EXTENDERS FOR VECTOR-VALUED FUNCTIONS

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Abstract. Given a subset A of a topological space X, a locally convex space Y, and a family $\mathcal{C}$ of subsets of Y we study the problems of the existence of a linear $\mathcal{C}$-extender $u : C_\infty(A,Y) \to C_\infty(X,Y)$, which is a linear operator extending bounded continuous functions $f : A \to C \subset Y$, $C \in \mathcal{C}$, to bounded continuous functions $\bar{f} = u(f) : X \to C \subset Y$. Two necessary conditions for the existence of such an extender are found in terms of a topological game, which is a modification of the classical strong Choquet game. The obtained results allow us to characterize reflexive Banach spaces as the only normed spaces Y such that for every closed subset A of a GO-space X there is a $\mathcal{C}$-extender $u : C_\infty(A,Y) \to C_\infty(X,Y)$ for the family $\mathcal{C}$ of closed convex subsets of Y. Also we obtain a characterization of Polish spaces and a characterization of weakly sequentially complete Banach lattices in terms of extenders.

Contents

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Introduction

In this paper, given a subspace A of a topological space X and a locally convex [ordered] space Y we study the problem of the existence (or rather non-existence) of a linear [monotone] operator that extends bounded continuous Y-valued functions from A to X. The obtained results have a dual nature: on one hand selecting a suitable pair (X, A) we can characterize certain important properties of locally convex spaces Y (like the reflexivity, finite-dimensionality of the weak sequential completeness) in terms of extenders (see Theorems 4.1, 5.1, 9.1) and on other hand, selecting a suitable locally convex space Y, we can characterize topological properties of the pair (X, A) in terms of extenders, see Theorems 3.2, 7.1.

By definition, a (linear) operator $u : F(A, Y) \to F(X, Y)$ defined on a (linear) subspace $F(A, Y) \subset Y^A$ and taking values in a (linear) subspace $F(X, Y) \subset Y^X$ is called a (*linear*) *extender* if for any function $f \in F(A, Y)$ the function $\bar{f} = u(f) \in F(X, Y)$ extends f in the sense that $\bar{f}|A = f$.

If the space Y is partially ordered, then so are the function spaces Y^A and Y^X. In this case we define an extender $u : F(A, Y) \to F(X, Y)$ to be *monotone* if $u(f) \le u(g)$ for any functions $f \le g$ in $F(A, Y)$.

Given a collection $\mathcal{C}$ of subsets of Y we define an extender $u : F(A, Y) \to F(X, Y)$ to be a $\mathcal{C}$-*extender* if $u(F(A, Y) \cap C^A) \subset C^X$ for every $C \in \mathcal{C}$. This is equivalent to saying that for every function $f : A \to Y$ the image $\bar{f}(X)$ of the extended function $\bar{f} : X \to Y$ lies in the $\mathcal{C}$-hull

$$\mathrm{hull}_{\mathcal{C}}\big(f(A)\big) = \bigcap\{C \in \mathcal{C} : f(A) \subset C\}$$

of $f(A)$ in Y (here we assume that $\bigcap\emptyset = Y$). Three collections $\mathcal{C}$ of subsets of Y will be of special importance for us:

- $\mathrm{conv}(Y)$, the collection of convex subsets of Y,
- $\overline{\mathrm{conv}}(Y)$, the collection of closed convex subsets of Y,
- $\mathrm{wcc}(Y)$, the collection of weakly compact convex subsets of Y.

If $Y = Z^*$ is a dual space, then we shall also consider the collection

- $\overline{\mathrm{conv}}^*(Y)$ of all convex subsets of Y, closed in the weak-star topology of Y.

The corresponding $\mathcal{C}$-extenders will be called conv-, $\overline{\mathrm{conv}}$-, wcc-, and $\overline{\mathrm{conv}}^*$-extenders.

The inclusions $\mathrm{wcc}(Y) \subset \overline{\mathrm{conv}}(Y) \subset \mathrm{conv}(Y)$ yield the trivial implications:

$$\text{conv-extender} \;\Rightarrow\; \overline{\text{conv}}\text{-extender} \;\Rightarrow\; \text{wcc-extender}.$$

In the role of linear subspaces $F(X, Y)$ we shall consider the spaces:

- $l_\infty(X, Y)$ of all bounded functions from X to Y,
- $C(X, Y)$ of all continuous functions from X to Y,
- $C_A(X, Y)$ of all functions $f : X \to Y$ that are continuous at a subset $A \subset X$,
- $C_\infty(X, Y) = C(X, Y) \cap l_\infty(X, Y)$ of all bounded continuous functions from X to Y.

A function $f : X \to Y$ is called *bounded* if its image $f(X)$ is bounded in Y. The latter means that for every neighborhood U of the origin in Y there is a real number r with $f(X) \subset rU$.

The space $l_\infty(X, Y)$ will be considered as a locally convex space endowed with the topology of uniform convergence. If Y is a Banach space with norm $\|\cdot\|$, then the topology of $l_\infty(X, Y)$ is generated by the sup-norm $\|f\|_\infty = \sup_{x \in X} \|f(x)\|$.

If Y is the real line $\mathbb{R}$, then we omit the symbol $\mathbb{R}$ and write $l_\infty(X)$, $C(X)$, $C_A(X)$, and $C_\infty(X)$ instead of $l_\infty(X,\mathbb{R})$, $C(X,\mathbb{R})$, $C_A(X,\mathbb{R})$, and $C_\infty(X,\mathbb{R})$.

A classical result on conv-extenders belongs to J.Dugundji [Dug].

Theorem 0.1 (Dugundji)**.** *For every closed subspace A of a metrizable space X and every locally convex space Y there is a linear* conv*-extender $u : Y^A \to Y^X$ such that $u(C(A,Y)) \subset C(X,Y)$.*

In [Bor] C.Borges has shown that the Dugundji's Theorem still is true for any closed subspace A of a stratifiable space X. On the other hand, R.W.Heath and D.J.Lutzer [HL] discovered that for the Michael line $\mathbb{R}_\mathbb{Q}$ and its closed subspace $\mathbb{Q}$ even a weaker form of the Dugundji Theorem is not true: no linear $\overline{\mathrm{conv}}$-extender $C(\mathbb{Q}) \to C(\mathbb{R}_\mathbb{Q})$ exists. Afterwards it was found that even a monotone extender $C(\mathbb{Q}) \to C(\mathbb{R}_\mathbb{Q})$ does not exist, see [vD$_1$], [SV], [GHO].

The Micheal line $\mathbb{R}_\mathbb{Q}$ is a particular case the following construction due to Bing [Bi] and Hanner [Han], see [Eng, 5.1.22]. Given a subspace A of a topological space X let X_A denote the set X endowed with the *Hanner topology*

$$\tau_A = \{D \cup U : D \subset X \setminus A \text{ and } U \text{ is open in } X\}$$

that is discrete on $X \setminus A$ but coincides with the original topology at A. The space X_A is sometimes called the *Hannerization* of X with respect to A.

Observe that each function $f : X \to Y$, continuous at points of the set A is (globally) continuous with respect to the Hanner topology τ_A. This is important because it allows us to reduce the study of extenders $C(A,Y) \to C_A(X,Y)$ to studying extenders of the form $C(A,Y) \to C(X_A,Y)$.

In spite of the fact that no linear $\overline{\mathrm{conv}}$-extender $C(\mathbb{Q}) \to C(\mathbb{R}_\mathbb{Q})$ exists, a linear $\overline{\mathrm{conv}}$-extender $C_\infty(\mathbb{Q}) \to C_\infty(\mathbb{R}_\mathbb{Q})$ for *bounded* continuous functions does exist. This is a particular case of the following result of R.W. Heath and D.J. Lutzer [HL].

Theorem 0.2 (Heath-Lutzer)**.** *For a closed subset A of a GO-space X there is a linear $\overline{\mathrm{conv}}$-extender $u : C_\infty(A) \to C_\infty(X)$.*

We recall that a topological space X is called *a generalized ordered space* (briefly, a *GO-space*) if X is Hausdorff and for a suitable linear order $\le$ on X the space X has a base of the topology consisting of order-convex sets, see [Lu]. The Michael line $\mathbb{R}_\mathbb{Q}$ is just a typical example of a GO-space.

In light of the Dugundji Theorem it was natural to ask about possible generalizations of the Heath-Lutzer Theorem to locally convex spaces, see Question (2) [HL]. In this paper we give many different answers to this question. Moreover, we shall show that various properties of locally convex (ordered) spaces Y and pairs (X,A) can be characterized with help of extenders, see Theorems 1.1, 1.4, 4.1, 5.1, 7.1, 9.1.

For the convenience of the reader we first survey the principal results of the paper and their interplay with known results, and also prove some easy immediate corollaries. Afterwards we present proofs of more difficult theorems.

In the sequel, working with different topologies on a set Y we shall write Y_τ to specify a chosen topology τ on Y.

1. CHARACTERIZING PAIRS (X,A) ADMITTING VARIOUS $\mathcal{C}$-EXTENDERS

In this section we search for conditions on a pair (X,A) guaranteeing the existence of a (linear) $\mathcal{C}$-extender $u : C_\infty(A,Y) \to C_\infty(X,Y)$ for a given locally convex space Y. We start with a [probably known] characterization of pairs (X, A) admitting a (linear) conv-extender $u : C_\infty(A, Y) \to C_\infty(X, Y)$ for every locally convex space Y.

For a Tychonov space X let $P(\beta X)$ denote the space of probability measures on the Stone-Čech compactification of X. The space $P(\beta X)$ can be identified with the set of all positive norm-one linear functionals on the Banach lattice $C(\beta X) = C_\infty(X)$ of bounded continuous functions on X. The space $P(\beta X)$ is endowed with the weak-star topology induced from $C^*(\beta X)$. It is well-known that this topology is generated by the sub-base consisting of the sets $\{\mu \in P(\beta X) : \mu(U) > a\}$ where $a \in \mathbb{R}$ and U runs over the topology of X. The support $\mathrm{supp}(\mu)$ of a measure $\mu \in P(\beta X)$ is the smallest closed subset $F \subset \beta X$ with $\mu(F) = 1$. Five subspaces of $P(\beta X)$ will be of interest:

- $P_2(\beta X) = \{\mu \in P(\beta X) : |\mathrm{supp}(\mu)| \le 2\}$;
- $P_\omega(X) = \{\mu \in P(\beta X) : \mu(F) = 1$ for some finite subset $F \subset X\}$;
- $P_R(X) = \{\mu \in P(\beta X) : \mu(K) = 1$ for some σ-compact subset $K \subset X\}$;
- $P_\tau(X) = \{\mu \in P(\beta X) : \mu(K) = 0$ for every compact subset $K \subset \beta X \setminus X\}$;
- $P_\sigma(X) = \{\mu \in P(\beta X) : \mu(K) = 0$ for every closed G_δ-subset $K \subset \beta X$ with $K \cap X = \emptyset\}$.

Measures from the sets $P_R(X)$, $P_\tau(X)$, and $P_\sigma(X)$ are called *Radon*, *τ-additive*, and *σ-additive* measures on X, respectively. By [Fe, §1], measures from the set $P_\sigma(X)$ can be identified with probability σ-additive measures on X. This justifies the choice of notation. A more detail information on the spaces $P_R(X)$ and $P_\tau(X)$ can be found in [Vr] and [Ba$_1$], [Ba$_2$].

Quite often, it happens that $P_\tau(X) = P_\sigma(X)$. In particular, this equality holds for all Lindelöf spaces X (or, more generally, for all paracompact spaces, not containing a closed discrete subset of Ulam-measurable cardinality), see [BCF, §2]. On the other hand, the equality $P_R(X) = P_\sigma(X)$ holds for Polish spaces X (more generally, for universally measurable spaces), see [BCF, §2].

