CLASSIFYING HOMEOMORPHISM GROUPS OF INFINITE GRAPHS

TARAS BANAKH, KOTARO MINE, AND KATSURO SAKAI

Abstract. In this paper, we classify topologically the homeomorphism groups $\mathcal{H}(\Gamma)$ of infinite graphs Γ with respect to the compact-open and the Whitney topologies.

Contents

Introduction

The aim of this paper is to classify the homeomorphism groups $\mathcal{H}(\Gamma)$ of infinite graphs Γ up to a homeomorphism with respect to the compact-open and the Whitney topologies.

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By a *graph*, we understand a 1-dimensional simplicial complex Γ with the Whitehead (or CW) topology. A point $v \in \Gamma$ is called a *topological vertex* of a graph Γ if v has no neighborhood homeomorphic to an open subset of the real line, that is, v is a branch point or an end-point or an isolated point. It follows from the definition of a graph that the set $\Gamma^{(0)}$ of topological vertices is closed and discrete in Γ and the complement $\Gamma \setminus \Gamma^{(0)}$ can be written as the disjoint union $\bigcup_{\alpha\in A} E_\alpha$ of connected components, homeomorphic to the open interval $(0,1)$ or the circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The closure $\overline{E}_\alpha$ of E_α in Γ is compact if and only if $\overline{E}_\alpha$ is homeomorphic to $\mathbb{I} = [0,1]$ or $\mathbb{T}$. In case $\overline{E}_\alpha = E_\alpha \approx \mathbb{T}$, we call E_α an *isolated circle* of the graph Γ. On the other hand, $\overline{E}_\alpha$ is non-compact if and only if $\overline{E}_\alpha$ is homeomorphic to $\mathbb{R}$ or $\mathbb{R}_+ = [0,\infty)$.

To each graph Γ we can assign four cardinal numbers:

- p_Γ — the number of isolated points of Γ,
- o_Γ — the number of isolated circles of Γ,
- κ_Γ — the number of components of $\Gamma \setminus \Gamma^{(0)}$ with compact closure in Γ,
- ν_Γ — the number of components of $\Gamma \setminus \Gamma^{(0)}$ with non-compact closure in Γ,

and put

$$e_\Gamma = \kappa_\Gamma + \nu_\Gamma \cdot \aleph_0.$$

As we shall see later, those cardinal numbers compose one of two ingredients determining the topological structure of the homeomorphism group $\mathcal{H}(\Gamma)$ of Γ and some its subgroups, in particular, the subgroup $\mathcal{H}_+(\Gamma)$ of orientation-preserving homeomorphisms.

A homeomorphism $h : \Gamma \to \Gamma$ of a graph Γ is called *orientation-preserving* if

- $h|\Gamma^{(0)} = \mathrm{id}$;
- for each connected component E_α of $\Gamma \setminus \Gamma^{(0)}$, the restriction $h|\overline{E}_\alpha$ is an orientation-preserving homeomorphism of $\overline{E}_\alpha$.

By the *support* of a homeomorphism $h : \Gamma \to \Gamma$, we mean the set

$$\mathrm{supp}(h) = \mathrm{cl}\{x \in \Gamma : h(x) \neq x\}.$$

Besides the group $\mathcal{H}(\Gamma)$, we shall be interested in the following subgroups of $\mathcal{H}(\Gamma)$:

- $\mathcal{H}_+(\Gamma)$ — the subgroup of orientation-preserving homeomorphisms of Γ;
- $\mathcal{H}_c(\Gamma)$ — the subgroup of homeomorphisms with compact support;
- $\mathcal{H}^{\mathrm{PL}}(\Gamma)$ — the subgroup of PL homeomorphisms of Γ.

Intersecting those subgroups, we obtains other four subgroups of $\mathcal{H}(\Gamma)$:

$$\mathcal{H}_0(\Gamma) = \mathcal{H}_+(\Gamma) \cap \mathcal{H}_c(\Gamma), \ \ \mathcal{H}_+^{\mathrm{PL}}(\Gamma) = \mathcal{H}_+(\Gamma) \cap \mathcal{H}^{\mathrm{PL}}(\Gamma)$$
$$\mathcal{H}_c^{\mathrm{PL}}(\Gamma) = \mathcal{H}_c(\Gamma) \cap \mathcal{H}^{\mathrm{PL}}(\Gamma) \ \text{ and } \ \mathcal{H}_0^{\mathrm{PL}}(\Gamma) = \mathcal{H}_0(\Gamma) \cap \mathcal{H}^{\mathrm{PL}}(\Gamma).$$

Those subgroups are important because of the following their property:

Proposition 1. *For an arbitrary graph Γ,*

(i) *the subgroup $\mathcal{H}_+(\Gamma)$ (resp. $\mathcal{H}_+^{\mathrm{PL}}(\Gamma)$) coincides with the connected component of the topological group $\mathcal{H}(\Gamma)$ (resp. $\mathcal{H}^{\mathrm{PL}}(\Gamma)$) endowed with the compact-open topology;*

(ii) *the subgroup $\mathcal{H}_0(\Gamma)$ (resp. $\mathcal{H}_0^{\mathrm{PL}}(\Gamma)$) coincides with the connected component of the topological group $\mathcal{H}(\Gamma)$ (resp. $\mathcal{H}^{\mathrm{PL}}(\Gamma)$) endowed with the Whitney topology.*

Consequently, $\mathcal{H}_+(\Gamma)$ *and* $\mathcal{H}_0(\Gamma)$ *(resp.* $\mathcal{H}_+^{\mathrm{PL}}(\Gamma)$ *and* $\mathcal{H}_0^{\mathrm{PL}}(\Gamma)$*) are normal subgroups of* $\mathcal{H}(\Gamma)$ *(resp.* $\mathcal{H}^{\mathrm{PL}}(\Gamma)$*).*

We recall that the *Whitney* (or else *graph*) topology on the space $C(X,Y)$ of continuous maps from a topological space X to a topological space Y is generated by the base consisting of the sets $\Gamma_U = \{f \in C(X,Y) : \Gamma_f \subset U\}$, where U runs over open subsets of $X \times Y$ and $\Gamma_f = \{(x, f(x)) : x \in X\}$ stands for the graph of a map $f \in C(X,Y)$.

It is known that for each paracompact space X, the homeomorphism group $\mathcal{H}(X) \subset C(X,X)$ of X endowed with the Whitney topology is a topological group (see Proposition 4.14 of [4]). In particular, the homeomorphism group $\mathcal{H}(\Gamma)$ of any graph Γ is a topological group with respect to the Whitney topology.

In contrast, for a paracompact (even locally compact) space X, the inverse operation on the homeomorphism group $\mathcal{H}(X)$ needs not be continuous with respect to the compact-open topology, see [7]. However, in case of a graph Γ, the homeomorphism group $\mathcal{H}(\Gamma)$ is a topological group with respect to the compact-open topology, see Proposition 15.

As we shall see in Proposition 17, the normal subgroup $\mathcal{H}_+(\Gamma)$ is open in the homeomorphism group $\mathcal{H}(\Gamma)$ endowed with the Whitney topology. Consequently, $\mathcal{H}(\Gamma)$ is homeomorphic to $\mathcal{H}_+(\Gamma) \times (\mathcal{H}(\Gamma)/\mathcal{H}_+(\Gamma))$. Because of that, it is important to study the structure of the quotient group $\mathcal{H}(\Gamma)/\mathcal{H}_+(\Gamma)$.

In Section 13, we shall show that the quotient group $\mathcal{H}(\Gamma)/\mathcal{H}_+(\Gamma)$ is isomorphic to the automorphism group $\mathrm{Aut}(\vec{\Gamma})$ of a certain directed graph (briefly, digraph) $\vec{\Gamma}$ associated to the graph Γ so that the geometric realization of $\vec{\Gamma}$ is homeomorphic to Γ. The automorphism group $\mathrm{Aut}(\vec{\Gamma})$ can be realized as a closed subgroup of $\mathcal{H}^{\mathrm{PL}}(\Gamma)$ which implies that $\mathcal{H}(\Gamma) = \mathcal{H}_+(\Gamma) \ltimes \mathrm{Aut}(\vec{\Gamma})$ is a semi-direct product of $\mathcal{H}_+(\Gamma)$ and $\mathrm{Aut}(\vec{\Gamma})$. Also $\mathcal{H}^{\mathrm{PL}}(\Gamma)$ is a semi-direct product of $\mathcal{H}_+^{\mathrm{PL}}(\Gamma)$ and $\mathrm{Aut}(\vec{\Gamma})$. By $\mathrm{Aut}_c(\vec{\Gamma})$, we denote the subgroup of $\mathrm{Aut}(\vec{\Gamma})$ consisting of automorphisms with compact support.

