to U \) is perfect. Take any compact subset \( C \subset U \) and find \( n \geq 0 \) such that \( \rho(C, X \setminus U) > 2^{-n} \). We claim that \( f_\infty^{-1}(C) \subset K_{n+1} \). Fix any \( x \in K \setminus K_{n+1} \) and find a unique number \( m \) such that \( x \in K_m \setminus K_{m-1} \). It follows that \( m \geq n + 2 \) and \( \rho(f_\infty(x), f_0(x)) < 2^{-(m+1)} \leq 2^{-(n+3)} \). By the definition of the set \( K_{m-1} \), we get \( \rho(f_0(x), X \setminus U) < 2^{-(m-1)} \leq 2^{-(n+1)} \) and thus

\[
\rho(f_0(x), C) \geq \rho(C, X \setminus U) - \rho(f_0(x), X \setminus U) > 2^{-n} - 2^{-(n+1)} = 2^{-(n+1)}.
\]

Then \( \rho(f_\infty(x), C) \geq \rho(f_0(x), C) - \rho(f_\infty(x), f_0(x)) > 2^{-(n+1)} - 2^{-(n+3)} > 0 \) and thus \( f_\infty(x) \not\in C \). Therefore \( f_\infty^{-1}(C) \subset K_{n+1} \). Since the map \( f_\infty|K_{n+1} = f_{n+2}|K_{n+1} \) is perfect we conclude that the preimage \( f_\infty^{-1}(C) = (f_\infty|K_{n+1})^{-1}(C) \) is compact. This means that the map \( f_\infty : K \to U \) is perfect. \( \square \)

**Lemma 5.** A space \( X \) has \( n \)-PHAP provided \( X \) is a union of two open subspaces with \( n \)-PHAP.

**Proof.** Suppose \( X = U_0 \cup U_1 \) where \( U_0, U_1 \) are open subspaces of \( X \) having \( n \)-PHAP. Find two open subsets \( V_0, V_1 \subset X \) such that \( V_0 \cup V_1 = X \) and \( \overline{V}_i \subset U_i \) for \( i = 0, 1 \).

To show that \( X \) has \( n \)-PHAP, fix an open cover \( \mathcal{U} \) of \( X \) and a map \( f : K \to X \) of a simplicial complex \( K \) with \( \dim K \leq n \). Pick an open cover \( \mathcal{V} \) of \( X \) such that \( St \mathcal{V} \prec \mathcal{U} \) and \( cl_X(St(\overline{V}_i \cup St(V_i))) \subset U_i \) for \( i = 0, 1 \).

Let \( W_i = f^{-1}(V_i) \) and \( W'_i = f^{-1}(U_i) \) for \( i = 0, 1 \). Taking a sufficiently fine triangulation of \( K \), we can assume that each simplex of \( K \) lies in \( W_0 \) or \( W_1 \). Then the union \( K_i \) of simplexes lying in \( W_i \) is a subcomplex of \( K \) and \( K_0 \cup K_1 = K \).

Since the space \( W_0 \subset K \) is triangulable, the \( n \)-PHAP of \( U_0 \) allows us to find a proper map \( f_0 : W_0 \to U_0 \) which is \( \mathcal{V} \)-homotopic to \( f|W_0' \) via a \( \mathcal{V} \)-homotopy \( h_0 : W_0 \times [0, 1] \to X \).
$U_0$ such that $h_0(x,0) = f(x)$ and $h_0(x,1) = f_0(x)$ for $x \in W'_0$. Note that $f_0(K_0) \subset \text{St}(f(K),\mathcal{V}) \subset \text{St}(\overline{V}_0,\mathcal{V}) \subset \text{cl}_X(\text{St}(\overline{V}_0,\mathcal{V})) \subset U_0$ which implies that the map $f_0|K_0 : K_0 \rightarrow X$ is perfect.

Let $\lambda : K \rightarrow [0,1]$ be a continuous map such that $\lambda^{-1}(1) \supset K_0$ and $\lambda^{-1}(0) \supset K \setminus W_0$. Since $\overline{W_0} \subset W'_0$, we can define a homotopy $\tilde{h}_0 : K \times [0,1] \rightarrow X$ letting $\tilde{h}_0(x,t) = h_0(x,\lambda(x) \cdot t)$ for $(x,t) \in W'_0 \times [0,1]$ and $\tilde{h}_0(x,1) = f(x)$ for $x \notin W_0$ and $t \in [0,1]$. Let $\tilde{f}_0(x) = \tilde{h}_0(x,1)$. Since $f_0|K_0 = f_0|K_0$ the map $\tilde{f}_0|K_0 : K_0 \rightarrow X$ is perfect.