A Tychonov subspace A of a topological space X is called a *$P\beta$-valued retract of* X if there is a continuous map $r : X \to P(\beta A)$ such that $\mathrm{supp}(r(a)) = \{a\}$ for every $a \in A$. The latter means that $r(a)$ coincides with the Dirac measure δ_a at a. If $r(X) \subset P_2(\beta A)$ then we say that A is a *$P_2\beta$-valued retract* of X. If $r(X) \subset P_\omega(A)$, then A is called a *P_ω-valued retract* of X. By analogy we define *P_R-valued*, *P_τ-valued*, and *P_σ-valued retracts* of X.

We recall that $C_A(X, Y)$ stands for the space of functions from X to Y that are continuous at each point of the subset $A \subset X$.

Theorem 1.1. *For a Tychonov subspace A of a topological space X the following conditions are equivalent:*

1. *for every linear space Y there is a linear* conv-*extender $u : Y^A \to Y^X$ such that $u(C_\infty(A, Y_\tau)) \subset C_\infty(X, Y_\tau)$ for any locally convex linear topology τ on Y;*
2. *there is a* conv-*extender $u : C_\infty(A, Y) \to C_\infty(X, Y)$ for the dual Banach space $Y = C^*_\infty(A)$ endowed with the weak-star topology;*
3. *A is a P_ω-valued retract of X;*

Proof. The implication (1) ⇒ (2) is trivial.

To prove that (2) $\Rightarrow$ (3), fix a conv-extender $u : C_\infty(A,Y) \to C_\infty(X,Y)$ where $Y = C^*_\infty(A)$ is the dual Banach space endowed with the weak-star topology. Consider the bounded continuous map $\delta : A \to C^*_\infty(A)$ assigning to each point $a \in A$ the Dirac measure δ_a supported by a. Let $r = u(\delta) : X \to Y$ be the continuous extension of δ given by the conv-extender u. It follows that $r(X) \subset \mathrm{conv}(\delta(A)) = P_\omega(A)$, which means that $r : X \to P_\omega(A)$ is the required P_ω-valued retraction of X onto A.

(3) $\Rightarrow$ (1) Fix a P_ω-valued retraction $r : X \to P_\omega(A)$. For every point $x \in X$ the measure $\mu_x = r(x) \in P_\omega(X)$ can be uniquely written as the convex combination $\mu_x = \sum_{a \in S_x} \mu_x(a)\delta_a$ where $S_x = \{a \in A : \mu_x(a) > 0\}$ is the (finite) support of the measure μ_x.

Now given a linear space Y, define a linear conv-extender $u : Y^A \to Y^X$ assigning to each function $f : A \to Y$ the function $\bar f : X \to Y$ defined by

$$\bar f(x) = \int_A f\,d\mu_x = \sum_{a \in S_x} \mu_x(a) \cdot f(a)$$

for $x \in X$.

It is a standard exercise to check that for every locally convex linear topology τ on Y, we get $u(C_\infty(A, Y_\tau)) \subset C_\infty(X, Y_\tau)$ (see also the proof of the the corresponding implication in Theorem 1.4). □

For P_σ-valued retracts we have a bit weaker result that will be applied in the proof of Theorem 7.1.

Proposition 1.2. *If a Tychonov subspace A of a topological space X is a P_σ-valued retract of X, then for every separable Banach space Y there is a linear $\overline{\mathrm{conv}}$-extender $u : C_\infty(A,Y) \to C_\infty(X,Y)$.*

Proof. Let $r : X \to P_\sigma(A)$ be a P_σ-valued retraction of X onto A. Given any bounded continuous function $f : A \to Y$, consider the closed convex hull K of the image $f(A)$ in Y. For the Polish space K we have an equality $P_R(K) = P_\sigma(K)$ and we can also consider the continuous map $b : P_R(K) \to K$ assigning to each measure $\mu \in P_R(K)$ its barycenter $b(\mu_x) \in K$, see [Kh$_1$], [Kh$_2$], [Ba$_2$]. The continuous map $f : A \to K$ induces a continuous map $P_\sigma(f) : P_\sigma(A) \to P_\sigma(K) = P_R(K)$. Then the composition $b \circ P_\sigma(f) : P_\sigma(A) \to K \subset Y$ is continuous and so is the composition $\bar f = b \circ P_\sigma(f) \circ r : X \to Y$. Observe that for every $a \in A$ we get $\bar f(a) = b \circ P_\sigma(f) \circ r(a) = a$, which means that the operator $u : C_\infty(A,Y) \to C_\infty(X,Y)$, $u : f \mapsto \bar f$, is a $\overline{\mathrm{conv}}$-extender. The linearity of u implies from the observation that

$$b \circ P_\sigma(f)(\mu) = \int_A f\,d\mu, \quad \mu \in P_\sigma(A),$$

and the linearity of the vector integral. □

Question 1.3. *Is a Tychonov subspace A of a topological space X a P_σ-valued retract in X if for each (separable) Banach space Y there is a linear $\overline{\mathrm{conv}}$-extender $u : C_\infty(A,Y) \to C_\infty(X,Y)$?*

Next, in terms of $P\beta$-valued retracts we characterize pairs (X, A) admitting a (linear) $\overline{\mathrm{conv}}$-extender $u : C_\infty(A,Y) \to C_A(X,Y)$ for each semireflexive locally convex space Y.

A locally convex space Y is called *semireflexive* if each bounded closed convex subset of Y is compact in the weak topology of Y. For a Banach space the semireflexivity is equivalent to the reflexivity, see [HHZ, Th.65]. By the Banach-Steinhaus Uniform Boundedness Principle each dual Banach space Y^* endowed with the weak-star topology is semireflexive.

We define a linear topological space Y to be *countably semireflexive* if for any decreasing sequence $(C_n)_{n\in\omega}$ of non-empty bounded closed convex subsets of Y the intersection $\bigcap_{n\in\omega} C_n$ is not empty. In is clear that each semireflexive locally convex space is countably semireflexive. By the Smulian Theorem 1.13.6 in [Me], the converse is true for normed spaces: *A normed space is (semi)reflexive if and only if it is countably semireflexive.*

A linear topology τ on a locally convex space Y will be called *admissible* if τ is stronger than the weak topology and for each neighborhood $U \in \tau$ of zero in Y there is a convex neighborhood $W \in \tau$ whose closure in Y lies in U. The space Y endowed with an admissible topology τ will be denoted by Y_τ.

Theorem 1.4. *For a Tychonov subspace A of a topological space X the following conditions are equivalent:*

1. *for every semireflexive locally convex space Y there is a linear $\overline{\mathrm{conv}}$-extender $u : l_\infty(A, Y) \to l_\infty(X, Y)$ such that $u(C_\infty(A, Y_\tau)) \subset C_A(X, Y_\tau)$ for every admissible topology τ on Y;*
2. *there is a $\overline{\mathrm{conv}}$-extender $u : C_\infty(A, Y) \to C_A(X, Y)$ for the dual Banach space $Y = C^*_\infty(A)$ endowed with the weak-star topology;*
3. *A is a $P\beta$-valued retract of X_A.*

Since the norm topology is admissible for the weak-star topology on a dual Banach space, Theorem 1.4 implies

Corollary 1.5. *Assume that a Tychonov subspace A of a topological space X is a $P\beta$-valued retract of X. Then for every dual Banach space Y^* there is a linear $\overline{\mathrm{conv}}^*$-extender $u : l_\infty(A, Y^*) \to l_\infty(X, Y^*)$ such that $u(C_\infty(A, Y^*)) \subset C_A(X, Y^*)$.*

In its turn, the above corollary will be applied to construct linear wcc-extenders for functions with values in Banach spaces Y that are norm-one complemented in their biduals Y^{**}. The class of such Banach spaces includes all dual Banach spaces [Me, 3.2.23] and also some non-dual spaces like L_1. The latter fact follows from Theorem 1.c.4 [LT] asserting that each weakly sequentially complete Banach lattice (in particular, each Banach lattice of the form $L_1(\mu)$) is norm one complemented in its bidual space. On the other hand, the Banach space c_0 is not complemented in $l_\infty = (c_0)^{**}$, see [Me, 3.2.22].

Theorem 1.6. *Assume that a Banach space Y is norm-one complemented in its bidual space Y^{**} and a Tychonov subspace $A \subset X$ is a $P\beta$-valued retract of X. Then there is a linear* wcc*-extender $u : l_\infty(A, Y) \to l_\infty(X, Y)$ with norm $\|u\| = 1$ such that $u(C_\infty(A, Y)) \subset C_A(X, Y)$.*

Proof. Let $P : Y^{**} \to Y$ be a linear projector with norm $\|P\| = 1$. This projector induces a norm-one linear operator $P^X : l_\infty(X, Y^{**}) \to l_\infty(X, Y)$ assigning to each bounded function $f : X \to Y^{**}$ the function $P^X(f) = P \circ f$. The continuity of P implies that $P^X(C_\infty(X_A, Y^{**})) \subset C_\infty(X_A, Y)$.

By Corollary 1.5, there is a linear $\overline{\mathrm{conv}}^*$-extender $u : l_\infty(A, Y^{**}) \to l_\infty(X, Y^{**})$ with unit norm such that $u(C_\infty(A, Y^{**})) \subset C_\infty(X_A, Y^{**})$. Now consider the norm-one linear operator $v = P^X \circ u : l_\infty(A, Y) \to l_\infty(X, Y)$, which assigns to each function $f \in l_\infty(A, Y) \subset l_\infty(A, Y^{**})$ the function $P^X \circ u(f) : X \to Y$. It follows that $v(C_\infty(A,Y)) \subset C_\infty(X_A,Y)$. If K is a weakly compact convex subset of Y, then K is $*$-weakly closed in Y^{**} and consequently, $u(l_\infty(A,K)) \subset l_\infty(X,K)$. Since $P^X(f) = f$ for each function $f \in l_\infty(X,Y)$, we conclude that $v(l_\infty(A,K)) \subset l_\infty(X,K)$, which means that v is a wcc-extender. $\square$

Question 1.7. *Is there a linear (continuous) extender $u : C_\infty(\mathbb{Q}, c_0) \to C_\infty(\mathbb{R}_\mathbb{Q}, c_0)$?*

2. Linear extenders on ordered spaces

In this section we shall construct nice linear extenders on linearly ordered topological spaces (briefly LOTS). Those are topological spaces X carrying the interval topology with respect to some linear order $\le$ on X. The interval topology is generated by the sub-base consisting of left and right rays $(\leftarrow, a) = \{x \in X : x < a\}$ and $(a, \to) = \{x \in X : x > a\}$ for $a \in X$. A Hausdorff topology on $(X, \le)$ having a base consisting of order-convex sets is called a GO-topology. It can be shown that the interval topology is the weakest GO-topology on $(X, \le)$.

A set A with the discrete topology will be denoted by A_d.