1. Main Results

The main results of this paper are Theorems 1 and 2 below, which treat the compact-open and the Whitney topologies of the homeomorphism groups of graphs. These theorems have similar form but recognining the Whitney topology on homeomorphism groups turned out to be much difficult comparing to the compact-open topology.

Below, by l_2 we denote the separable Hilbert space and by l_2^f the linear hull of the orthonormal basis in l_2.

For a group H with the neutral element 0 and a cardinal number κ, the small σ-product of κ many copies of H is denoted by

$$\boxdot\!\!\prod^{\kappa} H = \{(g_\alpha)_{\alpha\in\kappa} \in H^\kappa : |\{\alpha \in \kappa : g_\alpha \neq 0\}| < \aleph_0\} \subset H^\kappa.$$

To uniformize notation, we shall write $\prod^\kappa H$ instead of H^κ.

Theorem 1. *Let* Γ *be a graph and* $\vec{\Gamma}$ *be the associated digraph. For the groups* $\mathcal{H}(\Gamma)$ *and* $\mathrm{Aut}(\vec{\Gamma})$ *endowed with the compact-open topology, there is a homeomorphism*

$$\Phi_c : \mathcal{H}(\Gamma) \to \mathrm{Aut}(\vec{\Gamma}) \times \prod^{o_\Gamma} \mathbb{T} \times \prod^{e_\Gamma} l_2$$

such that

(1) $\Phi_{\mathrm{c}}(\mathcal{H}^{\mathrm{PL}}(\Gamma)) = \mathrm{Aut}(\vec{\Gamma}) \times \prod^{or}\mathbb{T} \times \prod^{er} l_2^f$;
(2) $\Phi_{\mathrm{c}}(\mathcal{H}_+(\Gamma)) = \{\mathrm{id}\} \times \prod^{or}\mathbb{T} \times \prod^{er} l_2$;
(3) $\Phi_{\mathrm{c}}(\mathcal{H}_c(\Gamma)) = \mathrm{Aut}_c(\vec{\Gamma}) \times \dot{\prod}^{or}\mathbb{T} \times \dot{\prod}^{er} l_2^f$;
(4) $\Phi_{\mathrm{c}}(\mathcal{H}_0(\Gamma)) = \{\mathrm{id}\} \times \dot{\prod}^{or}\mathbb{T} \times \dot{\prod}^{er} l_2$.

A similar result holds for the Whitney topology on $\mathcal{H}(\Gamma)$. However instead of the product topology on $\prod^{or}\mathbb{T}$ and $\prod^{er} l_2$ we should consider the box-topology. For a topological group H and a cardinal κ by $\square^\kappa H$ we denote the power H^κ endowed with the box-topology generated by the base consisting of the boxes $\prod_{\alpha\in\kappa} U_\alpha$ where U_α, $\alpha \in H$, are open sets in H. By $\boxdot^\kappa H$ we denote the set $\dot{\prod}^\kappa H$ endowed with the box-topology. If the cardinal κ is at most countable then $\boxdot^\kappa H$ coincides with the direct sum $\oplus^\kappa H$ of κ many copies of H in the category of topological groups.

Observe that the box-topology coincides with the Whitney topology on $\prod^\kappa H$ seen as the space of all functions from the discrete space κ to H. Because of that sometimes we shall refer to the box-topology on $\prod^\kappa H$ as the Whitney topology.

If the homeomoprhism group $\mathcal{H}(\Gamma)$ is endowed with the Whitney topology, then $\mathcal{H}_+(\Gamma)$ is an open normal subgroup in $\mathcal{H}(\Gamma)$ and hence the quotient group $\mathcal{H}(\Gamma)/\mathcal{H}_+(\Gamma) = \mathrm{Aut}(\vec{\Gamma})$ is discrete.

The following theorem resembles Theorem 1 (but has much more difficult proof) and is our principal tool in recognining the Whitney topology on the homeomorphism groups of graphs.

Theorem 2. *Let Γ be a graph and $\vec{\Gamma}$ be the associated digraph. For the groups $\mathcal{H}(\Gamma)$ and $\mathrm{Aut}(\vec{\Gamma})$ endowed with the Whitney topology there is a homeomorphism*

$$\Phi_{\mathrm{w}} : \mathcal{H}(\Gamma) \to \mathrm{Aut}(\vec{\Gamma}) \times \square^{or}\mathbb{T} \times \square^{er} l_2$$

such that

(1) $\Phi_{\mathrm{w}}(\mathcal{H}^{\mathrm{PL}}(\Gamma)) = \mathrm{Aut}(\vec{\Gamma}) \times \square^{or}\mathbb{T} \times \square^{er} l_2^f$;
(2) $\Phi_{\mathrm{w}}(\mathcal{H}_+(\Gamma)) = \{\mathrm{id}\} \times \square^{or}\mathbb{T} \times \square^{er} l_2$;
(3) $\Phi_{\mathrm{w}}(\mathcal{H}_c(\Gamma)) = \mathrm{Aut}_c(\vec{\Gamma}) \times \boxdot^{or}\mathbb{T} \oplus \boxdot^{er} l_2^f$;
(4) $\Phi_{\mathrm{w}}(\mathcal{H}_0(\Gamma)) = \{\mathrm{id}\} \times \boxdot^{or}\mathbb{T} \times \boxdot^{er} l_2$.

As we shall see from the proofs of Theorems 1 and 2, the maps Φ_{c} and Φ_{w} are distinct. This suggests the following open

Problem 1. *Let Γ be a graph and $\vec{\Gamma}$ be the associated digraph. Is there is bijective map $\Phi : \mathcal{H}(\Gamma) \to \mathrm{Aut}(\vec{\Gamma}) \times \prod^{or}\mathbb{T} \times \prod^{er} l_2$, which is a homeomorphism in both compact-open and Whitney topologies on $\mathcal{H}(\Gamma)$ and $\mathrm{Aut}(\vec{\Gamma}) \times \prod^{or}\mathbb{T} \times \prod^{er} l_2$?*

Theorems 1 and 2 imply the following topological classification.

Corollary 1. *For any graph Γ the triple $\big(\mathcal{H}_+(\Gamma), \mathcal{H}_+^{\mathrm{PL}}(\Gamma), \mathcal{H}_0(\Gamma)\big)$ of the homeomorphism groups endowed with the compact-open and Whitney topology is homeomorphic to*

$$\big(\prod^{or}\mathbb{T} \times \prod^{er} l_2, \prod^{or}\mathbb{T} \times \prod^{er} l_2^f, \dot{\prod}^{or}\mathbb{T} \times \dot{\prod}^{er} l_2\big)$$

and

$$\big(\square^{or}\mathbb{T} \times \square^{er} l_2, \square^{or}\mathbb{T} \times \square^{er} l_2^f, \boxdot^{or}\mathbb{T} \times \boxdot^{er} l_2\big),$$

respectively.

A similar classification holds also for the triple $(\mathcal{H}(\Gamma), \mathcal{H}^{\mathrm{PL}}(\Gamma), \mathcal{H}_0(\Gamma))$.

Theorem 3. *Let Γ be a non-compact graph. The triple $\big(\mathcal{H}(\Gamma), \mathcal{H}^{\mathrm{PL}}(\Gamma), \mathcal{H}_0(\Gamma)\big)$ of homeomorphism groups endowed with the Whitney topology is homeomorphic to*

(i) $(2^{p_\Gamma+e_\Gamma} \times \square^{o_\Gamma}\mathbb{T} \times \square^{e_\Gamma} l_2, 2^{p_\Gamma+e_\Gamma} \times \square^{o_\Gamma}\mathbb{T} \times \square^{e_\Gamma} l_2, 1 \times \boxdot^{o_\Gamma}\mathbb{T} \times \boxdot^{e_\Gamma} l_2)$*;*

(ii) $(\square^{o_\Gamma}\mathbb{T} \times \square^{e_\Gamma} l_2, \square^{o_\Gamma}\mathbb{T} \times \square^{e_\Gamma} l_2^f, \square^{o_\Gamma}\mathbb{T} \times \square^{e_\Gamma} l_2)$ *if* $2^{p_\Gamma} \le 2^{e_\Gamma}$.