Observe that $\tilde{f}_0(K_1) \subset \text{St}(f(K_1),\mathcal{V}) \subset \text{St}(\overline{V}_1,\mathcal{V}) \subset U_1$ and applying Lemma 3, find a perfect map $f_1 : K_1 \rightarrow U_1$ which is $\mathcal{V}$-homotopic to the restriction $\tilde{f}_0|K_1$ via a $\mathcal{V}$-homotopy $h_1 : K_1 \times [0,1] \rightarrow U_1$ such that $h_1(x,1) = f_1(x)$ and $h_1(x,t) = f_0(x)$ for $(x,t) \in K_1 \times \{0\} \cup (K_0 \cap K_1) \times [0,1]$. Then $f_1(K_1) \subset \text{St}(\tilde{f}_0(K_1),\mathcal{V}) \subset \text{St}(\text{St}(f(K_1),\mathcal{V}),\mathcal{V}) \subset \text{cl}_X\text{St}(\overline{V}_1,\text{St}\mathcal{V}) \subset U_1$ and hence the map $f_1|K_1 : K_1 \rightarrow X$ is perfect.

Finally, consider the map $g : K \rightarrow X$ defined by $g|K_0 = \tilde{f}_0|K_0$ and $g|K_1 = f_1$. The map $g$ is perfect because so are its restrictions onto the closed sets $K_0$ and $K_1$. It is easy to show that $g$ is $\mathcal{V}$-homotopic to $\tilde{f}_0$ and hence is $\text{St}\mathcal{V}$-homotopic to $f$. \hfill $\square$

Now we can prove the second item of Theorem 1. We shall exploit the classical Michael result [Mi] on local properties. Following E. Michael we call a property $\mathcal{P}$ of topological spaces to be local if a space $X$ has $\mathcal{P}$ if and only if each point of $X$ has an open neighborhood with the property $\mathcal{P}$. According to [Mi] (see also Proposition 4.1 of [BP, Ch.II]) a property $\mathcal{P}$ is local if and only if $\mathcal{P}$ is open-hereditary (open subspaces of a space with the property $\mathcal{P}$ have that property), open-additive (a space has the property $\mathcal{P}$ if it is a union of two open subspaces with that property), and discrete additive (a space has $\mathcal{P}$ provided it is the union of a discrete family of open subspaces with the property $\mathcal{P}$).

Lemmas 4 and 5 imply that the $n$-PHAP is an open-hereditary and open-additive property. It is trivial to check that the discrete union of spaces with $n$-PHAP has $n$-PHAP. Applying the Michael Theorem, we conclude that $n$-PHAP is a local property. In other words the following lemma implying Theorem 1(2) is true.

**Lemma 6.** A space $X$ has $n$-PHAP provided $X$ admits an open cover by subspaces with $n$-PHAP.

The third statement of Theorem 1 follows from

**Lemma 7.** If a space $X$ has $n$-PHAP, then $X$ has the discrete $n$-cells property.

*Proof.* This lemma trivially follows from a result of [Cu] asserting that a space $X$ has the discrete $n$-cells property if and only if each map $f : I^n \times \omega \rightarrow X$ can be approximated by a map $g$ sending $\{I^n \times \{i\}\}_{i \in \omega}$ onto a locally finite collection in $X$. \hfill $\square$

To reverse the preceding lemma we will need one classical result concerning $LC^n$-spaces.

**Lemma 8.** ([Hu, V.5.1]) For any cover $U \in \text{cov}(X)$ of an $LC^n$-space $X$ there is a cover $\mathcal{V} \in \text{cov}(X)$ such that any two $\mathcal{V}$-near maps $f, g : K \rightarrow X$ from a space $K$ with $\dim K \leq n$ are $U$-homotopic.

Now we are able to prove the item 4 of Theorem 1.

**Lemma 9.** An $LC^n$-space has $n$-PHAP if and only if it has the discrete $n$-cells property.

*Proof.* The “only if” part follows from Lemma 7. The “if” part will be proven by induction. Fix any finite $n \geq 0$ and assume that Lemma 9 has been proved for all $k < n$. To show that an $LC^n$-space $X$ with the discrete $n$-cells properties has $n$-PHAP, fix a cover $U \in \text{cov}(X)$ and a map $f : K \rightarrow X$ from an $n$-dimensional simplicial complex $K$. 

Let \( \mathcal{U}_1 \in \text{cov}(X) \) be an open cover with \( \text{St}\mathcal{U}_1 < \mathcal{U} \). Let \( K^{(n-1)} \) denote the \((n-1)\)-dimensional skeleton of \( K \). By the inductive hypothesis, the space \( X \) has \((n-1)\)-PHAP which allows us to find a perfect map \( g : K^{(n-1)} \to X \) which is \( \mathcal{U}_1 \)-homotopic to \( f|K^{(n-1)} \). Since the pair \((K, K^{(n-1)})\) has the homotopy extension property, the map \( g \) admits a continuous extension \( \bar{g} : K \to X, \mathcal{U}_1 \)-homotopic to \( f \).