The principal result of this section is the following

Theorem 2.1. *Let A be a non-empty subset of a linearly ordered space $(X, \le)$, and $i : A_d \to A$ denote the identity map.*

1. *There is a function $r : X \to P_2(\beta A_d)$ such that $r(a) = \delta_a$, $a \in A$, and for every GO-topology $g \ni X \setminus A$ on X, the map $P(\beta i) \circ r : X_g \to P_2(\beta A_g)$ is continuous.*
2. *For any semireflexive locally convex space Y there is a linear $\overline{\mathrm{conv}}$-extender $u : l_\infty(A,Y) \to l_\infty(X,Y)$ such that $u(C_\infty(A_g, Y_\tau)) \subset C_\infty(X_g, Y_\tau)$ for every GO-topology $g \ni X \setminus A$ on X and every admissible topology τ on Y;*
3. *For every dual Banach space Y^* there is a linear $\overline{\mathrm{conv}}^*$-extender $u : l_\infty(A, Y^*) \to l_\infty(X, Y^*)$ such that $u(C_\infty(A_g, Y^*)) \subset C_\infty(X_g, Y^*)$ for every GO-topology $g \ni X \setminus A$ on X;*
4. *For any Banach space Y that is norm-one complemented in its bi-dual space Y^{**} there is a linear* wcc*-extender $u : l_\infty(A,Y) \to l_\infty(X,Y)$ with norm $\|u\| = 1$ such that $u(C_\infty(A_g, Y)) \subset C_\infty(X_g, Y)$ for every GO-topology $g \ni X \setminus A$ on X.*

Applying this theorem to GO-spaces, we obtain a less complicated corollary generalizing the Heath-Lutzer Theorem 0.2.

Corollary 2.2. *Let A be a closed subset of a GO-space X. Then*

1. *A is a $P_2\beta$-valued retract of X;*
2. *For any semireflexive locally convex space Y there is a linear $\overline{\mathrm{conv}}$-extender $u : l_\infty(A,Y) \to l_\infty(X,Y)$ such that $u(C_\infty(A,Y)) \subset C_\infty(X,Y)$;*
3. *For every dual Banach space Y^* there is a linear $\overline{\mathrm{conv}}^*$-extender $u : l_\infty(A,Y^*) \to l_\infty(X,Y^*)$ such that $u(C_\infty(A,Y^*)) \subset C_\infty(X,Y^*)$;*
4. *For any Banach space Y that is norm-one complemented in its bi-dual space Y^{**} there us a linear* wcc*-extender $u : l_\infty(A,Y) \to l_\infty(X,Y)$ such that $\|u\| = 1$ and $u(C_\infty(A,Y)) \subset C_\infty(X,Y)$.*

3. The strong Choquet properties and games

In this section we shall introduce the so-called strong Choquet property of a subset A in a topological space X, which is necessary for the existence of a linear $\overline{\mathrm{conv}}$-extender $u : C_\infty(A, Y) \to C_\infty(X_A, Y)$ for functions with values in non-reflexive Banach spaces. This will prove that the semireflexivity cannot be removed from Theorems 1.4 and 2.1.

We shall need two modifications of the classical strong Choquet game introduced by G.Choquet to give a convenient game characterization of Polish spaces, see [Ch, Th. 8.7] (see also [Ke, §8]).

Our modifications, called the *strong Choquet game* $G_s(A, X)$ and the *relative strong Choquet game* $G_r(A, X)$, are player by two players, I and II, at a subset A of a topological space X. The games $G_s(A, X)$ and $G_r(A, X)$ are played in the same manner and differ only by the estimation of the outcome.

The player I starts the game selecting a point $a_0 \in A$ and a neighborhood U_0 of a_0 in X. The player II responds with a neighborhood $V_0 \subset U_0$ of a_0. Continuing in this fashion, at the nth inning the player I selects a point $a_n \in V_{n-1} \cap A$ and a neighborhood $U_n \subset V_{n-1}$ of a_n while the player II responds with a neighborhood $V_n \subset U_n$ of a_n. In the process of the game the players construct a sequence of points $\{a_n\}_{n\in\omega} \subset A$ and two sequences of open subsets $(U_n)_{n\in\omega}$ and $(V_n)_{n\in\omega}$ of X such that $a_n \in V_n \subset U_n \subset V_{n-1}$ for all $n \in \mathbb{N}$. The player I is declared the winner in the game $G_s(A, X)$ (resp. in the game $G_r(A, X)$) if $\bigcap_{n\in\omega} U_n = \emptyset$ (resp. $\emptyset \ne \bigcap_{n\in\omega} U_n \subset X \setminus A$). Otherwise, the player II wins the game.

Definition 3.1. If the player II has a winning strategy in the game $G_r(A, X)$ (resp. $G_s(A, X)$), then we shall say that the subset A is *strong Choquet in* X (resp. the space X is *strong Choquet at* A). A topological space X is *strong Choquet* if X is strong Choquet at X.

Let us observe that our definition of a strong Choquet space is equivalent to the classical definition from [Ke, 8.14]. This justifies our choice of the terminology.

According to Choquet's Theorem 8.18 in [Ke], a Tychonov (metrizable separable) space X is strong Choquet if (and only if) X is Čech complete. The latter means that X is a G_δ-set in its Stone-Čech compactification βX.

The following theorem, which is one of the main results of this article, shows that the (countable) semi-reflexivity necessarily appears as soon as we consider linear $\overline{\mathrm{conv}}$-extenders.

Theorem 3.2. *If for a Tychonov subspace A of a topological space X and a linear topological space Y there is a linear $\overline{\mathrm{conv}}$-extender $u : C_\infty(A, Y) \to C_A(X, Y)$, then either Y is countably semi-reflexive or the subset A is strong Choquet in X.*

In light of Theorem 3.2 it is important to study strong Choquet subsets in more details. This is done in the following

Theorem 3.3. *Let A be a subspace of a topological space X.*

1. *If X is strong Choquet, then X is strong Choquet at A;*
2. *X is strong Choquet at A if and only if X_A is strong Choquet at A if and only if X_A is strong Choquet;*
3. *If A is strong Choquet, then A is strong Choquet in X;*
4. *The space A is strong Choquet if X is strong Choquet at A and A is strong Choquet in X.*

Now we give a simple condition guaranteeing that a space X is strong Choquet at a subset $A \subset X$.

Definition 3.4. We shall say that a space X is *complete at* A if there is a countable family $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of covers of A by open subsets of X such that a decreasing sequence $(V_n)_{n\in\mathbb{N}}$ of open subsets of X has non-empty intersection $\bigcap_{n\in\mathbb{N}} V_n$ provided for every $n \in \mathbb{N}$ the following conditions are satisfied: (i) $V_n \cap A \neq \emptyset$, (ii) $\overline{V}_{n+1} \subset V_n$, and (iii) $\overline{V}_n \subset U$ for some $U \in \mathcal{U}_n$.

Proposition 3.5. *Assume that a Tychonov space X is complete at a subset $A \subset X$. Then*

1. *the space X is strong Choquet at A;*
2. *the space A is strong Choquet if and only if A is strong Choquet in X;*
3. *the space A is strong Choquet if there is a linear $\overline{\mathrm{conv}}$-extender $u : C_\infty(A, Y) \to C_A(X, Y)$ for a linear topological space Y that fails to be countably semi-reflexive.*

Proof. 1. We need to describe a winning strategy for the player II in the game $G_s(A, X)$. Let $(\mathcal{U}_n)_{n\in\mathbb{N}}$ be the sequence of cover of A witnessing that X is complete at A. To win the game $G_s(A, X)$, the player II at an n-th inning should select a neighborhood V_n of the point a_n given by the player I so that $a_n \in V_n \subset \overline{V}_n \subset U_n \cap U$ for some set $U \in \mathcal{U}_n$. Such a choice of the sets V_n, $n \in \mathbb{N}$, will guarantee the victory of the player II because $\bigcap_{n\in\omega} V_n \neq \emptyset$.

2. The second item follows from the first one and Proposition 3.3(3,4).

3. The third item follows from the second one and Theorem 3.2. □

Next, we show that the notion of a total π-base considered in [SV] also lead to strong Choquet subsets. Following [SV, 1.3], we say that a family $\mathcal{B}$ of open subsets of a topological space X is a *total π-base* at a subset $A \subset X$ if

1. each $B \in \mathcal{B}$ meets the subset A;
2. each open subset $U \subset X$ meeting A contains a set $B \in \mathcal{B}$;
3. each decreasing sequence $B_1 \supset B_2 \supset B_3 \supset \dots$ of elements of $\mathcal{B}$ has non-empty intersection.

Proposition 3.6. *Assume that a topological space X has a total π-base at a subset $A \subset X$. If the space A is not Baire, then the player I has a winning strategy in the game $G_r(A, X)$ and hence A fails to be strong Choquet in X.*

Proof. The space A is not Baire and hence contains an open non-empty subspace $W \subset A$ of the first Baire category. Write $W = \bigcup_{n\in\omega} W_n$ where $(W_n)_{n\in\omega}$ is an increasing sequence of nowhere dense subsets in W.

Now we describe a wining strategy of the player I in the game $G_r(A, X)$. To start the game she selects an open set $U_0 \in \mathcal{B}$ and a point $x_0 \in A$ such that $x_0 \in U_0 \cap A \subset W \setminus W_0$. At the n-th inning the player I receives an open neighborhood $V_{n-1} \subset U_{n-1}$ of x_{n-1} from the player II and then choose a set $U_n \in \mathcal{B}$ and a point $x_n \in U_n \cap A$ such that $U_n \subset V_{n-1} \setminus W_n$. The existence of such a set U_n follows from the nowhere density of W_n in W and the definition of the total π-base $\mathcal{B}$.

Now we see that described strategy of the player I is winning because $\bigcap_{n\in\omega} U_n$ is not empty and avoids the set A. □

Applying this proposition and [SV] to the Michael line, we obtain

Corollary 3.7. *The subset $\mathbb{Q}$ is not strong Choquet in the Michael line $\mathbb{R}_{\mathbb{Q}}$.*

Remark 3.8. Proposition 3.6 shows that various spaces X, besides GO-spaces, have no linear $\overline{\mathrm{conv}}$-extender $u : C_\infty(A, Y) \to C_A(X, Y)$ for a non-reflexive Banach space Y. For example, the set $X = 2^{\omega_1}$ with the countable box topology has a total π-base at the closed subset $A = \{(t_\alpha)_{\alpha<\omega_1} : |\{\alpha < \omega_1 : t_\alpha \neq 0\}| < \aleph_0\}$ which is of the first Baire category [SV] and hence fails to be strong Choquet in X. This implies that the linear $\overline{\mathrm{conv}}$-extender property for bounded vector-valued functions can fail in ω_μ-metrizable spaces X.

4. Characterizing reflexive Banach spaces with help of linear $\overline{\mathrm{conv}}$-extenders

Since for normed spaces the (countable) semireflexivity coincides with the usual reflexivity (see [Me, 1.13.6]), we can combine Theorems 1.4, 3.2 and Corollary 2.2 to obtain the following characterization of reflexivity in Banach spaces.

Theorem 4.1. *For a normed space Y the following conditions are equivalent:*

1. *Y is reflexive;*
2. *for every GO-space X and a closed subspace $A \subset X$ there is a linear $\overline{\mathrm{conv}}$-extender $u : l_\infty(A, Y) \to l_\infty(X, Y)$ such that $u(C_\infty(A, Y)) \subset C_\infty(X, Y)$;*
3. *for every topological space X and a Tychonov subspace $A \subset X$ that is a $P\beta$-valued retract of X there is a linear $\overline{\mathrm{conv}}$-extender $u : l_\infty(A, Y) \to l_\infty(X, Y)$ such that $u(C_\infty(A, Y)) \subset C_A(X, Y)$;*
4. *there is a linear $\overline{\mathrm{conv}}$-extender $u : C_\infty(A, Y) \to C_A(X, Y)$ for some topological space X and some Tychonov subspace $A \subset X$ that is not strong Choquet in X;*
5. *there is a linear $\overline{\mathrm{conv}}$-extender $u : C_\infty(\mathbb{Q}, Y) \to C(\mathbb{R}_{\mathbb{Q}}, Y)$.*

Proof. The implication (1) $\Rightarrow$ (2, 3) follows from Corollary 2.2 and Theorem 1.4; (2) $\Rightarrow$ (5) is trivial and (3) $\Rightarrow$ (5) follows from Corollary 2.2(1). The implication (5) $\Rightarrow$ (4) follows from Corollary 3.7 while (4) $\Rightarrow$ (1) follows from Theorem 3.2 and the reflexivity of countably semireflexive normed spaces guaranteed by the Smulian Theorem 1.13.6 [Me]. □

5. Characterizing finite-dimensional Banach spaces with help of extenders

By Theorem 4.1, the reflexivity of a Banach space Y is equivalent to the existence of a linear $\overline{\mathrm{conv}}$-extender $u : C_\infty(\mathbb{Q}, Y) \to C_\infty(\mathbb{R}_{\mathbb{Q}}, Y)$. Now we shall construct a space Π containing a countable closed discrete subset $N \subset \Pi$ for which the existence of a linear extender $u : C_\infty(N, Y) \to C_\infty(\Pi, Y)$ characterizes finite-dimensional Banach spaces Y.