Here we identify cardinals with the discrete spaces of ordinals of smaller cardinality. Under such a convention, the number $1 = \{0\}$ is a singleton.

It is known that the pair $(\dot{\prod}^\omega l_2, \dot{\prod}^\omega l_2^f)$ is homeomorphic to $(l_2 \times l_2^f, l_2^f \times l_2^f)$, see [24, Theorem 1.2]. On the other hand, $(\boxdot^\omega l_2, \boxdot^\omega l_2^f)$ is homeomorphic to $(l_2 \times \mathbb{R}^\infty, l_2^f \times \mathbb{R}^\infty)$, see [2]. Here $\mathbb{R}^\infty$ stands for the space l_2^f endowed with the strongest linear topology. This topology coincides with the direct limit topology with respect to the tower

$$\mathbb{R} \subset \mathbb{R}^2 \subset \mathbb{R}^n \subset \cdots$$

where $\mathbb{R}^n$ is identified with the subspace $\{(x_i)_{i\in\omega} \in l_2^f : \forall i \ge n,\ (x_i = 0)\} \subset l_2^f$.

Corollary 2. *If Γ is a non-compact separable graph without isolated circles, then the pair $(\mathcal{H}_0(\Gamma), \mathcal{H}_0^{\mathrm{PL}}(\Gamma))$ of the homeomorphism groups endowed with the compact-open or Whitney topology is homeomorphic to $(l_2 \times l_2^f, l_2^f \times l_2^f)$ or $(l_2 \times \mathbb{R}^\infty, l_2^f \times \mathbb{R}^\infty)$, respectively.*

Remark 1. Some particular cases of the results about the compact-open and Whitney topologies on homeomorphism groups of graphs obtained in this paper had already appeared in the literature. In particular, due to Anderson [1], we known that the homeomorphism group $\mathcal{H}(\Gamma)$ of a compact graph Γ, is an l_2-manifold. On the other hand, endowed with the Whitney topology, the homeomorphism group $\mathcal{H}_0(\mathbb{R})$ of the real line is homeomorphic to $l_2 \times \mathbb{R}^\infty \cong \boxdot^\omega l_2$, see [2].

2. Recovering the cardinals e_Γ, o_Γ and $2^{p_\Gamma+e_\Gamma}$

In this section we show how to recover the values of the cardinals e_Γ, o_Γ and $2^{p_\Gamma+e_\Gamma}$ from the topological structure of the homeomorphism groups of Γ. For a cardinal number κ let

$$\lfloor \kappa \rfloor = \begin{cases} 1 & \text{if } 1 \le \kappa < \aleph_0, \\ \kappa & \text{otherwise,} \end{cases} \quad \text{and} \quad \lceil \kappa \rceil = \begin{cases} \aleph_0 & \text{if } 1 \le \kappa < \aleph_0, \\ \kappa & \text{otherwise.} \end{cases}$$

Proposition 2. *For a graph Γ, the following hold:*

(i) *The cardinal $\lceil e_\Gamma \rceil$ is equal to the weight of the spaces $\mathcal{H}_+(\Gamma)$, $\mathcal{H}_+^{\mathrm{PL}}(\Gamma)$, $\mathcal{H}_0(\Gamma)$, and $\mathcal{H}_0^{\mathrm{PL}}(\Gamma)$ endowed with the compact-open topology.*

(ii) *The cardinal $\lceil e_\Gamma \rceil$ is equal to the density of the spaces $\mathcal{H}_0(\Gamma)$ and $\mathcal{H}_0^{\mathrm{PL}}(\Gamma)$ endowed with the Whitney topology.*

(iii) *If Γ is not compact, then $2^{p_\Gamma+e_\Gamma}$ is equal to the number of connected components of the spaces $\mathcal{H}(\Gamma)$ and $\mathcal{H}^{\mathrm{PL}}(\Gamma)$ endowed with the Whitney topology.*

Proof. The three items of this proposition follow from the inequality $o_\Gamma \le e_\Gamma$ and the topological equivalences

$$\big(\mathcal{H}_+(\Gamma), \mathcal{H}_0^{\mathrm{PL}}(\Gamma)\big) \overset{C}{\approx} (\prod^{o_\Gamma}\mathbb{T} \times \prod^{e_\Gamma} l_2,\ \dot{\prod}^{o_\Gamma}\mathbb{T} \times \dot{\prod}^{e_\Gamma} l_2^f),$$

$$\big(\mathcal{H}_0(\Gamma), \mathcal{H}_0^{\mathrm{PL}}(\Gamma)\big) \overset{W}{\approx} (\boxdot^{o_\Gamma}\mathbb{T} \times \boxdot^{e_\Gamma} l_2, \boxdot^{o_\Gamma}\mathbb{T} \times \boxdot^{e_\Gamma} l_2^f) \ \text{ and}$$

$$\big(\mathcal{H}(\Gamma), \mathcal{H}^{\mathrm{PL}}(\Gamma)\big) \overset{W}{\approx} (2^{p_\Gamma+e_\Gamma} \times \square^{o_\Gamma}\mathbb{T} \times \square^{e_\Gamma} l_2, 2^{p_\Gamma+e_\Gamma} \times \square^{o_\Gamma}\mathbb{T} \times \square^{e_\Gamma} l_2^f)$$

established in Theorems 1–3, respectively. □

Recovering the cardinal o_Γ is more delicate and requires some algebraic topology. We recall that the *Betti number* $b_1(X)$ of a topological space X equals the vector space dimension $b_1(X) = \dim_{\mathbb{Q}} H_1(X;\mathbb{Q})$ of the singular homology group of X with rational coefficients. Replacing the first homology group $H_1(X;\mathbb{Q})$ with the first Čech cohomology group $\check{H}^1(X;\mathbb{Q})$ we obtain the definition of the *co-Betti number* $b^1(X) = \dim_{\mathbb{Q}} \check{H}^1(X;\mathbb{Q})$.

We shall also need a compact modification of the co-Betti number suggested by Robert Cauty in a personal communication with the first author. The *compact co-Betti number* $cb^1(X)$ of a topologial space X is defined as

$$cb^1(X) = \sup_{f:K\to X} \dim_{\mathbb{Q}} f^*(\check{H}^1(X;\mathbb{Q}))$$

where the supremum is taken over all continuous maps $f : K \to X$ from a compact Hausdorff space K to X and $f^* : \check{H}^1(X;\mathbb{Q}) \to \check{H}^1(K;\mathbb{Q})$ is the induced linear operator between the first cohomology groups. It is easy to deduce from the definition that the compact co-Betti number is a homotopic invariant of a topological space, which means that $cb^1(X) = cb^1(Y)$ for any homotopically equivalent spaces X, Y.

Proposition 3. *For every cardinal κ, the following equalities hold:*

$$b_1(\boxdot^\kappa \mathbb{T}) = b^1(\textstyle\prod^\kappa \mathbb{T}) = cb^1(\dot{\prod}^\kappa \mathbb{T}) = \kappa.$$

This proposition will be proved in Section 16. Now we apply it to recover the value of o_Γ from the topological structure of homeomorphism groups of a graph Γ.

Proposition 4. *The number o_Γ of isolated circles of a graph Γ is equal to:*

(i) *the Betti number of the homeomorphism groups $\mathcal{H}_0(\Gamma)$ and $\mathcal{H}_0^{\mathrm{PL}}(\Gamma)$ endowed with the Whitney topology;*
(ii) *the co-Betti number of the homeomorphism groups $\mathcal{H}_+(\Gamma)$ and $\mathcal{H}_+^{\mathrm{PL}}(\Gamma)$ endowed with the compact-open topology;*
(iii) *the compact co-Betti number of the homeomorphism groups $\mathcal{H}_+(\Gamma)$, $\mathcal{H}_+^{\mathrm{PL}}(\Gamma)$, $\mathcal{H}_0(\Gamma)$ and $\mathcal{H}_0^{\mathrm{PL}}(\Gamma)$ endowed with the compact-open topology.*

Proof. (i) By Theorem 2, the spaces $\mathcal{H}_0(\Gamma)$ and $\mathcal{H}_0^{\mathrm{PL}}(\Gamma)$ endowed with the Whitney topology are homeomorphic to $\boxdot^{o_\Gamma}\mathbb{T} \times \boxdot^{e_\Gamma} l_2$ and $\boxdot^{o_\Gamma}\mathbb{T} \times \boxdot^{e_\Gamma} l_2^f$, respectively. By Proposition 1.10 of [4], the contractibility of the groups l_2, l_2^f imply the contractibility of the small box products $\boxdot^{e_\Gamma} l_2$ and $\boxdot^{e_\Gamma} l_2^f$. Consequently, the spaces $\mathcal{H}_0(\Gamma)$ and $\mathcal{H}_0^{\mathrm{PL}}(\Gamma)$ are homotopically equivalent to $\boxdot^{o_\Gamma}\mathbb{T}$ and hence

$$b_1(\mathcal{H}_0(\Gamma)) = b_1(\mathcal{H}_0^{\mathrm{PL}}(\Gamma)) = b_1(\boxdot^{o_\Gamma}\mathbb{T}) = o_\Gamma$$

according to Proposition 3.