By Lemma 2, the restriction \( \bar{g}|\overline{W} \) is perfect for some closed neighborhood \( \overline{W} \) of \( K^{(n-1)} \) in \( K \). By Lemma 1, there is a cover \( \mathcal{U}_2 \in \text{cov}(X) \) such that \( \mathcal{U}_2 < \mathcal{U}_1 \) and any map \( p : \overline{W} \to X, \mathcal{U}_2\)-near to \( \bar{g}|\overline{W} \) is perfect. By Lemma 8 there is a cover \( \mathcal{U}_3 \in \text{cov}(X) \) such that any two \( \mathcal{U}_3\)-near maps from a space \( D \) with \( \dim D \leq n \) into \( X \) are \( \mathcal{U}_2\)-homotopic.

Write the complement \( K \setminus K^{(n-1)} = \bigcup_{i \in I} \sigma_i \) as the disjoint union of open \( n \)-dimensional simplexes of \( K \) and consider the discrete topological sum \( D = \bigcup_{i \in I} \partial\sigma_i \) of their closures in \( K \). Denote by \( i : K \setminus K^{(n-1)} \to D \) the natural embedding. There is a natural surjective perfect map \( \pi : D \to K \) such that \( \pi(\bigcup_{i \in I} \partial\sigma_i) = K^{(n-1)} \).

Since \( X \) has the discrete \( n \)-cells property, there is a perfect map \( q : D \to X \) such that \( (g, \bar{g} \circ \pi) < \mathcal{U}_3 \). By the choice of the cover \( \mathcal{U}_3 \), there is a \( \mathcal{U}_2\)-homotopy \( h : D \times [0, 1] \to X \) connecting the maps \( \bar{g} \circ \pi \) and \( q \) in the sense that \( h(x, 0) = \bar{g} \circ \pi(x) \) and \( h(x, 1) = q(x) \) for \( x \in D \). Let \( \lambda : K \to [0, 1] \) be a continuous map such that \( \lambda^{-1}(0) \) is a neighborhood of \( K^{(n-1)} \) and \( \lambda \) is \( K \)-\( W \subset \lambda^{-1}(1) \). Finally, consider the map \( p : K \to X \) defined by

\[
p(x) = \begin{cases} g(x) & \text{if } x \in K^{(n-1)}, \\ h(i(x), \lambda(x)) & \text{otherwise.} \end{cases}
\]

It is easy to see that the map \( p \) is continuous and \( \mathcal{U}_2\)-homotopic to \( \bar{g} \). Taking into account that \( \mathcal{U}_2 < \mathcal{U}_1, \text{St}\mathcal{U}_1 < \mathcal{U} \), and \( \bar{g} \) is \( \mathcal{U}_1\)-homotopic to \( f \), we conclude that the map \( p \) is \( \mathcal{U}\)-homotopic to \( f \).

Finally, let us show that the map \( p \) is perfect. For this observe that the restriction \( p|\overline{W} \), being \( \mathcal{U}_2\)-homotopic to \( \bar{g} \), is perfect while the restriction \( p|K \setminus W \), being equal to \( q \circ i|K \setminus W \), is perfect too. \( \square \)

For the proof of Theorem 1(5) we shall need

**Lemma 10.** Let \( K \) be a simplicial complex and \( \emptyset = L_0 \subset L_1 \subset \cdots \) be a tower of subcomplexes of \( K \) such that \( K = \bigcup_{i \in \omega} L_i \) and each \( L_i \) lies in the interior of \( L_{i+1} \). Then for any map \( f : K \to X \) into a metric space \((X, d)\) with \( n \)-PHAP and any sequence \((\varepsilon_i)_{i \in \omega} \) in \((0, 1] \) there exists a map \( \tilde{f} : K \to X \) and a homotopy \( H : K \times [0, 1] \to X \) satisfying the following conditions:

(a) \( H(z, 0) = f(z), H(z, 1) = \tilde{f}(z) \) for all \( z \in K \);

(b) \( \text{diam } H(\{z\} \times [0, 1]) < \varepsilon_k \) for all \( z \in L_k \setminus L_{k-1} \) and \( k \in \omega \);

(c) \( \tilde{f}|L_k^{(n)} \) is perfect for every \( k \in \omega \).