In the Stone-Čech compactification $\beta\mathbb{N}$ of the set $\mathbb{N}$ of positive integers, take any free ultrafilter $p \in \beta\mathbb{N} \setminus \mathbb{N}$ and consider the subspace $\mathbb{N} \cup \{p\}$ with a unique non-isolated point p.

Let $[0, \omega_1)$ stand for the space of all countable ordinals with the order topology. Let $N = \mathbb{N} \times \{\omega_1\}$ and $\Pi = \mathbb{N} \times [0, \omega_1] \cup \{p\} \times [0, \omega_1)$ be subspaces of the product $(\mathbb{N} \cup \{p\}) \times [0, \omega_1]$.

Theorem 5.1. *For a normed space Y the following conditions are equivalent:*

1. *there is a linear $\overline{\mathrm{conv}}$-extender $u : C_\infty(N, Y) \to C_\infty(\Pi, Y)$;*

(2) *there is an extender* $u : C_\infty(N, Y) \to C(\Pi, Y)$*;*
(3) Y *is finite-dimensional.*

Proof. The implication (1) ⇒ (2) is trivial.

(2) ⇒ (3) If Y is infinite-dimensional, then we can find a homeomorphism $f : N \to Y$ to a bounded closed discrete subset of Y. We claim that there is no continuous map $\bar f : \Pi \to Y$ with $\bar f|N = f$. Assuming that such a continuous map exists, we can use the first axiom of Y to find a countable ordinal α such that $\bar f(n, \alpha) = \bar f(n, \omega_1)$ for all $n \in \mathbb{N}$. Now the continuity of $\bar f$ at the point (p, α) would imply that $\bar f(p, \alpha)$ is a limit point of the set $\bar f(\mathbb{N} \times \{\alpha\}) = f(N)$, which contradicts the choice of $f(N)$ as a closed discrete subset of Y.

(3) ⇒ (1) If Y is finite-dimensional, then each bounded map $f : N \to Y$ can be extended to a continuous map $\beta f : \beta N \to Y$ defined on the Stone-Čech compactification $\beta N = \beta\mathbb{N} \times \{\omega_1\}$ of $N = \mathbb{N} \times \{\omega_1\}$. Now extend f to a continuous map $\bar f : \Pi \to Y$ letting $\bar f(x, \alpha) = \beta f(x)$ for $(x, \alpha) \in \Pi \subset \beta\mathbb{N} \times [0, \omega_1]$. Observe that $\bar f(\Pi) \subset \overline{f(N)}$. Consequently, the operator $u : C_\infty(N, Y) \to C_\infty(\Pi, Y)$, $u : f \mapsto \bar f$, is a linear $\overline{\text{conv}}$-extender. □

Remark 5.2. Since the space $\Pi = \mathbb{N} \times [0, \omega_1] \cup \{p\} \times [0, \omega_1)$ is the union of two orderable spaces, we see that Corollary 2.2 cannot be generalized from GO-spaces to spaces X that are unions of two orderable spaces.

6. Monotone extenders for functions with values in pospaces

It turns out that the method of the proof of Theorem 3.2 can be modified to prove the non-existence of monotone extenders for functions with values in pospaces. As a result we obtain a general theorem that generalizes many known results on the non-existence of extenders, see [vD$_1$], [HL], [SV], [GHO].

By a *pospace* we understand a topological space Y endowed with a partial order $\le$. For a point $y \in Y$ and a subset B of a pospace Y let

$$\uparrow y = \{x \in Y : x \ge y\} \text{ and } \uparrow B = \bigcup_{b \in B} \uparrow b$$

be the upper cones of y and B in Y.

Observe that a subset $B \subset Y$ has an upper bound in Y if and only if $\bigcap_{b \in B} \uparrow b \ne \emptyset$.

We shall say that a subset $B \subset Y$ is *almost upper bounded* in Y if for any family $\{G_b\}_{b \in B}$ of G_δ-subsets of Y with $b \in G_b$, $b \in B$, the intersection $\bigcap_{b \in B} \uparrow G_b$ is not empty.

It is clear that each upper bounded set $B \subset Y$ is almost upper bounded while the converse is true if each point $b \in B$ has countable pseudocharacter in Y. In particular, each almost upper bounded subset in a metrizable space is upper bounded.

By an *ω-increasing ray* in a pospace Y we shall understand a continuous map $\gamma : [0, \infty) \to Y$ such that $\gamma(n) \le \gamma(t)$ for any integer number $n \in \omega$ and a real number $t \ge n$.

Theorem 6.1. *If for a subspace* Y_0 *of a pospace* Y *and a Tychonov subspace* A *of a topological space* X *there is a monotone extender* $u : C(A, Y_0) \to C_A(X, Y)$*, then either* A *is strong Choquet in* X *or else for each ω-increasing ray* $\gamma : [0, \infty) \to Y_0$ *the set* $\gamma(\omega)$ *is almost upper bounded in* Y*.*

Applying this theorem to the real line $\mathbb{R}$, we obtain the following corollary generalizing Theorem 1.4 of [SV].

Corollary 6.2. *A Tychonov subspace A of a topological space X is strong Choquet in X if there is a monotone extender $u : C(A) \to C_A(X)$.*

Applying Theorem 6.1 to the Banach lattice c_0 (endowed with the natural partial order), we obtain another non-existence result. Here we remark that each linear $\overline{\mathrm{conv}}$-extender $u : C_\infty(A, c_0) \to C_\infty(X, c_0)$ is monotone.

Corollary 6.3. *A Tychonov subspace A of a topological space X is strong Choquet in X if there is a monotone extender $u : C_\infty(A, c_0) \to C_A(X, c_0)$.*

Proof. Denote by $(\mathbf{e}_n)_{n\in\mathbb{N}}$ the standard basis of the Banach space c_0. Let $\gamma : [0, \infty) \to c_0$ be the increasing piece-linear function such that $\gamma(n) = \sum_{i=1}^{n} \mathbf{e}_i$ for all $n \in \mathbb{N}$. It is clear that the set $\gamma(\omega)$ is norm-bounded but has no upper bound in c_0. Applying Theorem 6.1 we conclude that A is strong Choquet in X. □

Surprisingly, but we do not know if the same result is true for the Banach lattice c of all convergent sequences.

Question 6.4. *Is there a monotone extender $u : C_\infty(\mathbb{Q}, c) \to C(\mathbb{R}_\mathbb{Q}, c)$?*

7. CHARACTERIZING POLISH SPACES WITH HELP OF EXTENDERS

In this section we unify all results proved in the preceding sections and obtain the following characterization of Polish spaces.

Theorem 7.1. *For a metrizable separable space A the following conditions are equivalent:*

1. *A is Polish;*
2. *A is strong Choquet;*
3. *A is a P_ω-valued retract in each normal space X containing A as a closed subspace;*
4. *A is a P_σ-valued retract in some topological space $X \supset A$ that is strong Choquet at A;*
5. *for every locally convex linear topological space Y and every normal space X containing A as a closed subspace there is a linear* conv*-extender $u : C(A, Y) \to C(X, Y)$;*
6. *for some infinite-dimensional Banach space Y and some topological space $X \supset A$ that is strong Choquet at A there is a* conv*-extender $u : C_\infty(A, Y) \to C_A(X, Y)$;*
7. *for some topological space $X \supset A$ that is strong Choquet at A and some separable non-reflexive Banach space Y there is a linear $\overline{\mathrm{conv}}$-extender $u : C_\infty(A, Y) \to C_A(X, Y)$.*
8. *for some topological space $X \supset A$ that is strong Choquet at A there is a monotone extender $u : C(A) \to C_A(X)$;*
9. *for some topological space $X \supset A$ that is strong Choquet at A there is a monotone extender $u : C_\infty(A, c_0) \to C_A(X, c_0)$.*

Proof. We shall establish the implications (2) ⇒ (1) ⇒ (5) ⇒ (3, 6) ⇒ (4) ⇒ (7) ⇒ (2) and (5) ⇒ (7, 8, 9) ⇒ (2).

The implication (2) ⇒ (1) is due to G.Choquet, see [Ke, 8.18].

(1) ⇒ (5) Assume that A is a Polish space. By [Ke, 4.17], A admits a closed embedding $e : A \to \mathbb{R}^\omega$. Given any normal subspace X containing A as a closed

subset, we can apply the Tietze-Urysohn Theorem to find a continuous map $g : X \to \mathbb{R}^\omega$ extending the map e. By the Dugundji Theorem 0.1, for every locally convex space Y there is a linear conv-extender $v : C(e(A), Y) \to C(\mathbb{R}^\omega, Y)$. Now define a linear conv-extender $u : C(A, Y) \to C(X, Y)$ by the formula $u(f) = v(f \circ e^{-1}) \circ g : X \to Y$.

(5) ⇒ (3) Let X be a normal space containing A as a closed subspace. Let $Y = C^*_\infty(A)$ be the dual Banach space endowed with the weak-star topology. By (5), there is a conv-extender $u : C_\infty(A, Y) \to C_\infty(X, Y)$. Then the embedding $\delta : A \to P_\omega(A) \subset C^*_\infty(A)$ assigning to each point $a \in A$ the Dirac measure supported at a has a continuous extension $r = u(\delta) : X \to Y$ given by the conv-extender u. It follows that $r(X) \subset \mathrm{conv}(\delta(A)) = P_\omega(A)$, which means that $r : X \to P_\omega(A)$ is the required P_ω-valued retraction of X onto A.

The implications (3) ⇒ (4) and (5) ⇒ $(6-9)$ will follow as soon as we find a normal space $X \supset A$ that is strong Choquet at A. For this take any metrizable compactification K of A and consider the space K_A. The compactness of K implies the completeness of K_A at A. By Proposition 3.5(1), the space K_A is strong Choquet at A. The normality of K_A follows from [Eng, 5.1.22].

To prove the implication (6) ⇒ (4), assume that for some infinite-dimensional Banach space Y and some topological space $X \supset A$ that is strong Choquet at A there exists a conv-extender $u : C_\infty(A, Y) \to C(X_A, Y)$. Let K be any metrizable compactification of the separable metrizable space A and let $P(K)$ be the space of probability measures on K. Let $\delta : A \to P(K)$ be the embedding assigning to each point $x \in A$ the Dirac measure $\delta_x \in P(K)$ supported by x. Observe that $P_\omega(A)$ coincides with the convex hull of the set $\delta(A)$ in $P(K)$.

According to [BP, §III.2], there is a continuous affine embedding $e : P(K) \to Y$. Consider the map $g = e \circ \delta : A \to Y$ and its continuous extension $\bar{g} = u(g) : X_A \to Y$. Since u is a conv-extender, $\bar{g}(X) \subset \mathrm{conv}(g(A)) = e(P_\omega(A))$. It is clear that the map $r = e^{-1} \circ \bar{g} : X_K \to P_\omega(A)$ is continuous and $r|A = \delta$, which means that A is a P_ω-valued retract of X_A. By Theorem 3.3(2), the space X_A is strong Choquet at A.