(ii) It follows from Theorem 1 that the spaces $\mathcal{H}_+(\Gamma)$ and $\mathcal{H}_+^{\mathrm{PL}}(\Gamma)$ endowed with the compact-open topology are homotopically equivalent to $\prod^{o_\Gamma}\mathbb{T}$. Consequently,

$$b^1(\mathcal{H}_+(\Gamma)) = b^1(\mathcal{H}_+^{\mathrm{PL}}(\Gamma)) = b^1(\textstyle\prod^{o_\Gamma}\mathbb{T}) = o_\Gamma$$

according to Proposition 3.

(iii) It follows from Theorem 1 that the spaces $\mathcal{H}_0(\Gamma)$ and $\mathcal{H}_0^{\mathrm{PL}}(\Gamma)$ endowed with the compact-open topology are homotopically equivalent to $\dot{\prod}^{o_\Gamma}\mathbb{T}$. Consequently,

$$cb^1(\mathcal{H}_0(\Gamma)) = cb^1(\mathcal{H}_0^{\mathrm{PL}}(\Gamma)) = cb^1(\dot{\prod}^{o_\Gamma}\mathbb{T}) = o_\Gamma$$

according to Proposition 3. □

3. Classifying homeomorphism groups of graphs

In this section we discuss the problem of detecting pairs of graphs Γ, Γ' having topologically equivalent homeomorphism groups. For subgroups $H \subset \mathcal{H}(\Gamma)$ and $H' \subset \mathcal{H}(\Gamma')$ we write $H \overset{C}{\approx} H'$ (resp. $H \overset{W}{\approx} H'$) if there is a homeomorphism $h : H \to H'$ with respect to the compact-open (resp. Whitney) topologies on H and H'. The same convention concerns also homeomorphisms of tuples.

Theorem 4. *Let Γ, Γ' be two graphs. Then*

(i) $\mathcal{H}(\Gamma) \overset{C}{\approx} \mathcal{H}(\Gamma')$ *if and only if* $\mathrm{Aut}(\vec{\Gamma}) \overset{C}{\approx} \mathrm{Aut}(\vec{\Gamma}')$ *and* $\mathcal{H}_+(\Gamma) \overset{C}{\approx} \mathcal{H}_+(\Gamma')$*;*

(ii) $\mathcal{H}^{\mathrm{PL}}(\Gamma) \overset{C}{\approx} \mathcal{H}^{\mathrm{PL}}(\Gamma')$ *if and only if* $\mathrm{Aut}(\vec{\Gamma}) \overset{C}{\approx} \mathrm{Aut}(\vec{\Gamma}')$ *and* $\mathcal{H}^{\mathrm{PL}}_+(\Gamma) \overset{C}{\approx} \mathcal{H}^{\mathrm{PL}}_+(\Gamma')$.

Proof. Assume that $\mathcal{H}(\Gamma) \overset{C}{\approx} \mathcal{H}(\Gamma')$. Then the identity component $\mathcal{H}_+(\Gamma)$ of $\mathcal{H}(\Gamma)$ is homeomorphic to the identity component $\mathcal{H}_+(\Gamma')$ of $\mathcal{H}(\Gamma')$. Observe that the automorphism group $\mathrm{Aut}(\vec{\Gamma}) = \mathcal{H}(\Gamma)/\mathcal{H}_+(\Gamma)$ can be identified with the space of quasi-components of the space $\mathcal{H}(\Gamma)$ (endowed with the compact-open topology).

The spaces $\mathcal{H}(\Gamma)$ and $\mathcal{H}(\Gamma')$, being homeomorphic, have homeomorphic spaces of quasi-components. Conseqently, $\mathrm{Aut}(\Gamma) \overset{C}{\approx} \mathrm{Aut}(\Gamma')$. This proves the "if" part of the first item. The "only if" part follows from Theorem 1.

By analogy, we can prove the second item of the theorem. $\square$

Theorem 5. *For two graphs Γ, Γ' the following conditions are equivalent:*

(a) $o_\Gamma = o_{\Gamma'}$ *and* $\lceil e_\Gamma \rceil = \lceil e_{\Gamma'} \rceil$*;*

(b) $\mathcal{H}_+(\Gamma) \overset{C}{\approx} \mathcal{H}_+(\Gamma')$*;*

(c) $\mathcal{H}^{\mathrm{PL}}_0(\Gamma) \overset{C}{\approx} \mathcal{H}^{\mathrm{PL}}_0(\Gamma')$*;*

(d) $(\mathcal{H}_+(\Gamma), \mathcal{H}^{\mathrm{PL}}_0(\Gamma)) \overset{C}{\approx} (\mathcal{H}_+(\Gamma'), \mathcal{H}^{\mathrm{PL}}_0(\Gamma'))$.

Proof. The implication (a) $\Rightarrow$ (d) follows from Theorem 1 and the known topological equivalence $(\prod^\omega l_2, \boxdot^\omega l_2^f) \approx (l_2, l_2^f)$. The implications (d) $\Rightarrow$ (b) and (d) $\Rightarrow$ (c) are trivial. Finally, (b) $\Rightarrow$ (a) and (c) $\Rightarrow$ (a) follow from Propositions 2 and 4(ii), (iii). $\square$

Theorem 6. *For two graphs Γ, Γ' the following conditions are equivalent:*

(a) $o_\Gamma = o_{\Gamma'}$ *and* $\lfloor e_\Gamma \rfloor = \lfloor e_{\Gamma'} \rfloor$*;*

(b) $\mathcal{H}_0(\Gamma) \overset{W}{\approx} \mathcal{H}_0(\Gamma')$*;*

(c) $\mathcal{H}^{\mathrm{PL}}_0(\Gamma) \overset{W}{\approx} \mathcal{H}^{\mathrm{PL}}_0(\Gamma')$*;*

(d) $\mathcal{H}_+(\Gamma) \overset{W}{\approx} \mathcal{H}_+(\Gamma')$*;*

(e) $\mathcal{H}^{\mathrm{PL}}_+(\Gamma) \overset{W}{\approx} \mathcal{H}^{\mathrm{PL}}_+(\Gamma')$*;*

(f) $\mathcal{H}^{\mathrm{PL}}_+(\Gamma) \overset{C}{\approx} \mathcal{H}^{\mathrm{PL}}_+(\Gamma')$*;*

(g) $(\mathcal{H}_+(\Gamma), \mathcal{H}^{\mathrm{PL}}_+(\Gamma), \mathcal{H}_0(\Gamma)) \overset{W}{\approx} (\mathcal{H}_+(\Gamma'), \mathcal{H}^{\mathrm{PL}}_+(\Gamma'), \mathcal{H}_0(\Gamma'))$*;*

(h) $(\mathcal{H}_+(\Gamma), \mathcal{H}^{\mathrm{PL}}_+(\Gamma), \mathcal{H}_0(\Gamma)) \overset{C}{\approx} (\mathcal{H}_+(\Gamma'), \mathcal{H}^{\mathrm{PL}}_+(\Gamma'), \mathcal{H}_0(\Gamma'))$.

Proof. We shall show the implications in the following diagram:

$$\begin{tikzcd}
& \text{(h)} \arrow[r, Rightarrow, "\text{triv.}"] & \text{(f)} \arrow[dl, Rightarrow] \\
\text{(b)} \arrow[r, Rightarrow] & \text{(a)} \arrow[u, Rightarrow, "\text{Thm.1}"] \arrow[d, Rightarrow, "\text{Thm.2}"'] & \text{(c)} \arrow[l, Rightarrow] \\
\text{(d)} \arrow[u, Rightarrow, "\text{Prop.1}"] & \text{(g)} \arrow[l, Rightarrow, "\text{triv.}"] \arrow[r, Rightarrow] & \text{(e)} \arrow[u, Rightarrow]
\end{tikzcd}$$

The implication (a) $\Rightarrow$ (h) follows from Theorem 1, while (h) $\Rightarrow$ (f) is trivial. The implication (a) $\Rightarrow$ (g) follows from Theorem 2, while (g) $\Rightarrow$ (d) is trivial. The implication (d) $\Rightarrow$ (b) follows from Proposition 1 and the fact that the identity components of homeomorphic topological groups are homeomorphic.