**Proof.** Without loss of generality, \( \varepsilon_{k+1} < \varepsilon_k/2 \) for all \( k \in \omega \). Put \( f_0 = f \). By induction, for every \( k \in \mathbb{N} \) we shall construct a map \( f_k : K \to X \) and a homotopy \( H_k : K \times [0, 1] \to X \) satisfying the following conditions:

\( (1_k) \) \( H_k(z, 0) = f_{k-1}(z) \) and \( H_k(z, 1) = f_k(z) \) for all \( z \in K \);

\( (2_k) \) \( H_k(z, t) = f_{k-1}(z) \) for all \( z \in L_{k-1} \cup K \setminus L_{k+1} \) and \( t \in [0, 1] \);

\( (3_k) \) \( \text{diam } H_k(\{z\} \times [0, 1]) < \varepsilon_{k+1} \) for all \( z \in K \);

\( (4_k) \) \( f_k|L_k^{(n)} \) is perfect.

Suppose that functions \( f_i \) and homotopies \( H_i \) have been constructed for \( i \leq k \). Take any open cover \( \mathcal{U} \) of \( X \) with mesh \( \mathcal{U} < \varepsilon_{k+2} \). Using Lemma 3, find a perfect map \( g : K^{(n)} \to X \), \( \mathcal{U}\)-homotopic to \( f_k \) via a homotopy \( h : K^{(n)} \times [0, 1] \to X \) such that \( h(z, 1) = g(z) \)
for \( z \in K(n) \) and \( h(z, t) = f_k(z) \) for \((z, t) \in K(n) \times \{0\} \cup L_k(n) \times [0, 1] \). Then \( M = L_k(n) \cup L_{k+1}(n) \cup K \setminus L_{k+2} \) is a simplicial subcomplex of \( K \) and the homotopy extension property of the simplicial pair \((K, M)\) allows us to find a \( \mathcal{U}\)-homotopy \( H_{k+1} : K \times [0, 1] \rightarrow X \) such that \( H_{k+1}(z, t) = f_k(z) \) if \((z, t) \in K \times \{0\} \cup (L_k(n) \cup K \setminus L_{k+2}) \times [0, 1] \) and \( H_{k+1}(z, t) = h(z, t) \) if \((z, t) \in L_k(n) \times [0, 1] \). Letting \( f_{k+1}(z) = H_{k+1}(z, 1) \) for \( z \in K \) we finish the inductive step.

The conditions (1\(_k\))–(3\(_k\)) imply that the limit map \( \tilde{f} = \lim_{k \to \infty} f_k \) is well-defined and continuous. Using the homotopies \( H_k \) it is easy to compose a homotopy \( H \) connecting the maps \( f \) and \( \tilde{f} \) and satisfying the conditions (a)–(c) of the lemma. \( \square \)

With Lemma 10 in disposition we can prove the fifth item of Theorem 1. It should be mentioned that a particular case of Lemma 11 was proven by P. Bowers in [Bo2, 4.6].

**Lemma 11.** If \( X_1 \) is a space with \( n_1\)-PHAP and \( X_2 \) is a space with \( n_2\)-PHAP, then the product \( X_1 \times X_2 \) has \((n_1 + n_2 + 1)\)-PHAP.

**Proof.** Let \( n = n_1 + n_2 + 1 \), \( K \) be a simplicial complex with \( \dim K \leq n \), \( \mathcal{U} \in \operatorname{cov}(X_1 \times X_2) \), and \( f = (f_1, f_2) : K \rightarrow X_1 \times X_2 \) be a map. For every \( i \in \{1, 2\} \) fix an admissible metric \( d_i \leq 1 \) on \( X_i \). On the product \( X_1 \times X_2 \) consider the metric \( d((x_1, x_2), (x_1', x_2')) = \max\{d_1(x_1, x_1'), d_2(x_2, x_2')\} \). Find a continuous map \( \varepsilon : X_1 \times X_2 \rightarrow (0, 1] \) such that \( \{B(x, 6\varepsilon(x)) : x \in X_1 \times X_2\} \subset \mathcal{U} \). Replacing \( K \) by its sufficiently fine subdivision, we can assume that for any simplex \( \sigma \) of \( K \) we have

1. \( \min\{\varepsilon \circ f(z) : z \in \sigma\} > \frac{1}{4} \max\{\varepsilon \circ f(z) : z \in \sigma\} \)
2. \( \operatorname{diam} f(\sigma) < \min\{\varepsilon \circ f(z) : z \in \sigma\} \)

For every \( k \in \omega \) let \( F_k = (\varepsilon \circ f)^{-1}([2^{-k}, 1]) \). It follows from (1) that any simplex of \( K \) meeting \( F_k \) lies in the interior of \( F_{k+1} \). Consequently, the simplicial subcomplex \( L_k \) of \( K \), composed by simplices meeting \( F_k \) lies in the interior of the subcomplex \( L_{k+1} \). Evidently, the subcomplexes \( L_k \), \( k \in \omega \), cover the complex \( K \).