The implication (4) ⇒ (7) follows from Proposition 1.2 and $(7, 8, 9) \Rightarrow (2)$ from Theorem 3.2 and Corollaries 6.2, 6.3, respectively. □

8. Monotone extenders for functions with values in Banach lattices

In light of Corollary 6.3 it is natural to ask about the existence of linear monotone extenders $u : C_\infty(A, Y) \to C_\infty(X, Y)$ for functions taking their values in a (non-reflexive) Banach lattice Y. Many classical Banach spaces like c_0, l_p, L_p, $C(K)$ have the natural structure of a Banach lattice.

We recall that a Banach lattice is a real Banach space $(Y, \|\cdot\|)$ endowed with a partial order $\le$ satisfying the following four axioms (see [LT]):

- $x \le y$ implies $x + z \le y + z$ for all $x, y, z \in Y$;
- $a \cdot x \ge 0$ for any $x \ge 0$ in Y and any real number $a \ge 0$;
- any two points $x, y \in Y$ have the largest lower and smallest upper bounds $x \wedge y$ and $x \vee y$ in Y;
- $\|x\| \le \|y\|$ whenever $|x| \le |y|$ where the absolute value $|x|$ of $x \in Y$ is defined by $|x| = -x \vee x$.

Let us remark that the dual Banach space Y^* to a Banach lattice Y is a Banach lattice with respect to the partial order $\le$ defined by the formula: $x^* \le y^*$ for $x^*, y^* \in Y^*$ iff $x^*(z) \le y^*(z)$ for all $z \ge 0$ in Y.

We shall say that a Banach lattice Y is *positively norm-one complemented in its bidual* Y^{**} if there is a linear monotone projector $P : Y^{**} \to Y$ with norm $\|P\| = 1$. The class of such Banach lattices includes all dual Banach lattices and also all weakly sequentially complete Banach lattices (like $L_1(\mu)$), see [LT, 1.c.4].

The following theorem is a monotone version of Theorem 1.6 and can be proved by analogy.

Theorem 8.1. *For any $P\beta$-valued retract A of a topological space X and every Banach lattice Y that is positively norm-one complemented in its bidual Y^{**} there is a linear monotone* wcc*-extender $u : l_\infty(A, Y) \to l_\infty(X, Y)$ such that $\|u\| = 1$ and $u(C_\infty(A, Y)) \subset C_A(X, Y)$.*

The same concerns the following corollary that can be derived from Theorem 2.1(2).

Corollary 8.2. *For every subset A of a linearly ordered space $(X, \le)$ and every Banach lattice Y that is positively norm-one complemented in its bidual Y^{**} there is a linear monotone* wcc*-extender $u : l_\infty(A, Y) \to l_\infty(X, Y)$ such that $\|u\| = 1$ and $u(C_\infty(A_g, Y)) \subset C_\infty(X_g, Y)$ for any GO-topology $g \ni X \setminus A$ on X.*

Now we are going to characterize σ-complete Banach lattices Y admitting a linear monotone extender $u : C_\infty(A, Y) \to C_\infty(X, Y)$ for any closed subset A of a GO-space X. The characterization Theorem 9.1 below relies on the notion of a $\perp$-extender defined as follows.

Two elements x, y of a Banach lattice Y are called *disjoint* if $|x| \wedge |y| = 0$. For any point $y \in Y$ the set

$$y^\perp = \{x \in Y : |x| \wedge |y| = 0\}$$

of elements disjoint with y is a closed linear subspace in Y called the *polar* of y, see [LT]. For a subset $B \subset Y$ the closed linear subspace

$$B^\perp = \bigcap_{b \in B} b^\perp$$

is called the *polar* of B and $B^{\perp\perp} = (B^\perp)^\perp$ is called the *bi-polar* of B. It is clear that $B^{\perp\perp}$ is a closed linear subspace of Y, containing B.

An extender $u : C_\infty(A, Y) \to Y^X$ will be called a $\perp$-*extender* if u is a $\mathcal{C}$-extender for the collection $\mathcal{C} = \{B^\perp : B \subset Y\}$ of the polar sets. It is easy to see that an extender $u : C_\infty(A, Y) \to C_\infty(X, Y)$ is a $\perp$-extender if and only if for every bounded function $f : A \to Y$ the image $\bar f(X)$ of the extended function $\bar f = u(f) : X \to Y$ lies in the bi-polar $f(A)^{\perp\perp}$ of $f(A)$. An extender which is simultaneously a $\perp$-extender and a wcc-extender will be called a $\perp$-wcc-*extender*.

Now we shall derive from Theorem 6.1 a necessary condition of the existence of a monotone extender.

Theorem 8.3. *Assume that a Tychonov subspace A of a topological space X is not strong Choquet in X. If for a Banach lattice Y there is a monotone extender $u : C_\infty(A, Y) \to C_A(X, Y)$ (which is a $\perp$-extender), then each countable norm-bounded upward directed subset $D \subset Y$ has an upper bound in Y (in $D^{\perp\perp}$).*

Proof. Let $D = \{y_n : n \in \mathbb{N}\} \subset Y$ be a countable norm-bounded upward directed subset. Since D is upward directed, by induction we can construct an increasing sequence $\{z_n\}_{n\in\omega} \subset D$ such that $z_n \geq y_i$ for all $i \leq n$. Then each upper bound for the set $E = \{z_n\}_{n\in\omega}$ is also an upper bound for D. Consider the piece-linear map $\gamma : [0, \infty) \to Y$ defined by $\gamma(k) = z_k$ for all $k \in \omega$. It follows that γ is an ω-increasing ray with bounded range $B = \gamma([0, \infty))$. Consequently, $C(A, B) \subset C_\infty(A, Y)$ and the restriction $v = u|C(A, B) : C(A, B) \to C_A(X, Y)$ is a well-defined monotone extender. Applying Theorem 6.1, we conclude that the set $\gamma(\omega) = E$ is (almost) upper bounded in Y.

Now assume that the extender $u : C_\infty(A, Y) \to C(X, Y)$ is a $\perp$-extender and thus $u(C_\infty(A, D^{\perp\perp})) \subset C(X, D^{\perp\perp})$.

Observe that $E^{\perp\perp} \supset E$ is a linear subspace of Y and thus $B = \gamma([0, \infty)) \subset \mathrm{conv}(\gamma(\omega)) \subset \mathrm{conv}(E) \subset E^{\perp\perp} \subset D^{\perp\perp}$. Then $C(A, B) \subset C_\infty(A, D^{\perp\perp})$ and we can consider the monotone extender $v = u|C(A, B) : C(A, B) \to C(X, D^{\perp\perp})$. By Theorem 6.1 the set $E = \gamma(\omega)$ is (almost) upper bounded in $D^{\perp\perp}$. □

9. Characterizing weakly sequentially complete Banach lattices

In this section we characterize weakly sequentially complete Banach lattices with help of monotone extenders.

We recall that a Banach lattice Y is called

- *σ-complete* if each upper bounded increasing sequence $\{y_n\}_{n\in\omega} \subset Y$ has the smallest upper bound $\vee_{n\in\omega} y_n$.
- *order continuous* if each downward directed subset $D \subset Y$ with $\wedge D = 0$ contain zero in its closure.

For example, the Banach lattice c_0 is σ-complete but not order continuous while c is not σ-complete. By [LT, page 7] each order continuous Banach lattice is σ-complete.

Theorem 9.1. *For a Banach lattice Y the following conditions are equivalent:*

1. *Y is weakly sequentially complete;*
2. *Y does not contain a copy of c_0;*
3. *norm-bounded increasing sequences in Y converge;*
4. *Y is order continuous and there is a monotone extender $u : C_\infty(A, Y) \to C_A(X, Y)$ for some topological space X and a Tychonov subspace $A \subset X$, which is not strongly Choquet in X;*
5. *Y is σ-complete, does not contain a copy of l_∞, and there is a monotone extender $u : C_\infty(A, Y) \to C_A(X, Y)$ for some topological space X and a Tychonov subspace $A \subset X$, which is not strongly Choquet in X.*

Moreover, if $\mathrm{dens}(Y) < \mathfrak{c}$*, then the conditions (1)–(5) are equivalent to:*

6. *for every subset A of a linearly ordered space $(X, \leq)$ there is a linear monotone $\perp$-wcc-extender $u : l_\infty(A, Y) \to l_\infty(X, Y)$ such that $\|u\| = 1$ and $u(C_\infty(A_g, Y)) \subset C_\infty(X_g, Y)$ for every GO-topology $g \ni X \setminus A$ on X;*
7. *there is a monotone $\perp$-extender $u : C_\infty(A, Y) \to C_A(X, Y)$ for some topological space X and a Tychonov subspace $A \subset X$, which is not strongly Choquet in X.*

Proof. The equivalence of the first three conditions is well-known and can be found in [LT, 1.c.4].

(2) ⇒ (5). Assume that the Banach lattice Y does not contain a copy of c_0. Then it also does not contain a copy of l_∞. By [LT, 1.c.4, 1.a.8], Y is σ-complete and by [LT, 1.c.4], Y is positively norm-one complemented in the bidual space Y^{**}. By Corollary 8.2, there is a monotone extender $u : C_\infty(\mathbb{Q}, Y) \to C_\infty(\mathbb{R}_\mathbb{Q}, Y)$. By Corollary 3.7, $\mathbb{Q}$ is not strong Choquet in $\mathbb{R}_A$. Thus (5) follows.

(5) ⇒ (4) Assume that Y is σ-complete but contains no copy of l_∞. By Propositions 1.a.7 and 1.a.8 of [LT], Y is order continuous.

(4) ⇒ (3) By Theorem 8.3, norm-bounded increasing sequences in Y are upper-bounded and thus converge by the order continuity of Y.

Now assume that $\mathrm{dens}(Y) < \mathfrak{c}$.

(1) ⇒ (6) Assume that Y is weakly sequentially complete, and let A be a subset of a linearly ordered space $(X, \le)$.

By Theorem 2.1(2), there is a linear $\overline{\mathrm{conv}}^*$-extender $u : l_\infty(A, Y^{**}) \to l_\infty(X, Y^{**})$ such that $u(C_\infty(A_g, Y^{**})) \subset C_\infty(X_g, Y^{**})$ for every GO-topology $g \ni X \setminus A$ on $(X, \le)$. Since the positive cone $Y^{**}_+ = \{y^{**} \in Y^{**} : y^{**} \ge 0\}$ is convex and closed in the $*$-weak topology of Y^{**}, we conclude that $u(l_\infty(A, Y^{**}_+)) \subset l_\infty(X, Y^{**}_+)$ which implies that u is monotone.

By Theorem 1.c.4 of [LT], the weak sequential completeness of Y implies the existence of a monotone norm-one projector $P : Y^{**} \to Y$ whose kernel coincides with $Y^\perp \subset Y^{**}$. Consequently, Y^{**} can be identified with the direct sum $Y \oplus Y^\perp$. By analogy with the proof of Theorem 1.6, we can consider the extender $v : l_\infty(A, Y) \to l_\infty(X, Y)$ assigning to each bounded function $f \in l_\infty(A, Y)$ the function $P \circ u(f) : X \to Y$ and prove that v is a wcc-extender with $v(C_\infty(A, Y)) \subset C_\infty(X, Y)$. Being the composition of two monotone norm one operators, the extender v is monotone and has norm $\|v\| = 1$.