(f) $\Rightarrow$ (a): Assume that $\mathcal{H}_+^{\mathrm{PL}}(\Gamma) \overset{C}{\approx} \mathcal{H}_+^{\mathrm{PL}}(\Gamma')$. By Propositions 4(ii) and 2, $o_\Gamma = o_{\Gamma'}$ and $\lceil e_\Gamma \rceil = \lceil e_{\Gamma'} \rceil$. If $\lceil e_\Gamma \rceil = \lceil e_{\Gamma'} \rceil \neq \aleph_0$, then $\lfloor e_\Gamma \rfloor = \lfloor e_{\Gamma'} \rfloor$ by the definitions of $\lceil \kappa \rceil$ and $\lfloor \kappa \rfloor$. Now we see that the inequality $\lfloor e_\Gamma \rfloor \neq \lfloor e_{\Gamma'} \rfloor$ is possible only if $1 \leq \min\{e_\Gamma, e_{\Gamma'}\} < \max\{e_\Gamma, e_{\Gamma'}\} = \aleph_0$. Without loss of generality, $1 \leq e_\Gamma < \aleph_0 = e_{\Gamma'}$. Then the spaces $\mathcal{H}_+^{\mathrm{PL}}(\Gamma) \overset{C}{\approx} \prod^{o_\Gamma} \mathbb{T} \times l_2^f$ and $\mathcal{H}_+^{\mathrm{PL}}(\Gamma') \overset{C}{\approx} \prod^{o_{\Gamma'}} \mathbb{T} \times (l_2^f)^\omega$ cannot be homeomorphic because the first of them is σ-compact while the second is not. This contradiction completes the proof.

(b) $\Rightarrow$ (a): Assume that $\mathcal{H}_0(\Gamma) \overset{W}{\approx} \mathcal{H}_0(\Gamma')$. By Propositions 4(i) and 2(ii), $o_\Gamma = o_{\Gamma'}$ and $\lceil e_\Gamma \rceil = \lceil e_{\Gamma'} \rceil$. If $\lceil e_\Gamma \rceil = \lceil e_{\Gamma'} \rceil \neq \aleph_0$, then $\lfloor e_\Gamma \rfloor = \lfloor e_{\Gamma'} \rfloor$ by the definitions of $\lceil \kappa \rceil$ and $\lfloor \kappa \rfloor$. Now we see that the inequality $\lfloor e_\Gamma \rfloor \neq \lfloor e_{\Gamma'} \rfloor$ is possible only if $1 \leq \min\{e_\Gamma, e_{\Gamma'}\} < \max\{e_\Gamma, e_{\Gamma'}\} = \aleph_0$. Without loss of generality, $1 \leq e_\Gamma < \aleph_0 = e_{\Gamma'}$. In this case $o_{\Gamma'} = o_\Gamma \leq e_\Gamma$ is finite. Then the spaces $\mathcal{H}_0(\Gamma) \overset{W}{\approx} \boxdot^{o_\Gamma} \mathbb{T} \times l_2$ and $\mathcal{H}_0(\Gamma') \overset{W}{\approx} \boxdot^{o_{\Gamma'}} \mathbb{T} \times \boxdot^\omega l_2$ cannot be homeomorphic because the first of them is metrizable while the second is not. This contradiction completes the proof.

By analogy we can prove the implications (g) $\Rightarrow$ (e) $\Rightarrow$ (c) $\Rightarrow$ (a). $\square$

Theorem 7. *For two non-compact graphs Γ and Γ', the following conditions are equivalent:*

(a) $2^{p_\Gamma + e_\Gamma} = 2^{p_{\Gamma'} + e_{\Gamma'}}$, $o_\Gamma = o_{\Gamma'}$, *and* $\lfloor e_\Gamma \rfloor = \lfloor e_{\Gamma'} \rfloor$;

(b) $\mathcal{H}(\Gamma) \overset{W}{\approx} \mathcal{H}(\Gamma')$;

(c) $\mathcal{H}^{\mathrm{PL}}(\Gamma) \overset{W}{\approx} \mathcal{H}^{\mathrm{PL}}(\Gamma')$;

(d) $(\mathcal{H}(\Gamma), \mathcal{H}^{\mathrm{PL}}(\Gamma), \mathcal{H}_0(\Gamma)) \overset{W}{\approx} (\mathcal{H}(\Gamma'), \mathcal{H}^{\mathrm{PL}}(\Gamma'), \mathcal{H}_0(\Gamma'))$.

Proof. The implication (a) $\Rightarrow$ (d) follows from Theorem 3, while (d) $\Rightarrow$ (b), (c) are trivial.

(b) $\Rightarrow$ (a): Assume that $\mathcal{H}(\Gamma) \overset{W}{\approx} \mathcal{H}(\Gamma')$. Then $\mathcal{H}_0(\Gamma) \overset{W}{\approx} \mathcal{H}_0(\Gamma')$ (because homeomorphic groups have homeomorphic identity components). By Theorem 6, $o_\Gamma = o_{\Gamma'}$, and $\lfloor e_\Gamma \rfloor = \lfloor e_{\Gamma'} \rfloor$. By Proposition 2(iii), $2^{p_\Gamma + e_\Gamma} = 2^{p_{\Gamma'} + e_{\Gamma'}}$.

By analogy we can prove that (c) $\Rightarrow$ (a). $\square$

4. Normality and the paracompactness of homeomorphism groups

In this section, we discuss some applications of the results from the preceding section to the problem of detecting normal and paracompact groups among the homeomorphism groups of graphs.

Corollary 3. *Let Γ be a graph. Endowed with the compact-open topology the homeomorphism group*

(i) $\mathcal{H}_0(\Gamma)$ *is paracompact;*
(ii) $\mathcal{H}(\Gamma)$ *is metrizable iff* $\mathcal{H}^{\mathrm{PL}}(\Gamma)$ *is normal iff* $e_\Gamma + p_\Gamma \le \aleph_0$.

Proof. (i) By Theorem 1, the group $\mathcal{H}_0(\Gamma)$ endowed with the compact-open topology is homeomorphic to the product $\boxdot^{o_\Gamma}\mathbb{T} \times \boxdot^{e_\Gamma} l_2$, which is paracompact according to Theorem 3 of [15].

(ii) Let $\vec{\Gamma}$ be the digraph associated to Γ. If $e_\Gamma + p_\Gamma \le \aleph_0$, then the graph $\vec{\Gamma}$ is countable and its automorphism group $\mathrm{Aut}(\vec{\Gamma})$ is metrizable. Taking into account that $o_\Gamma \le e_\Gamma \le \aleph_0$, we conclude that the product $\mathrm{Aut}(\vec{\Gamma}) \times \prod^{o_\Gamma}\mathbb{T} \times \prod^{e_\Gamma} l_2$ is metrizable and so is its topological copy $\mathcal{H}(\Gamma)$.

If the group $\mathcal{H}(\Gamma)$ is metrizable, then its subgroup $\mathcal{H}^{\mathrm{PL}}(\Gamma)$ is metrizable and thus normal.

Finally, assuming that the group $\mathcal{H}^{\mathrm{PL}}(\Gamma)$ is normal, we shall show that $p_\Gamma + e_\Gamma \le \aleph_0$. By Theorem 1, $\mathcal{H}^{\mathrm{PL}}(\Gamma)$ is homeomorphic to the product $\mathrm{Aut}(\vec{\Gamma}) \times \prod^{o_\Gamma}\mathbb{T} \times \prod^{e_\Gamma} l_2^f$ and consequently, the latter product is normal.

Assuming that $e_\Gamma > \aleph_0$ we conclude that the power $\prod^{e_\Gamma} l_2$ is not normal because it contains a closed topological copy of the uncountable power $\mathbb{N}^{\aleph_1}$ which is non-normal by a famous Stone result, see [11, 2.7.16(a)]. Then $\mathcal{H}^{\mathrm{PL}}(\Gamma)$ is not normal because it contains a closed topological copy of the non-normal space $\prod^{o_\Gamma} l_2$.