Denote by \( K_1 \) the \( n_1\)-dimensional skeleton of \( K \) and let \( K_2 \) be the full subcomplex of the barycentric subdivision of \( K \), generated by the barycenters of simplexes of dimension \( > n_1 \). Then \( K_2 \) is a subcomplex of dimension \( \dim K - (n_1 + 1) \leq n_2 \) of the barycentric subdivision of \( K \). Applying Lemma 10 with \( \varepsilon_k = 2^{-(k+1)} \), for every \( i \in \{1, 2\} \) we can find a map \( \tilde{f}_i : K \rightarrow X_i \) and a homotopy \( H^1_k : K \times [0, 1] \rightarrow X_i \) such that the following conditions hold

3. \( H^1_k(z, 0) = f_i(z) \) and \( H^1_k(z, 1) = \tilde{f}_i(z) \) for \( z \in K \);
4. \( \operatorname{diam} H_i((z) \times [0, 1]) < \varepsilon \circ f(z) \) for \( z \in K \);
5. \( f_i|K_i \cap L_k \) is perfect for all \( k \in \omega \).

Observe that for points \( z, z' \) of a simplex \( \sigma \) of \( K \), the conditions (1), (2) and (4) imply

\[
\begin{align*}
d_i(\tilde{f}_i(z), \tilde{f}_i(z')) &\leq d_i(\tilde{f}_i(z), f_i(z)) + \operatorname{diam} f_i(\sigma) + d_i(f_i(z'), \tilde{f}_i(z')) \\
&< \varepsilon \circ f(z) + \operatorname{diam} f_i(\sigma) + \varepsilon \circ f(z') < 5 \min \varepsilon \circ f_i(\sigma),
\end{align*}
\]

which yields \( \operatorname{diam} \tilde{f}_i(\sigma) < 5 \min \varepsilon \circ f_i(\sigma) \).

Each point \( z \in K \) can be written as \( z = sz_1 + (1-s)z_2 \) with \( z_1 \in K_i \) and \( s \in [0, 1] \) and such a representation is unique if \( z \notin K_1 \cup K_2 \). The set \( C_1 \) (resp. \( C_2 \)) of points \( z \) for which \( s \geq \frac{1}{2} \) (resp. \( s \leq \frac{1}{2} \)) is closed in \( K \) and \( K = C_1 \cup C_2 \). For every \( i \in \{1, 2\} \) there is a homotopy \( \Phi_i : K \times [0, 1] \rightarrow K \) such that \( \Phi_i(z, 0) = z \), \( \Phi_i(C_i \times \{1\}) \subset K_i \) and \( \Phi_i(\sigma \times [0, 1]) \subset \sigma \) for each simplex \( \sigma \) of \( K \) (such a homotopy \( \Phi_i \) can be defined by \( \Phi_i(z, t) = \alpha_i(s, t)z_1 + (1-\alpha_i(s, t))z_2 \) for \( z = sz_1 + (1-s)z_2 \), where \( \alpha_1(s, t) = \min\{1, (1+t)s\} \) and \( \alpha_2(s, t) = \max\{0, (s+t)(s-1)\} \)).
For $i \in \{1, 2\}$, define a homotopy $H^2_i : K \times [0,1] \to X_i$ by $H^2_i(z, t) = \bar{f}_i \circ \Phi_i(z, t)$ and let $g_i(z) = H^2_i(z, 1)$. Let $z \in K$ and $\sigma$ be a simplex of $K$, containing the point $z$. Since $\Phi_i(\sigma \times [0,1]) \subset \sigma$ we get $\text{diam } H^2_i(\{z\} \times [0,1]) \leq \text{diam } \bar{f}_i(\sigma) < 5 \varepsilon \circ f(z)$. Since $H^2_i(z, 1) = \bar{f}_i(z) = H^2_i(z, 0)$, we can glue $H^2_i$ and $H^2_i$ together and define a homotopy $H_i$ linking $f_i$ and $g_i$ and such that $\text{diam } H_i(\{z\} \times [0,1]) < 6 \varepsilon \circ f(z)$ for all $z \in K$. Then $H = (H_1, H_2)$ is a homotopy between $f$ and $g = (g_1, g_2)$ such that $\text{diam } h_i(\{z\} \times [0,1]) < 6 \varepsilon \circ f(z)$ for all $z \in K$. The choice of $\varepsilon$ guarantees that $H$ is a $U$-homotopy.