It remains to prove that v is a $\perp$-extender. Given any subset $B \subset Y$ we should prove that $v(f)(X) \subset B^\perp \subset Y$ for every bounded function $f : A \to B^\perp$. Since $B^\perp = \bigcap_{b \in B} b^\perp$, it suffices to check that $v(f)(X) \subset b^\perp$ for every $b \in B$.

By Proposition 1.a.9 of [LT], $Y = Y_1 \oplus Y_2$ where $Y_1 = b^\perp$ and $Y_2 = b^{\perp\perp}$. Consequently, $Y^{**} = Y_1^{**} \oplus Y_2^{**}$ and Y_1^{**} is $*$-weakly closed in Y^{**}. The sublattices Y_1, Y_2 of Y are weakly sequentially complete and by Theorem 1.c.4 of [LT], their biduals decompose as $Y_i^{**} = Y_i \oplus Y_i^\perp$ where the polar set $Y_i^\perp$ is taken in Y_i^{**}. Consequently, $Y^{**} = Y_1 \oplus Y_2 \oplus Y_1^\perp \oplus Y_2^\perp = Y \oplus Y^\perp$ and $P(Y_1^{**}) \subset Y_1$. Since u is a $\overline{\mathrm{conv}}^*$-extender, $u(f)(X) \subset \overline{\mathrm{conv}}^*(f(A)) \subset \overline{\mathrm{conv}}^*(Y_1) \subset Y_1^{**}$ and then $v(f)(X) = P(u(f)(X)) \subset P(Y_1^{**}) \subset Y_1 = b^\perp$.

The implication (6) ⇒ (7) follows from Corollary 3.7.

It remains to prove that (7) ⇒ (2). By Theorem 8.3, the condition (7) implies that each norm-bounded countable upward directed subset $D \subset Y$ has an upper bound in $B^{\perp\perp}$. Assume that Y contains a copy of c_0. By the proof of Theorem 1.a.5 in [LT] (see remark after Theorem 1.c.4 in [LT]), the space Y contains c_0 as a sublattice. Denote by $(e_n)_{n \in \mathbb{N}}$ the standard basis of the Banach lattice c_0 and let $c = \inf_{n \in \mathbb{N}} \|e_n\| > 0$ where $\|\cdot\|$ stands for the norm of the Banach lattice Y.

For every subset $A \subset \mathbb{N}$ consider the countable upward directed subset $D_A = \{\sum_{i \in F} e_i : F \subset A \text{ is finite}\}$. By Theorem 8.3, this set has an upper bound $b_A \in D_A^{\perp\perp} \subset Y$.

Let us show that $\|b_A - b_B\| \ge c$ for every distinct sets $A, B \subset \mathbb{N}$. Without loss of generality, there is an element $n \in B \setminus A$. Since $e_n \in D_A^\perp$ and $b_A \in D_A^{\perp\perp}$, we get that b_A and e_n are disjoint and hence $b_A \wedge e_n = |b_A| \wedge |e_n| = 0$. Now we see that

$$(b_B - b_A) \geq (b_B - b_A) \wedge e_n = b_B \wedge e_n - b_A \wedge e_n = b_B \wedge e_n - 0 = e_n$$

and hence $\|b_B - b_A\| \geq \|e_n\| \geq c$. Consequently, $\{b_A : \emptyset \neq A \subset \mathbb{N}\}$ is a discrete subset of size continuum in Y, which is not possible because $\mathrm{dens}(Y) < \mathfrak{c}$. □

Remark 9.2. For a subset A of a topological space X and a Banach lattice Y consider the following three properties:

1. there is a linear $\overline{\mathrm{conv}}$-extender $u : C_\infty(A, Y) \to C_\infty(X, Y)$;
2. there is a norm-one linear extender $u : C_\infty(A, Y) \to C_\infty(X, Y)$;
3. there is a monotone linear extender $u : C_\infty(A, Y) \to C_\infty(X, Y)$.

It is clear that (1) ⇒ (2, 3). In [vD$_1$] and [vD$_2$] E.K. van Douwen asked if for $Y = \mathbb{R}$ there are other implications among the conditions (1), (2), and (3). The results of this paper allow us to conclude that (1) does not follow from (2,3). Indeed, by Theorem 9.1, for the Banach lattice $Y = l_1$ and the Michael line $X = \mathbb{R}_{\mathbb{Q}}$ there is a monotone norm-one linear extender $u : C_\infty(A, l_1) \to C_\infty(\mathbb{R}_{\mathbb{Q}}, l_1)$ for every closed subset $A \subset \mathbb{R}_{\mathbb{Q}}$. Yet, by Theorem 4.1, no linear $\overline{\mathrm{conv}}$-extender $u : C_\infty(\mathbb{Q}, l_1) \to C_\infty(\mathbb{R}_{\mathbb{Q}}, l_1)$ exists.

10. Proof of Theorem 1.4

Let A be a Tychonov subspace of a topological space X.

The implication (1) ⇒ (2) will follow as soon as we show that the dual Banach space $Y = C^*_\infty(A)$ endowed with the weak-star topology is semireflexive. But this follows from the Banach-Steinhaus Uniform Boundedness Principle, see [HHZ, Th.58].

To prove that (2) ⇒ (3), fix a $\overline{\mathrm{conv}}$-extender $u : C_\infty(A, Y) \to C_A(X, Y)$ where $Y = C^*_\infty(A)$ is the dual Banach space endowed with the weak-star topology. Since each function $f \in C_\infty(A)$ admits a unique continuous extension to βA, we can identify the Banach space $C_\infty(A)$ with the Banach space $C(\beta A)$ and Y with $C^*(\beta A)$. Consider the bounded continuous map $\delta : A \to C^*(\beta A)$ assigning to each point $a \in A$ the Dirac measure δ_a supported by a. Let $r = u(\delta) : X_A \to Y = C^*(\beta A)$ be the continuous extension of δ given by the $\overline{\mathrm{conv}}$-extender u. It follows that $r(X) \subset \overline{\mathrm{conv}}(\delta(A)) = P(\beta A)$, which means that $r : X_A \to P(\beta A)$ is the required $P\beta$-valued retraction of X_A onto A.

(3) ⇒ (1) Fix a $P\beta$-valued retraction $r : X_A \to P(\beta A)$. Denote by A_d the space A endowed with the discrete topology. The identity map $i : A_d \to A$ is continuous and hence extends to a continuous surjective map $\beta i : \beta A_d \to \beta A$ between the Stone-Čech compactifications. This map induces a surjective continuous map $P(\beta i) : P(\beta A_d) \to P(\beta A)$ between the spaces of probability measures. The surjectivity of the map $P(\beta i)$ allows us to select a (generally discontinuous) function $s : X \to P(\beta A_d)$ such that $P(\beta i) \circ s = r$ and $s(a) = \delta_a$ for all $a \in A$.

Now, given a locally convex semireflexive space Y, we are ready to define a linear $\overline{\mathrm{conv}}$-extender $u : l_\infty(A, Y) \to l_\infty(X, Y)$. Given any bounded function $f : A \to Y$, consider the closed convex hull $K_{\mathrm{w}} \subset Y_{\mathrm{w}}$ of $f(A)$, endowed with weak topology. The semireflexivity of Y guarantees that the space K_{w} is compact in the weak topology. Let $\beta f_d : \beta A_d \to K_{\mathrm{w}}$ be the continuous extension of the map $f_d = f \circ i : A_d \to K_{\mathrm{w}}$.

For every $x \in X$ consider the probability measure $\mu_x = s(x)$ and the Pettis integral

$$u(f)(x) = \int_{\beta A_d} (\beta f_d)\, d\,\mu_x \in K_{\mathrm{w}} \subset Y_{\mathrm{w}}, \tag{1}$$

which is well-defined because K_{w} weakly compact and convex, see [DU, §II.3].

The linearity of the Pettis integral implies that the formula $u : C_\infty(A, Y) \to Y^X$, $u : f \mapsto u(f)$, determines a well-defined linear $\overline{\mathrm{conv}}$-extender.

It remains to check that for any admissible topology τ on Y the function $u(f) : X \to Y_\tau$ is continuous at A provided $f : A \to Y_\tau$ is continuous. Given any point $a \in A$ and an open convex neighborhood $O \subset Y_\tau$ of zero we should find a neighborhood $W \subset X$ of a such that $u(f)(W) \subset f(a) + O$. Since τ is admissible, there is an open convex symmetric neighborhood $U \subset Y_\tau$ of zero such that its closure $\overline{U}$ in Y lies in $\frac{1}{3}O$. By the Hahn-Banach Theorem, the set $\overline{U}$, being closed and convex, is weakly closed. Now the continuity of the map $\beta f : \beta A \to Y_{\mathrm{w}}$ implies that the set $F = (\beta f)^{-1}(f(a) + \overline{U})$ is closed in βA. On the other hand, the continuity of $f : A \to Y_\tau$ implies that the set $V = f^{-1}(f(a) + U)$ is open in A. It follows from the regularity of βA that the closure $\overline{V} \subset F$ is a neighborhood of a in βA. Consequently, F is a neighborhood of a in βA and $F' = (\beta i)^{-1}(F) \subset \beta A_d$ is a neighborhood of a in βA_d.

Since the map $f : A \to Y_\tau$ is bounded, for the neighborhood $U \subset Y_\tau$ of zero there is a number m so large that $f(A) \subset mU$. Then $K_{\mathrm{w}} = \overline{\mathrm{conv}}(f(A)) \subset m\overline{U}$ and $K_{\mathrm{w}} - f(a) \in m\overline{U} - m\overline{U} = 2m\overline{U}$. It follows that the set $\mathcal{V} = \{\mu \in P(\beta A) : \mu(F) > 1 - \frac{1}{m}\}$ is a neighborhood of the Dirac measure δ_a in $P(\beta A)$. Since the $P\beta$-valued retraction r is continuous at a, there is a neighborhood $W \subset X$ of a such that $r(W) \subset \mathcal{V}$. We claim that $u(f)(W) \subset f(a) + O$.

Take any point $x \in W$ and consider the measures $r(x) \in \mathcal{V}$ and $\mu_x = s(x) \in P(\beta A_d)$. It follows from $P(\beta i)(\mu_x) = P(\beta i)(s(x)) = r(x)$ that $\mu_x(F') > 1 - \frac{1}{m}$. Then

$$\begin{aligned} u(f)(x) - f(a) = \int_{\beta A_d} (\beta f_d - f(a)) d\mu_x = \\ \int_{\beta A_d \setminus F'} (\beta f_d - f(a)) d\mu_x + \int_{F'} (\beta f_d - f(a))\, d\mu_x \in \\ \mu_x(\beta A_d \setminus F') \cdot (K_{\mathrm{w}} - f(a)) + \mu_x(F') \cdot \overline{U} < \\ \frac{1}{m} 2m\overline{U} + \overline{U} = 3\overline{U} \subset O. \end{aligned}$$

11. Proof of Theorem 2.1

Let A be a non-empty subset of a linearly ordered space $(X, \leq)$. By [Lu, 2.9], the linearly ordered topological space $(X, \leq)$ has a linearly ordered compactification $(\bar{X}, \leq)$.

1. Let A_d be A with the discrete topology and $i : A_d \to A$ be the identity map. We shall construct a function $r : \bar{X} \to P_2(\beta A_d)$ such that $r(a) = \delta_a$ and for every GO-topology $g \ni X \setminus A$ on X the composition $P(\beta i) \circ r : X_g \to P_2(\beta A_g)$ is continuous.

Let $\bar{A}$ be the closure of A in $\bar{X}$ and $\beta i : \beta A_d \to \bar{A}$ be the Stone-Čech extension of the identity map $i : A_d \to A$. We shall identify the space βA_d with the set of

Dirac measures in $P_2(\beta A_d)$. For every $a \in \bar{A}$ select an ultrafilter $u_a \in \beta A_d$ such that $\beta i(u_a) = a$ and $u_a = a$ if $a \in A$.