If $p_\Gamma > \aleph_0$, then the digraph $\vec{\Gamma}$ associated to Γ contains uncountably many isolated vertices. Consequently, its automorphism group $\mathrm{Aut}(\vec{\Gamma})$ contains a closed copy of the group $\mathrm{Aut}(\aleph_1)$ of all bijections of the uncountable discrete space $\aleph_1$, endowed with the compact-open topology. It is easy to see that the group $\mathrm{Aut}(\aleph_1)$ contains a closed subgroup topologically isomorphic to the group $\mathbb{Z}^{\aleph_1}$, which is not normal according to the Stone Theorem. Consequently, $\mathrm{Aut}(\aleph_1)$ and also $\mathrm{Aut}(\vec{\Gamma})$ and $\mathcal{H}^{\mathrm{PL}}(\Gamma)$ all are not normal. This contradition completes the proof of the inequality $p_\Gamma + e_\Gamma \le \aleph_0$. □

The situation with the Whitney topology is more interesting.

Corollary 4. *Let Γ be a graph. Endowed with the Whitney topology the homeomorphism group*

(i) $\mathcal{H}_0(\Gamma)$ *as well as* $\mathcal{H}_0^{\mathrm{PL}}$ *is paracompact;*
(ii) $\mathcal{H}(\Gamma)$ *is normal iff* $\mathcal{H}(\Gamma)$ *is metrizable iff* $e_\Gamma < \aleph_0$*;*
(iii) $\mathcal{H}^{\mathrm{PL}}(\Gamma)$ *is not normal if* $e_\Gamma > \aleph_0$ *and* $\mathcal{H}^{\mathrm{PL}}(\Gamma)$ *is metrizable if* $e_\Gamma < \aleph_0$*;*
(iv) $\mathcal{H}^{\mathrm{PL}}(\Gamma)$ *is paracompact if* $e_\Gamma = \aleph_0$ *and either* $\mathfrak{b} = \mathfrak{d}$ *or* $\mathfrak{d} = \mathfrak{c}$.

Proof. (i) By Theorem 3, the groups $\mathcal{H}_c(\Gamma)$ and $\mathcal{H}_c^{\mathrm{PL}}(\Gamma)$ endowed with the Whitney are homeomorphic to $\boxdot^{o_\Gamma}\mathbb{T} \times \boxdot^{e_\Gamma} l_2$ and $\boxdot^{o_\Gamma}\mathbb{T} \times \boxdot^{e_\Gamma} l_2^f$. By [19], the latter products are paracompact.

(ii) If $e_\Gamma < \aleph_0$, then $\mathcal{H}(\Gamma)$ is metrizable because it contain a metrizable open subgroup $\mathcal{H}_+(\Gamma) \approx \prod^{o_\Gamma}\mathbb{T} \times \prod^{e_\Gamma} l_2$. If $e_\Gamma \ge \aleph_0$, then $\mathcal{H}_+(\Gamma) \approx \square^{o_\Gamma}\mathbb{T} \times \square^{e_\Gamma} l_2$ is not normal because it contains a closed subspace homeomorphic to the non-normal space $\square^\omega l_2$, see [8].

(iii) By Theorem 2, the space $\mathcal{H}^{\mathrm{PL}}(\Gamma)$ contains a closed subspace homeomorphic to $\square^{e_\Gamma} l_2^f$. The latter space is not normal if $e_\Gamma > \aleph_0$, see [17]. If $e_\Gamma < \aleph_0$, then the group $\mathcal{H}^{\mathrm{PL}}(\Gamma)$ is metrizable because it contains an open metrizable subgroup $\mathcal{H}^{\mathrm{PL}}_+(\Gamma) \approx \prod^{o_\Gamma} \mathbb{T} \times \prod^{e_\Gamma} l_2^f$.

(iv) Assume that $e_\Gamma = \aleph_0$. Recently Scott Williams in [26] announced that under the Continuum Hypothesis (more generally, under $\mathfrak{b} = \mathfrak{d}$ or $\mathfrak{d} = \mathfrak{c}$) the box power $\square^\omega X$ of any metrizable σ-compact space is paracompact. In particular, $\mathcal{H}^{\mathrm{PL}}_+(\Gamma) \approx \boxdot^{o_\Gamma} \mathbb{T} \times \boxdot^{e_\Gamma} l_2^f$, is paracompact. The group $\mathcal{H}^{\mathrm{PL}}(\Gamma)$ is paracompact because it contains the open paracompact subgroup $\mathcal{H}_+(\Gamma)$. $\square$

5. The groups $\mathcal{H}_+(\mathbb{I})$, $\mathcal{H}_+(\mathbb{T},1)$, and $\mathcal{H}_+(\mathbb{T})$

In this section we recognize the topological structure of the homeomorphism groups $\mathcal{H}_+(\mathbb{I})$, $\mathcal{H}_+(\mathbb{T})$, and $\mathcal{H}_+(\mathbb{T},1)$, endowed with the compact-open topology. It coincides with the Whitney topology because of the compactness of $\mathbb{I} = [0,1]$ and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

We start with considering the homeomorphism group $\mathcal{H}(\mathbb{I})$ and its subgroups $\mathcal{H}_+(\mathbb{I})$ and $\mathcal{H}^{\mathrm{PL}}_+(\mathbb{I})$ consisting of orientation-preserving and orientation-preserving PL homeomorphisms of $\mathbb{I}$, respectively. Note that $\mathcal{H}_+(\mathbb{I})$ is an open subgroup of index 2 in $\mathcal{H}(\mathbb{I})$. Now, we shall show the following propositions (cf. Geoghegan and Haver [12]).

Proposition 5. *The pair $\big(\mathcal{H}_+(\mathbb{I}), \mathcal{H}^{\mathrm{PL}}_+(\mathbb{I})\big)$ is homeomorphic to (l_2, l_2^f).*

Proof. Observe that the group $\mathcal{H}_+(\mathbb{I})$ consists of increasing homeomorphisms of $\mathbb{I}$ and is homeomorphic to the Hilbert space l_2 [1] (cf. [14]). The latter can be seen as follows. Being a convex subset of the Banach space $C[0,1]$ of continuous real-valued functions on $[0,1]$, $\mathcal{H}_+(\mathbb{I})$ is an absolute retract by the Dugundji Extension Theorem [5, Chapter II, Theorem 3.1]. Being a Polish non-locally compact topological AR-group, $\mathcal{H}_+(E, \partial E)$ is homeomorphic to the separable Hilbert space l_2 according to [9].

The group $\mathcal{H}^{\mathrm{PL}}_+(\mathbb{I})$ consists of increasing PL-homeomorphisms of $[0,1]$ and thus is a convex infinite-dimensional subset of $C[0,1]$. It is easy to see that $\mathcal{H}^{\mathrm{PL}}_+(\mathbb{I})$ can be written as the countable union of finite-dimensional compact subsets. Applying Theorem 4.1 of [6], we conclude that $\mathcal{H}_+(\mathbb{I})$ is homeomorphic to l_2^f.

Being a dense convex subset of $\mathcal{H}_+(\mathbb{I})$ the group $\mathcal{H}^{\mathrm{PL}}_+(\mathbb{I})$ is homotopy dense in $\mathcal{H}_+(\mathbb{I})$, see [3, Ex.12,13]. Since $\mathcal{H}^{\mathrm{PL}}_+(\mathbb{I}) \approx l_2^f$ is a σ-compact homotopy dense subset of $\mathcal{H}_+(\mathbb{I}) \approx l_2$, we can apply Theorem 3.1.10 of [3] to conclude that the pair $(\mathcal{H}_+(\mathbb{I}), \mathcal{H}^{\mathrm{PL}}_+(\mathbb{I}))$ is homeomorphic to (l_2, l_2^f). $\square$

Next we consider the homeomorphism group $\mathcal{H}(\mathbb{T})$ of the circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Let $\mathcal{H}(\mathbb{T},1)$ be the subgroup of $H(\mathbb{T})$ consisting of homeomorphisms with $h(1) = 1$ and let $\mathcal{H}_+(\mathbb{T},1) = \mathcal{H}_+(\mathbb{T}) \cap \mathcal{H}(\mathbb{T},1)$.