Let us show that the map $g$ is perfect. Assuming the converse we would find a sequence $\{z_r\}$ without limit points in $K$ and such that the sequence $\{g(z_r)\}$ converges to some point $x = (x_1, x_2) \in X$. Since $C_1 \cup C_2 = K$, we can suppose that $\{z_r\} \subset C_i$ for some $i \in \{1, 2\}$. The inclusion $\Phi_i(\sigma \times [0,1]) \subset \sigma$ for any simplex $\sigma$ of $K$ implies that the homotopy $\Phi_i$ is proper and $\Phi_i(L_k \times [0,1]) \subset L_k$ for all $k$. In particular, $\Phi_i((C_i \cap L_k) \times \{1\}) \subset K_i \cap L_k$ and since the restriction $\bar{f}_i|K_i \cap L_k$ is proper, we get that the restriction of $g_i$ onto the closed subset $C_i \cap L_k$ is proper. Then $C_i \cap L_k$ contains only finitely many points $z_r$ which yields $\varepsilon \circ f(z_r) < 2^{-k}$ for all sufficiently large $r$ and thus $\lim_{r \to \infty} \varepsilon \circ f(z_r) = 0$. Since $d(f(z_r), g(z_r)) < 6 \varepsilon \circ f(z_r)$, we get that the sequence $\{f(z_r)\}$ converges to $x$ and thus $\varepsilon(x) = \lim_{r \to \infty} \varepsilon \circ f(z_r) = 0$, which is impossible.

Let $X$ be a topological space and $U \in \text{cov}(X)$. We define a subset $B \subset X$ to be $U$-bounded, if $B \subset \bigcup \mathcal{F}$ for some finite subcollection $\mathcal{F}$ of $U$.

Lemma 12. Let $X$ be a space with $n$-PHAP and $U \in \text{cov}(X)$. Then for any simplicially approximable map $f : P \to X$ from a space $P$ with $\dim P \leq n$ and any open cover $V$ of $P$ there exists an open cover $W$ of $X$ and a map $g : P \to X$, $U$-homotopic to $f$ and such that $g^{-1}(A)$ is $V$-bounded in $P$ for any $W$-bounded subset $A \subset X$.

Proof. Given a cover $U \in \text{cov}(X)$ let $U' \in \text{cov}(X)$ be any cover with $\text{St}^2 U' \subsetneq U$. Since $f$ is simplicially approximable, there are a simplicial complex $K_0$ and two maps $p_0 : P \to K_0$ and $q_0 : K_0 \to X$ such that the map $q_0 \circ p_0$ is $U'$-homotopic to $f$. Replacing the triangulation of $K_0$ by a sufficiently fine subdivision, if necessary, we can assume that $\text{St}(K_0) \subsetneq q_0^{-1}(U')$.

Let $V_1 \subsetneq V$ be an open star-finite cover of $P$, $K_1$ be the nerve of $V_1$ and $p_1 : P \to K_1$ be a canonical map such that $p_1^{-1}(\text{St}(K_1)) \subsetneq V$. Let $K = K_0 \times K_1$, $p = (p_0, p_1) : P \to K$ and $\alpha = q_0 \circ \text{pr}_{K_0} : K \to X$. Endow $K$ with a triangulation such that the projections of $K$ onto $K_0$ and $K_1$ are simplicial maps. Then $\text{St}(K) \subsetneq (\text{pr}_{K_0})^{-1}(\text{St}(K_0)) \subsetneq \alpha^{-1}(U')$ while $p^{-1}(\text{St}(K)) \subset p_1^{-1}(\text{St}(K_1)) \subsetneq V$.

Since $\dim P \leq n$, there is a continuous function $\xi : P \to K^{(n)}$ such that for any $x \in P$ the point $\xi(x)$ belongs to the minimal simplex containing $p(x)$. Then $\xi$ is $\text{St}(K)$-homotopic to $p$ and hence $\alpha \circ \xi$ is $U'$-homotopic to $\alpha \circ p = q_0 \circ p_0$. On the other hand, for every vertex $v$ of $K$, $\xi^{-1}(\text{St}(v, K)) \subset p^{-1}(\text{St}(v, K))$ and thus $\xi^{-1}(\text{St}(K))$ refines $V$.

Using the $n$-PHAP of $X$, we can find a perfect map $\pi : K^{(n)} \to X$, $U'$-homotopic to $\alpha|K^{(n)}$. Then $g = \pi \circ \xi$ is $U'$-homotopic to $\alpha \circ \xi$ and consequently, $\text{St}^2(U')$-homotopic to $f$. 

\begin{center}
\begin{tikzpicture}
\node (X) at (0,0) {$X$};
\node (K) at (-1,1) {$K$};
\node (K0) at (-1,2) {$K_0$};
\node (P) at (-3,0) {$P$};
\node (K(n)) at (-1,3) {$K^{(n)}$};
\draw[->] (X) -- (K) node[pos=0.5, above] {$f$};
\draw[->] (K) -- (K0) node[pos=0.5, above] {$\pi$};
\draw[->] (K) -- (K(n)) node[pos=0.5, above] {$g_0$};
\draw[->] (P) -- (K) node[pos=0.5, left] {$p$};
\draw[->] (P) -- (K0) node[pos=0.5, left] {$p_0$};
\end{tikzpicture}
\end{center}
Since \( \pi \) is perfect and \( St(K) \) is locally finite, each point \( x \in X \) has an open neighborhood \( O(x) \) such that \( \pi^{-1}(O(x)) \) is \( St(K) \)-bounded. Then \( g^{-1}(O(x)) \) is \( \xi^{-1}(St(K)) \)-bounded and hence \( \mathcal{V} \)-bounded. Consequently, the cover \( \mathcal{W} = \{ O(x) : x \in X \} \) has the desired properties. 