Write the complement $\bar{X} \setminus \bar{A}$ as the disjoint union $\cup\mathcal{C}$ of the family $\mathcal{C}$ of order-convex components of $\bar{X} \setminus \bar{A}$. Those are maximal order-convex subsets of the complement $\bar{X} \setminus \bar{A}$. For each component $C \in \mathcal{C}$ we define an order-convex set $\tilde{C} \supset C$ and a continuous map $r_C : \tilde{C} \to P_2(\beta A_d)$ as follows.

If $C = (\leftarrow, \min \bar{A})$, then we put $\tilde{C} = (\leftarrow, \min \bar{A}]$, $b_C = u_{\min \bar{A}}$ and $r_C : \tilde{C} \to \{b_C\} \subset \beta A_d \subset P_2(\beta A)$ be the constant map.

If $C = (\max \bar{A}, \to)$, then we put $\tilde{C} = [\max \bar{A}, \to)$, $a_C = u_{\max \bar{A}}$, and $r_C : \tilde{C} \to \{a_C\} \subset \beta A_d \subset P_2(\beta A)$ be the constant map.

In the remaining case, C coincides with the open interval (a, b) for some points $a < b$ in $\bar{A}$. Let $a_C = u_a$ and $b_C = u_b$. Using the normality of the compact space $\bar{X}$, find a continuous function $\lambda_C : X \to [0, 1]$ such that $\lambda_C(a) = 0$ and $\lambda_C(b) = 1$. Finally, define the map $r_C : \tilde{C} \to P_2(\beta A_d)$ by the formula

$$r_C(x) = (1 - \lambda_C(x)) \cdot a_C + \lambda_C(x) \cdot b_C, \quad x \in C.$$

Unifying the maps r_C, $C \in \mathcal{C}$, define a function $r : \bar{X} \to P_2(\beta A_d)$ by the formula

$$r(x) = \begin{cases} u_x & \text{if } x \in \bar{A}, \\ r_C(x) & \text{if } x \in C \in \mathcal{C}. \end{cases}$$

Observe that for every component $C \in \mathcal{C}$ the map $r|\tilde{C}$ is continuous. It remains to prove that r has the continuity property required in the item (1) of Theorem 2.1. We shall return to this problem after establishing the item (2).

2. Given a semireflexive locally convex space Y, we shall define a linear $\overline{\text{conv}}$-extender $v : l_\infty(A, Y) \to l_\infty(\bar{X}, Y)$. Take any bounded function $f : A \to Y$ and consider the function $f_d = f \circ i : A_d \to Y$. The semireflexivity of the space Y guarantees that the closed convex hull K_{w} of $f(A)$ is compact in the weak topology of Y. Then the map $f_d : A_d \to K_{\mathrm{w}}$ has a continuous extension $\beta f_d : \beta A_d \to K_{\mathrm{w}}$. For every $x \in \bar{X}$ we can integrate this function by the measure $\mu_x = r(x) \in P_2(\beta A_d)$ and obtain the value $\bar{f}(x) = \int_{\beta A_d} (\beta f_d)\, d\mu_x$ of the function $v(f) = \bar{f}$ at the point $x \in \bar{X}$.

It is clear that the so-defined operator $v : l_\infty(A, Y) \to l_\infty(\bar{X}, Y)$, $v : f \mapsto \bar{f} = v(f)$, is a linear $\overline{\text{conv}}$-extender. It induces a linear $\overline{\text{conv}}$-extender $u : l_\infty(A, Y) \to l_\infty(X, Y)$, $u : f \mapsto u(f)|X$.

To finish the proof of the second item of Theorem 2.1, it remains to show that $u(C_\infty(A_g, Y_\tau)) \subset C_\infty(X_g, Y_\tau)$ for every GO-topology $g \ni X \setminus A$ on $(X, \le)$ and every admissible topology τ on Y. Take any bounded continuous map $f \in C_\infty(A_g, Y_\tau)$ and consider its extension $\bar{f} = v(f) : \bar{X} \to Y$. It follows from the definition of $\bar{f}$ that $\bar{f}|\tilde{C}$ is continuous for every component $C \in \mathcal{C}$. So it remains to check the continuity of $u(f) = \bar{f}|X$ at each point $x_0 \in \bar{A} \cap X$.

It suffices to prove that the restrictions of $\bar{f}$ to the closed subsets

$$X_g^- = (\leftarrow, x_0] \cap X_g \text{ and } X_g^+ = [x_0, \to) \cap X_g$$

are continuous at x_0. We shall do that for the right ray X_g^+ while for the left ray X_g^- the argument is analogous. If the point x_0 is isolated in X_g^+, then there is noting to prove: $\bar{f}|X_g^+$ is trivially continuous at x_0.

So assume that x_0 is not isolated in X_g^+. First we consider the case $x_0 \in \bar{A} \setminus A$. Since the set A is closed in X_g, there is an order-convex open set $U \subset X_g^+$ such that $x_0 \in U \subset X_g^+ \setminus A$. Since the point x_0 is not isolated in X_g^+, there is a point $x_1 \in U$. It follows that the open interval (x_0, x_1) does not intersect A and hence $(x_0, x_1) \subset C$ and $[x_0, x_1) \subset \tilde{C}$ for some component $C \in \mathcal{C}$. Now the continuity of $\bar{f}|[x_0, x_1)$ follows from the continuity of $\bar{f}|\tilde{C}$.

Next, assume that $x_0 \in A$. Given any open neighborhood $O \subset Y_\tau$ of zero, we should find a neighborhood $W \subset X_g^+$ of x_0 such that $\bar{f}(W) \subset f(a) + O$. Since the topology τ is admissible, there is an open convex neighborhood $U \subset Y_\tau$ whose closure $\overline{U}$ in Y lies in O.

The continuity of $f : A \to Y_\tau$ yields an open order-convex subset $V \subset X_g^+$ such that $x_0 \subset V \cap A \subset f^{-1}(U)$. Since x_0 is not isolated in X_g^+, there is a point $x_1 > x_0$ in V. If $[x_0, x_1) \subset \tilde{C}$ for some component $C \in \mathcal{C}$, then the continuity of $\bar{f}|X_g^+$ at x_0 follows from the continuity of $\bar{f}|\tilde{C}$. In the other case, (x_0, x_1) contains a point $x_2 \in A$. Let F be the closure of the set $[x_0, x_2] \cap A$ in βA_d. The continuity of the map $\beta i : \beta A_d \to \bar{A}$ implies that $(\beta i)^{-1}([x_0, x_2] \cap \bar{A}) = F$. Consequently, $u_a \in F$ for every $a \in [x_0, x_2] \cap \bar{A}$. Then $a_C, b_C \in F$ for every component $C \subset [x_0, x_2]$.

On the other hand, the continuity of $\beta f_d : \beta A_d \to Y_{\rm w}$ and the inclusion $\beta f_d([x_0, x_2] \cap A) \subset f(x_0) + U$ imply $\beta f_d(F) \subset f(x_0) + \bar{U} \subset f(x_0) + O$. Now looking at the definition of $\bar{f} = u(f)$, we see that $\bar{f}([x_0, x_2]) \subset \mathrm{conv}\{\beta f_d(u_a) : a \in [x_0, x_2] \cap \bar{A}\} \subset \mathrm{conv}(\beta f_d(F)) \subset f(x_0) + \overline{U} \subset f(x_0) + O$. Then $W = [x_0, x_2) \cap X$ is the required neighborhood of x_0 in X_g^+ with $\bar{f}(W) \subset f(x_0) + \overline{U} \subset f(x_0) + O$. This completes the proof of the second item of Theorem 2.1.

1'. Now we shall finish the proof of the item (1), establishing the continuity property of the function $r : \bar{X} \to P_2(\beta A_d)$. Let $g \ni X \setminus A$ be any GO-topology on $(X, \leq)$. Let $Y = C^*(\beta A_g)$ be the dual Banach space endowed with the weak-star topology and let $u : l_\infty(A, Y) \to l_\infty(X, Y)$ be the linear $\overline{\mathrm{conv}}$-extender constructed in the item (2). It has the property that $u(C_\infty(A_g, Y)) \subset C_\infty(X_g, Y)$.

Consider the Dirac embedding $\delta : A_g \to P(\beta A_g) \subset C^*(\beta A_g) = Y$ and its continuous extension $\bar{\delta} = u(\delta) : X_g \to Y$ given by the extender u. The definition of u implies that $\bar{\delta}$ is equal to the composition $P(\beta i) \circ r$ of the maps $r : X \to P_2(\beta A_d)$ and $P(\beta i) : P(\beta A_d) \to P(\beta A_g)$. Consequently, this composition $P(\beta i) \circ r : X_g \to P_2(\beta A_g)$ is continuous.

3. The third item follows from the second one and the fact that the norm-topology of a dual Banach space Y^* is admissible for the weak-star topology on Y^*.

4. The fourth item can be derived from the third one by the argument of the proof of Theorem 1.6.

12. PROOF OF THEOREM 3.2

Assume that for a subspace A of a Tychonov space X and a linear topological space Y there is a linear $\overline{\mathrm{conv}}$-extender $u : C(A, Y) \to C(X, Y)$. Assuming that Y is not countably semireflexive, we shall prove that the subset A is strong Choquet in X. We should describe a winning strategy for the player II in the game $G_r(A, X)$.

Since the space Y is not countably semireflexive, there is a decreasing sequence $(K_n)_{n=1}^\infty$ of non-empty closed bounded convex subsets of Y with $\bigcap_{n=1}^\infty K_n = \emptyset$. Let $y_0 = 0$ and $y_n \in K_n$ for every $n \in \mathbb{N}$. For every $m \in \mathbb{N}$ consider the finite-dimensional linear subspace L_m of Y spanned by the vectors $y_0, \dots, y_m$. Since the union $L = \bigcup_{m \in \omega} L_m$ has countable pseudocharacter, we can select a decreasing sequence $(O_n)_{n\in\omega}$ of open neighborhoods of zero in Y such that $L\cap\bigcap_{n\in\omega} O_n = \{0\}$. Since u is a linear $\overline{\mathrm{conv}}$-extender, for every continuous function $f : A \to L_n$ the extension $\bar f = u(f)$ has range $\bar f(X) \subset L_n$.

Now we (somewhat informally) describe a winning strategy of the player II in the game $G_r(A, X)$. The point is that at her nth inning the player II chooses a neighborhood $V_n \subset U_n$ of the point a_n given by the Ist player with help of a continuous function $\lambda_n : A \to [0, 1]$ such that

$$a_n \in V_n \cap A \subset \lambda_n^{-1}(1) \subset \lambda_n^{-1}(0, 1] \subset U_n \subset V_{n-1} \tag{2}$$

and keeps the functions λ_i from the previous innings in her memory.

Therefore the players I and II will consecutively choose the pairs

$$\begin{tikzcd}
\text{I:} & (U_1, a_1) \arrow[dr] & & (U_2, a_2) \arrow[dr] & & (U_3, a_3) \arrow[dr] & & \cdots \\
\text{II:} & & (V_1, \lambda_1) \arrow[ur] & & (V_2, \lambda_2) \arrow[ur] & & (V_3, \lambda_3) &
\end{tikzcd}$$

so that the condition (2) is satisfied.