Proposition 6. *The pair $\big(\mathcal{H}_+(\mathbb{T},1), \mathcal{H}^{\mathrm{PL}}_+(\mathbb{T},1)\big)$ is homeomorphic to (l_2, l_2^f) and the pair $\big(\mathcal{H}_+(\mathbb{T}), \mathcal{H}^{\mathrm{PL}}_+(\mathbb{T})\big)$ is homeomorphic to $(\mathbb{T} \times l_2, \mathbb{T} \times l_2^f)$.*

Proof. As is easily observed, the subgroup $\mathcal{H}_+(\mathbb{T})$ can be identified with $\mathcal{H}_+(\mathbb{I})$ by a homeomorphism

$$F_1 : (\mathcal{H}_+(\mathbb{I}), \mathcal{H}_+^{\mathrm{PL}}(\mathbb{I})) \to (\mathcal{H}_+(\mathbb{T}, 1), \mathcal{H}_+^{\mathrm{PL}}(\mathbb{T}, 1)).$$

For any $t \in \mathbb{T}$, let $h_t : \mathbb{T} \to \mathbb{T}$ be the rotational homeomorphism defined by $h_t(s) = st$ for any $s \in \mathbb{T}$. Let $S = \{h_t : t \in \mathbb{T}\}$ be the subgroup of $\mathcal{H}_+(\mathbb{T})$. It is clear that the map $F_2 : \mathbb{T} \ni t \mapsto h_t \in S$ is a homeomorphism. Thus, we have the homeomorphism

$$H : (H_+(\mathbb{I}) \times \mathbb{T}, H_+^{PL}(\mathbb{I}) \times \mathbb{T}) \to (H_+(\mathbb{T}), H_+^{PL}(\mathbb{T}))$$

by $H(f, t) = F_2(t) \circ F_1(f)$. By Proposition 5, it follows that

$$(\mathcal{H}_+(\mathbb{T}), \mathcal{H}_+^{\mathrm{PL}}(\mathbb{T})) \approx (H_+(\mathbb{I}) \times \mathbb{T}, H_+^{\mathrm{PL}}(\mathbb{I}) \times \mathbb{T}) \approx (l_2 \times \mathbb{T}, l_2^f \times \mathbb{T}).$$

The proof is completed. □

Note that $\mathcal{H}_+(\mathbb{I})$ and $\mathcal{H}_+(\mathbb{T})$ are open subgroups of index 2 in $\mathcal{H}(\mathbb{I})$ and $\mathcal{H}(\mathbb{T})$, respectively. Thus, we get

Corollary 5. *The pairs $(\mathcal{H}(\mathbb{I}), \mathcal{H}^{\mathrm{PL}}(\mathbb{I}))$ and $(\mathcal{H}(\mathbb{T}, 1), \mathcal{H}^{\mathrm{PL}}(\mathbb{T}, 1))$ are homeomorphic to $(2 \times l_2, 2 \times l_2^f)$ and $(\mathcal{H}(\mathbb{T}), \mathcal{H}^{\mathrm{PL}}(\mathbb{T}))$ is homeomorphic to $(2 \times \mathbb{T} \times l_2, 2 \times \mathbb{T} \times l_2^f)$.*

Here $2 = \{0, 1\}$ is the discrete two-point space.

6. The groups $\mathcal{H}_+(\mathbb{R})$ and $\mathcal{H}(\mathbb{R}_+)$ with the compact-open topology

In this section we shall recognize the topological structure of the triples

$$(\mathcal{H}_+(\mathbb{R}), \mathcal{H}_+^{\mathrm{PL}}(\mathbb{R}), \mathcal{H}_0(\mathbb{R})) \quad \text{or} \quad (\mathcal{H}(\mathbb{R}_+), \mathcal{H}^{\mathrm{PL}}(\mathbb{R}_+), \mathcal{H}_0(\mathbb{R}_+))$$

of the homeomorphism groups endowed with the compact-open topology.

Proposition 7. *The triple $(\mathcal{H}_+(\mathbb{R}), \mathcal{H}_+^{\mathrm{PL}}(\mathbb{R}), \mathcal{H}_0(\mathbb{R}))$ endowed with the compact-open topology is homeomorphic to the triple $(\prod^\omega l_2, \prod^\omega l_2^f, \boxdot^\omega l_2)$.*

Proof. Consider the closed subgroups

$$\mathcal{H}(\mathbb{R}, \mathbb{Z}) = \{f \in \mathcal{H}(\mathbb{R}) : f|\mathbb{Z} = \mathrm{id}\} \text{ and}$$
$$L = \{f \in \mathcal{H}_+(\mathbb{R}) : f \text{ is linear on each interval } [n, n+1],\ n \in \mathbb{Z}\}$$

of $\mathcal{H}(\mathbb{R})$ and observe that the map

$$\Psi_1 : L \times \mathcal{H}(\mathbb{R}, \mathbb{Z}) \to \mathcal{H}_+(\mathbb{R}), \quad \Psi_1 : (f, g) \mapsto f \circ g,$$

is a homeomorphism witnessing that

$$\big(\mathcal{H}_+(\mathbb{R}), \mathcal{H}_+^{\mathrm{PL}}(\mathbb{R}), \mathcal{H}_0(\mathbb{R})\big) \overset{C}{\approx} \big(L \times \mathcal{H}(\mathbb{R}, \mathbb{Z}), L \times \mathcal{H}^{\mathrm{PL}}(\mathbb{R}, \mathbb{Z}), L_0 \times \mathcal{H}_0(\mathbb{R}, \mathbb{Z})\big) \tag{1}$$

where

$$L_0 = L \cap \mathcal{H}_0(\mathbb{R}),\ \ \mathcal{H}^{\mathrm{PL}}(\mathbb{R}, \mathbb{Z}) = \mathcal{H}^{\mathrm{PL}}(\mathbb{R}) \cap \mathcal{H}(\mathbb{R}, \mathbb{Z}),\ \ \mathcal{H}_0(\mathbb{R}, \mathbb{Z}) = \mathcal{H}_0(\mathbb{R}) \cap \mathcal{H}(\mathbb{R}, \mathbb{Z}).$$

By analogy with the proof of Proposition 5 we can establish the topological equivalences of pairs

$$(L, L_0) \approx (l_2, l_2^f) \approx (\mathbb{R} \times l_2, \mathbb{R} \times l_2^f) \approx (\prod^\omega \mathbb{R}, \boxdot^\omega \mathbb{R}). \tag{2}$$

Next, consider the homeomorphism

$$\Psi_2 : \mathcal{H}(\mathbb{R}, \mathbb{Z}) \to \prod^{\mathbb{Z}} \mathcal{H}_+(\mathbb{I}), \quad \Psi_2 : h \mapsto (h_n)_{n \in \mathbb{Z}}, \quad h_n : t \mapsto h(t+n) - n,$$

witnessing that

$$\big(\mathcal{H}(\mathbb{R},\mathbb{Z}),\mathcal{H}^{\mathrm{PL}}(\mathbb{R},\mathbb{Z}),\mathcal{H}_0(\mathbb{R},\mathbb{Z})\big) \overset{C}{\approx} \big(\prod^{\mathbb{Z}}\mathcal{H}_+(\mathbb{I}),\prod^{\mathbb{Z}}\mathcal{H}_+^{\mathrm{PL}}(\mathbb{I}),\ \dot{\prod}^{\mathbb{Z}}\mathcal{H}_+(\mathbb{I})\big) \approx \big(\prod^{\omega} l_2,\prod^{\omega} l_2^f,\ \dot{\prod}^{\omega} l_2\big). \tag{3}$$

The last homeomorphism follows from the topological equivalence $(\mathcal{H}_+(\mathbb{I}),\mathcal{H}_+^{\mathrm{PL}}(\mathbb{I})) \approx (l_2,l_2^f)$ established in Proposition 5. Unifying the homeomorphisms (1)–(3) we see that

$$\begin{aligned}\big(\mathcal{H}_+(\mathbb{R}),\mathcal{H}_+^{\mathrm{PL}}(\mathbb{R}),\mathcal{H}_0(\mathbb{R})\big) &\overset{C}{\approx} \big(L\times\mathcal{H}(\mathbb{R},\mathbb{Z}),L\times\mathcal{H}^{\mathrm{PL}}(\mathbb{R},\mathbb{Z}),L_0\times\mathcal{H}_0(\mathbb{R},\mathbb{Z})\big)\\ &\overset{C}{\approx} \big(\prod^{\omega}\mathbb{R}\times\prod^{\omega} l_2,\prod^{\omega}\mathbb{R}\times\prod^{\omega} l_2^f,\ \dot{\prod}^{\omega}\mathbb{R}\times\dot{\prod}^{\omega} l_2\big)\\ &\approx \big(\prod^{\omega}(\mathbb{R}\times l_2),\prod^{\omega}(\mathbb{R}\times l_2^f),\ \dot{\prod}^{\omega}(\mathbb{R}\times l_2)\big)\\ &\approx \big(\prod^{\omega} l_2,\prod^{\omega} l_2^f,\ \dot{\prod}^{\omega} l_2\big).\end{aligned}$$