Next, we prove the sixth item of Theorem 1.

**Lemma 13.** For any simplicially approximable map \( f : P \rightarrow X \) from a Polish space \( P \) with \( \dim P \leq n \) into a Polish space \( X \) with \( n \)-PHAP and any open cover \( U \in \text{cov}(X) \) there is a perfect map \( g : P \rightarrow X, U \)-homotopic to \( f \).

**Proof.** We assume that the Polish spaces \( P \) and \( X \) are endowed with some complete metrics generating their topology.

Let \( f_{-1} = f \) and \( U_{-1} = U \). Using Lemma 12 we can construct by induction two sequences of star-finite open covers \( (V_n)_{n \in \omega} \subseteq \text{cov}(P) \) and \( (U_n)_{n \in \omega} \subseteq \text{cov}(X) \) and a sequence \( (f_n)_{n \in \omega} \) of continuous maps from \( P \) into \( X \) satisfying the following conditions:

(a) \( \lim_{n \to \infty} \text{mesh}(V_n) = 0 \);  
(b) \( \text{mesh}(U_n) \leq \frac{1}{2^n} \) for every \( n \in \omega \);  
(c) \( St(U_{n+1}) \subset U_n \) for every \( n \in \omega \);  
(d) \( f_{n-1}^{-1}(B) \) is \( V_n \)-bounded in \( P \) for any \( U_n \)-bounded subset \( B \subset X \);  
(e) \( f_n \) and \( f_{n-1} \) are \( U_{n-1} \)-homotopic for all \( n \in \omega \).

It follows from (b), (c) and (e) that the limit map \( g = \lim_{n \to \infty} f_n : P \to X \) is a well-defined continuous function, \( St(U_n) \)-homotopic to each \( f_n \).

We claim that the map \( g \) is proper. Indeed, let \( C \) be a compact subset of \( X \). We have to show that \( g^{-1}(C) \) is compact. Since \( g^{-1}(C) \) is closed in the complete metric space \( P \), we may prove the total boundedness of \( g^{-1}(C) \). Due to (a) it suffices to verify that for every \( n \in \omega \) the set \( g^{-1}(C) \) is \( V_n \)-bounded. Since \( (g, f_n) \prec St(U_n) \), we get \( g^{-1}(C) \subset f_n^{-1}(St(C, St(U_n))) \). Taking into account that the cover \( U_n \) is star-finite and the set \( C \) is compact, we conclude that the set \( St(C, St(U_n)) \) is \( U_n \)-bounded. Then (d) implies that \( f_n^{-1}(St(C, St(U_n))) \supset g^{-1}(C) \) is \( V_n \)-bounded.

For the proof of the two last items of Theorem 1 we need to recall some definitions from [BRZ]. Given two spaces \( X, Y \) denote by \( C(X, Y) \) the space of all continuous functions from \( X \) to \( Y \), endowed with the limitation topology whose neighborhood base at an \( f \in C(X, Y) \) consists of the sets \( B(f, \mathcal{U}) = \{ g \in C(X, Y) : (g, f) \prec \mathcal{U} \} \), where \( \mathcal{U} \) runs over all open covers of \( Y \), see [Bo3]. If the space \( Y \) is Polish, then the space \( C(X, Y) \) is Baire, see [To] or [BRZ, 3.2.1].

By a multivalued map \( F : X \Rightarrow Y \) we understand a function assigning to each point \( z \in Z \) a (possibly empty) subset \( F(z) \subseteq Y \). Such a multivalued map \( F : X \Rightarrow Y \) is called perfect if for any compact subsets \( A \subseteq Z \), \( B \subseteq Y \) the sets \( F(A) = \bigcup_{z \in A} F(z) \) and \( F^{-1}(B) = \{ z \in Z : F(z) \cap B \neq \emptyset \} \) are compact.

Following [BRZ, p.124] we define a map \( f : X \to Y \) to be \( F \)-injective if \( |f^{-1}(F(z))| \leq 1 \) for all \( z \in Z \). A map \( f : X \to Y \) is called a \( (\mathcal{U}, F) \)-map, where \( \mathcal{U} \) is an open cover of \( X \), if there is an open cover \( \mathcal{V} \) of \( Y \) such that \( \{ f^{-1}(St(F(z), \mathcal{V})) \}_{z \in Z} \prec \mathcal{U} \).