Now we explain how to select the function λ_n and the neighborhood V_n at the nth inning. After receiving the point $a_n \in A$ and the neighborhood $U_n \subset X$ of a_n from the Ist player, the player II uses the Tychonov property of A to find a continuous function $\lambda_n : A \to [0, 1]$ such that $\lambda_n(A \setminus U_n) \subset \{0\}$ and a_n lies in the interior W_n of $\lambda_n^{-1}(1)$ in A. Now for every $k \le n$ consider the continuous function $f_k = \sum_{i=1}^{k} \lambda_i \cdot (y_i - y_{i-1}) : A \to L_n$ and its extension $\bar f_k = u(f_k) : X \to L_k \subset Y$ given by the linear $\overline{\mathrm{conv}}$-extender u. It follows from $a_n \in W_n \subset V_i \cap A \subset \lambda_i^{-1}(1)$, $i < n$, that $\bar f_k(a_n) = f_k(a_n) \in f_k(W_n) \subset \{y_k\}$ for all $k \le n$. Using the continuity of the functions $\bar f_k$ at a_n, choose a neighborhood $V_n \subset U_n$ of a_n with $V_n \cap A \subset W_n$ such that $\bar f_k(V_n) \subset y_k + O_n$ for all $k \le n$. Finally the player II presents the set V_n to the player I as her n-th move.

We claim that the player II wins the game $G_r(A, X)$ if she chooses the sets V_n according to the strategy described above. Assuming the converse, we would get that player I wins, which means that $\emptyset \ne \bigcap_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} V_n \subset X \setminus A$. It follows from the last inclusion that the formula $f_\infty(x) = \sum_{i=1}^{\infty} \lambda_i(x)\cdot(y_i - y_{i-1})$ determines a well-defined continuous function $f_\infty : A \to Y$. Consider its extension $\bar f_\infty = u(f_\infty)$ and pick a point $c \in \bigcap_{n=1}^{\infty} V_n$.

We claim that $\bar f_k(c) = y_k$ for all $k \in \mathbb{N}$. Indeed, for every $n \ge k$, the choice of the set V_n guarantees that $\bar f_k(c) \in \bar f_k(V_n) \subset y_k + O_n$ and thus $\bar f_k(c) - y_k \in L_k \cap \bigcap_{n\ge k} O_n = \{0\}$.

For every $n \in \mathbb{N}$ consider the function

$$g_n = f_\infty - f_n + y_n = y_n + \sum_{i>n} \lambda_i \cdot (y_i - y_{i-1}) = (1 - \lambda_{n+1}) \cdot y_n + \sum_{i>n} (\lambda_i - \lambda_{i+1}) \cdot y_i$$

and observe that $g_n(A) \subset \mathrm{conv}\{y_i\}_{i\ge n} \subset K_n$. Since u is a linear $\overline{\mathrm{conv}}$-extender, we get $u(g_n)(c) \in K_n$ and by the linearity of u,

$$K_n \ni u(g_n)(c) = \bar f_\infty(c) - \bar f_n(c) + y_n = \bar f_\infty(c) + 0,$$

which implies that the intersection $\bigcap_{n=1}^{\infty} K_n$ contains the point $\bar f_\infty(c)$ and thus is not empty. This contradiction completes the proof of the Theorem 3.2.

13. Proof of Theorem 6.1

Assume that Y_0 is a subspace of a pospace Y, and A is a Tychonov subspace of a topological space X.

Assuming that there is a monotone extender $u : C(A, Y_0) \to C(X, Y)$, we should prove that A is strong Choquet in X or else for each ω-increasing ray $\gamma : [0, \infty) \to Y_0$ the set $\gamma(\omega)$ is almost upper bounded in Y. Assume that the latter condition does not hold, i.e., there is an ω-increasing ray $\gamma : [0, \infty) \to Y_0$ such that the set $\gamma(\omega)$ is not almost upper bounded in Y. The latter means that for some G_δ-sets $G_n \subset Y$ with $\gamma(n) \in G_n$, $n \in \omega$, the intersection $\bigcap_{n\in\omega} {\uparrow}G_n$ is empty. For every $n \in \omega$ select a decreasing sequence $(O_m(y_n))_{m\ge n}$ of open neighborhoods of the point $y_n = \gamma(n)$ such that $\bigcap_{m\ge n} O_m(y_n) \subset G_n$.

Now we modify the winning strategy constructed in the proof of Theorem 3.2 and describe a winning strategy of the player II in the game $G_r(A, X)$. The key idea is the same: in her nth inning the player II chooses a neighborhood $V_n \subset U_n$ of the point a_n with help of a continuous function $\lambda_n : A \to [0, 1]$ such that

$$a_n \in V_n \cap A \subset \lambda_n^{-1}(1) \subset \lambda_n^{-1}(0, 1] \subset U_n \subset V_{n-1} \tag{3}$$

and keeps the functions λ_i from the previous innings in her memory.

The choice of the function $\lambda_n : A \to [0, 1]$ is the same as in the proof of Theorem 3.2 while the choice of the neighborhood $V_n \subset \lambda^{-1}(1)$ of a_n is a bit different. For every $0 \le k \le n$ consider the continuous function $s_k = \sum_{i=1}^k \lambda_i : A \to [0, \infty)$ (for $k = 0$, we put $s_0 \equiv 0$). Then the function $f_k = \gamma \circ s_k : A \to Y_0$ is continuous and so is its extension $\bar f_k = u(f_k) : X \to Y$ given by the monotone extender u. It follows from $a_n \in W_n \subset V_i \cap A \subset \lambda_i^{-1}(1)$, $i < n$, that $\bar f_k(a_n) = f_k(a_n) = \gamma(k) = y_k$ for all $k \le n$. Using the continuity of the functions $\bar f_k$ at a_n, choose a neighborhood $V_n \subset U_n$ of a_n with $V_n \cap A \subset W_n$ such that $\bar f_k(V_n) \subset O_n(y_k)$ for all $k \le n$. Finally the player II presents the set V_n as her n-th move in the game $G_r(A, X)$.

We claim that the player II wins the game $G_r(A, X)$ if she chooses the sets V_n according to the strategy described above. Assuming the converse, we would get that player I wins, which means that $\emptyset \ne \bigcap_{n=1}^\infty U_n = \bigcap_{n=1}^\infty V_n \subset X \setminus A$. It follows from the last inclusion that the formula $s_\infty(a) = \sum_{i=1}^\infty \lambda_i(a)$, $a \in A$, determines a well-defined continuous function $s_\infty : A \to [0, \infty)$. Then the function $f_\infty = \gamma \circ s_\infty : A \to Y_0$ is also continuous. Consider its extension $\bar f_\infty = u(f_\infty) : X \to Y$ and pick a point $c \in \bigcap_{n=1}^\infty V_n$.

We claim that $\bar f_k(c) \in G_k$ for all $k \in \mathbb{N}$. Indeed, for every $n \ge k$, the choice of the set V_n guarantees that $\bar f_k(c) \in \bar f_k(V_n) \subset O_n(y_k)$ and thus $\bar f_k(c) \in \bigcap_{n\ge k} O_n(y_k) \subset G_k$.

The ω-increasing property of the ray γ implies that $f_\infty \ge f_k$ for all $k \ge 0$. Now the monotonicity of the extender u guarantees that $\bar f_\infty(c) \ge \bar f_k(c) \in G_k$ and thus $\bar f_\infty(c) \in \bigcap_{n\in\omega} {\uparrow}G_n$, which contradicts the choice of the G_δ-sets G_n. This contradiction completes the proof of the Theorem 6.1.

14. Proof of Theorem 3.3

1,2. The first two items easily follow from the definitions.

3. We need to prove that a subset A of a topological space X is strong Choquet in X provided A is strong Choquet as a topological space.

The latter means that the player II has a winning strategy in the game $G_s(A, A)$. We shall prove that this winning strategy induces a winning strategy in the game

$G_r(A,X)$ and even in a more difficult (for the player II) game $G'_r(A,X)$ which differs from $G_r(A,X)$ by the estimation of the result of the game. In the game $G'_r(A,X)$ the player II is declared the winner if the intersection $\bigcap_{n\in\omega} V_n$ meets the set A. Otherwise the player I wins the game $G'_r(A,X)$. It is clear that of the player II wins the game $G'_r(A,X)$, then she wins also the game $G_r(A,X)$.

To win the game $G'_r(A,X)$ the player II simultaneously plays the game $G_s(A,A)$ for itself and for the player I and transforms her moves in the game $G_s(A,A)$ suggested by the winning strategy into the moves in the game $G'_r(A,X)$.

Namely, after receiving the nth move (U_n,a_n) of the player II in the nth inning, the player I declares that $(U_n\cap A,a_n)$ in the nth move of the player I in the auxiliary game $G_s(A,A)$. Then the winning strategy in the game $G_s(A,A)$ instructs the player II to select a neighborhood $V_n\subset U_n\cap A$ of the point a_n in A. The player II enlarges the set V_n to an open subset $\tilde V_n\subset U_{n-1}$ in X such that $\tilde V_n\cap A=V_n$ and suggests the set $\tilde V_n$ as her move in the n-th inning of the game $G'_r(A,X)$.

In such a way the players I and II choose sequences (U_n,a_n), $(U_n\cap A,a_n)$, (V_n) and $(\tilde V_n)$. Since the sets V_n, $n\in\mathbb N$, are chosen according to the winning strategy of the player II in the game $G_s(A,A)$, we get $\bigcap_{n\in\mathbb N} V_n\ne\emptyset$. Then $A\cap\bigcap_{n\in\mathbb N}\tilde V_n=\bigcap_{n\in\mathbb N} V_n\ne\emptyset$, so we conclude that the player II wins also the game $G'_r(A,X)$.

4. Assume that a topological space X is strong Choquet at A and A is strong Choquet in X. We should prove that A is strong Choquet. On the language of strategies this means that given winning strategies of the player II in the games $G_s(A,X)$ and $G_r(A,X)$ we should describe a winning strategy for the player II in the game $G_s(A,A)$.

Repeating the argument from the preceding item, we can prove that the player II has a winning strategy in the game $G_s(A,A)$ if and only if she has a winning strategy in the game $G'_r(A,X)$ describes above. So it suffices to describe a winning strategy for the player II in the game $G'_r(A,X)$.

Fix winning strategies of the player II in the games $G_s(A,X)$ and $G_r(A,X)$. To win the game $G'_r(A,X)$, the player II plays simultaneously two auxiliary games $G_s(A,X)$ and $G_r(A,X)$ as follows. In the n-th inning of the game $G'_r(A,X)$ she receives from the Ist player a point $a_n\in A$ and a neighborhood $U_n\subset X$ of a_n and declares that (U_n,a_n) is the nth move of the player I in the auxiliary game $G_s(X,A)$. The winning strategy of the player II in the game $G_s(A,X)$ instructs her how to make the n-th move by choosing a neighborhood $W_n\subset U_n$ of a_n.

Then the player II declares that (W_n,a_n) is the n-th move of the Ist player in the game $G_r(A,X)$ and selects a neighborhood $V_n\subset W_n$ of a_n according to the winning strategy in the game $G_r(A,X)$.

The neighborhood V_n is the n-th move of the second player in the game $G'_r(A,X)$. Let us show that if the player II plays according to the described strategy, then she will win the game $G'_r(A,X)$. Playing three games simultaneously, the players I and II construct the sequences (U_n,a_n), (W_n), and (V_n). The choice of the sets W_n according to the winning strategy in the game $G_s(A,X)$ guarantees that $\bigcap_{n=1}^\infty W_n$ is not empty. Taking into account that $W_n\subset U_n\subset V_n\subset W_{n-1}$ for all n, we conclude that the intersection $\bigcap_{n=1}^\infty V_n$ is not empty too. Since the player II won the game $G_r(A,X)$, the intersection $\bigcap_{n=1}^\infty V_n$, being non-empty, must meet the set A. This means that the player II have won the game $G'_r(A,X)$ too.

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