**Lemma 14.** Let \( U \subset \mathbb{R}^\omega \) be an open subspace of the countable product of lines and \( F : Z \Rightarrow U \) be a perfect multivalued map. For any Polish space \( P \) the set of all perfect \( F \)-injective maps is dense in the function space \( C(P, U) \).

**Proof.** Fix a complete metric on the Polish space \( P \) and let \( (U_n)_{n \in \omega} \) be a sequence of open covers of \( P \) with \( \text{mesh} U_n < 2^{-n} \) for all \( n \in \omega \).
By [To] the set $\mathcal{E}$ of closed embeddings is dense $G_δ$ in $C(P, U)$. By Lemma 3.2.14 of [BRZ] for every $n ∈ ω$ the set $\mathcal{H}_n$ of $(U_n, F)$-maps is open and dense in $C(P, U)$. Since the function space $C(P, U)$ is Baire (see [To, 1.1]), the intersection $\mathcal{I} = \mathcal{E} \cap \bigcap_{n ∈ ω} \mathcal{H}_n$ is dense in $C(P, U)$. It is clear that each function $f ∈ \mathcal{I}$ is perfect and $F$-injective. 

Our final lemma proves the item (7) of Theorem 1 and (8) follows from (7) applied to a constant map.

**Lemma 15.** If a Polish space $X$ has $n$-PHAP, then for any open cover $U$ of $X$ and any simplicially approximable map $f : P → X$ from a Polish space $P$ with $\dim P ≤ n$ there is a closed embedding $g : P → X$, $U$-near to $f$.

**Proof.** Let $V ∈ \text{cov}(X)$ be any cover with $\text{St}(V) \prec U$. The map $f : P → X$, being simplicially approximable, is $V$-homotopic to the composition $p ∘ q$ of maps $q : P → K$, $p : K → X$, where $K$ is a simplicial complex. Identify the Polish space $P$ with a closed subset of $s = (−1, 1)^ω$, the pseudo-interior of the Hilbert cube $Q = [−1, 1]^{ω}$. Since $K$ is an ANR, the map $q$ admits a continuous extension $\bar{q} : U → K$ onto some open neighborhood $U$ of $P$ in $s$.

According to a result of Dranishnikov [Dr] (see also [BRZ, 2.3.5]), there is an map $µ : N → Q$ from an $n$-dimensional compactum $N$ onto $Q$, which is $n$-invertible in the sense that for any map $α : A → Q$ from a space $A$ with $\dim A ≤ n$ there is a map $β : A → N$ such that $α = µ ∘ β$. It follows that $µ^{-1}(U)$ is a Polish space with $\dim µ^{-1}(U) ≤ \dim N ≤ n$.

Consider the simplicially approximable map $p ∘ \bar{q} ∘ µ : µ^{-1}(U) → X$. By Lemma 13, it is $V$-near to a perfect map $π : µ^{-1}(U) → X$. It is easy to see that for any $t ∈ U$ we get $π(µ^{-1}(t)) ⊂ \text{St}(p ∘ \bar{q}(t), V)$. Since the map $µ|µ^{-1}(U)$ is perfect, we can find an open cover $W$ of $U$ such that $π(µ^{-1}(\text{St}(t, W))) ⊂ \text{St}(p ∘ \bar{q}(t), V)$ for all $t ∈ U$.

Now consider the multivalued map $F : U → U$ defined by $F(x) = µ ∘ π^{-1} ∘ π ∘ µ^{-1}(x)$ for $x ∈ U$ and observe that it is perfect (in the sense that for any compact set $C ⊂ U$ the sets $F(C)$ and $F^{-1}(C)$ are compact in $U$). By Lemma 14, there is a perfect $F$-injective map $α : P → U$ which is $W$-near to the inclusion $P ⊂ U$. By the choice of the map $µ$, there is a map $β : P → µ^{-1}(U)$ such that $α = µ ∘ β$. The perfectness of the maps $α$ and $π$ implies the perfectness of the maps $β$ and $g = π ∘ β : P → X$. Moreover, the $F$-injectivity of the map $α$ implies the injectivity of the map $g$. Thus $g$, being injective and perfect, is a closed embedding.

Observe that for each $t ∈ P$ we get

$$g(t) = π ∘ β(t) ∈ π(µ^{-1}(α(t))) ⊂ π(µ^{-1}(\text{St}(t, W))) ⊂ \text{St}(p ∘ q(t), V),$$

which means that the maps $g$ and $p ∘ q$ are $V$-near. Since $f$ and $p ∘ q$ are $V$-near and $\text{St} V \prec U$ we get that $f$ and $g$ are $U$-near. \□
References


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