$\square$

By analogy we can prove

Proposition 8. *The triple $\big(\mathcal{H}(\mathbb{R}_+),\mathcal{H}^{\mathrm{PL}}(\mathbb{R}_+),\mathcal{H}_0(\mathbb{R}_+)\big)$ endowed with the compact-open topology is homeomorphic to the triple $(\prod^{\omega} l_2,\prod^{\omega} l_2^f,\ \dot{\prod}^{\omega} l_2)$.*

To prove a similar result for the Whitney topology we need first to establish

7. A multiplication property of the triple $(\square^\omega l_2,\square^\omega l_2^f,\boxdot^\omega l_2^f)$.

Observe that the triple $\big(\square^\omega l_2,\square^\omega l_2^f,\boxdot^\omega l_2\big)$ is a particular case of a triple of the form

$$\big(\square^\kappa G,\square^\kappa H,\boxdot^\kappa G\big), \tag{4}$$

where κ is a cardinal, G is a topological group, and H is a dense subgroup of G.

It is easy to see that for each topological group G the box power $\square^\kappa G$ is a topological group with respect to the operation of coordinatewise multiplication.

Proposition 9. *Let G be a non-discrete topological group and H be a dense subgroup of G. For any infinite cardinal κ and any non-zero cardinal $\lambda \le 2^\kappa$, the triple $\big(\square^\kappa G,\square^\kappa H,\boxdot^\kappa G\big)$ is homeomorphic to*

$$\big(\lambda\times\square^\kappa G,\lambda\times\square^\kappa H,1\times\boxdot^\kappa G\big).$$

Proof. Let e denote the neutral element of the group G. By [23, Theorem 2.2] (cf. [23, Proof of Theorem 5.16]), the topological group G admits a continuous left-invariant pseudo-metric ρ such that $\rho(x_0,e)\ne 0$ for some point x_0 which can be chosen in the dense subgroup H of G.

The cardinal κ, being infinite, can be identified with the product $P=\kappa\times\omega$. So, the triple (4) is homeomorphic to the triple

$$\big(\square^P G,\square^P H,\boxdot^P G\big). \tag{5}$$

In the group $\square^P G$, consider the open subgroup

$$\square^P_0 G = \{(x_{\alpha,n})_{(\alpha,n)\in P}\in\square^P G : \forall\alpha\in\kappa\ \lim_{n\to\infty}\rho(x_{\alpha,n},e)=0\}$$

and let $\square^P_0 H=\square^P_0 G\cap\square^P H$.

Using the density of the subgroup $\square^P H$ in $\square^P G$ (which follows from the density H in G), choose a subset $D \subset \square^P H$ meeting each left coset $\vec{x} \cdot \square^P_0 G$, $\vec{x} \in \square^P G$, at a single point. Moreover, the set D can be chosen so that $D \cap \square^P_0 G = \{e\}$.

Since the subgroup $\square^P_0 G$ is open in $\square^P G$, the set D is closed and discrete in $\square^P G$. We claim that the cardinality $|D| \geq 2^\kappa$. To prove this fact, for every subset $A \subset \kappa$ consider the vector $\vec{x}_A = (x^A_{\alpha,n})_{(\alpha,n)\in P} \in \square^P H$ such that

$$x^A_{\alpha,n} = \begin{cases} x_0 & \text{if } \alpha \in A \\ e & \text{if } \alpha \notin A. \end{cases}$$

Observe that for different subsets $A, B \subset \kappa$, we get $\vec{x}_A \cdot (\vec{x}_B)^{-1} \notin \square^P_0 G$, which implies that the subgroup $\square^P G$ has at least 2^κ cosets in $\square^P G$ and hence

$$|D| = |\square^P G / \square^P_0 G| \geq 2^\kappa.$$

One can readily check that the map

$$\Psi : D \times \square^P_0 G \to \square^P G,\ \Psi : (x, y) \mapsto x \cdot y,$$

is a homeomorphism mapping the triple

$$\big(D \times \square^P_0 G, D \times \square^P_0 H, \{e\} \times \boxdot^P G\big)$$

onto the triple (5).

Given any non-zero cardinal $\lambda \leq 2^\kappa$, we see that $|D \times \lambda| = |D|$ and hence

$$\begin{aligned} &\big(\lambda \times \square^P G, \lambda \times \square^P H, 1 \times \boxdot^P G,\big) \\ &\qquad \approx \big(\lambda \times D \times \square^P_0 G, \lambda \times D \times \square^P_0 H, 1 \times \{e\} \times \boxdot^P G\big) \\ &\qquad \approx \big(D \times \square^P_0 G, D \times \square^P_0 H, \{e\} \times \boxdot^P G\big) \\ &\qquad \approx \big(\square^P G, \square^P H, \boxdot^P G\big). \end{aligned}$$

Thus, we have the result. $\square$

Applying the preceding proposition to the triple $(\square^\omega l_2, \square^\omega l_2^f, \boxdot^\omega l_2)$, we obtain

Corollary 6. *The triple* $(\square^\omega l_2, \square^\omega l_2^f, \boxdot^\omega l_2)$ *is homeomorphic to*

$$(\lambda \times \square^\omega l_2, \lambda \times \square^\omega l_2^f, 1 \times \boxdot^\omega l_2)$$

for every non-zero cardinal $\lambda \leq \mathfrak{c}$.

8. The topological structure of some subspaces of $\square^\omega \mathbb{R}$

In this section, we recognize the topological structure of the pair (M, M_0) of subspaces $M_0 \subset M \subset \square^\omega \mathbb{R}$ defined below. By definition, M is the subspace of the box power $\square^\omega \mathbb{R}$ consisting of all sequences $(x_n)_{n\in\omega}$ such that

- $x_0 = 0$;
- $x_n < x_{n+1}$ for all $n \in \omega$;
- $\lim_{n\to\infty} |x_n - n| = 0$.

By M_0, we denote the subspace of M consisting of all sequences $(x_n)_{n\in\omega} \in M$ such that $x_n = n$ for all but finitely many numbers n.

Our aim is to prove that the pair (M, M_0) is homeomorphic to $(\square^\omega \mathbb{R}, \boxdot^\omega \mathbb{R})$. This will be done with help of the following three lemmas.

Let us consider the difference operator $\partial : M \to \square^\omega \mathbb{R}$ assigning to each sequence $(x_n)_{n\in\omega} \in M$ the sequence $(x_{n+1} - x_n - 1)_{n\in\omega}$. Observe that the image $\partial(M)$

coincides with the subspace S of $\square^\omega\mathbb{R}$ consisting of sequences $(y_n)_{n\in\omega} \in (-1,\infty)^\omega$ such that $\sum_{n\in\omega} y_n = 0$. Moreover, the image $\partial(M_0)$ coincides with the subspace S_0 of S consisting of eventually zero sequences.

Lemma 1. *The difference operator ∂ is a homeomorphism of the pair (M, M_0) onto (S, S_0).*

Proof. The inverse operator $\partial^{-1} : S \to M$ assigns to a sequence $\vec{y} = (y_n)_{n\in\omega} \in S$ the sequence $\vec{x} = (x_n)_{n\in\omega}$ defined inductively as follows:

$$x_0 = 0 \text{ and } x_{n+1} = x_n + y_n + 1 = n + 1 + \sum_{i=0}^{n} y_i \text{ for } n \geq 0. \tag{6}$$

To prove the continuity of the map ∂^{-1} at $\vec{y}$, take any box neighborhood $O(\vec{x})$ of $\vec{x} = \partial^{-1}(\vec{y})$ and find a sequence $\vec{\varepsilon} = (\varepsilon_n)_{n\in\omega}$ of positive reals such that

$$M \cap \square_{n\in\omega}(x_n - \varepsilon_n, x_n + \varepsilon_n) \subset O(\vec{x}).$$

Choose a sequence $\vec{\delta} = (\delta_n)_{n\in\omega}$ of positive reals such that $\delta_n < \frac{1}{2}\min\{\delta_{n-1}, \varepsilon_n\}$ for all $n \in \omega$ (here we put $\delta_{-1} = 1$). Then, we get

$$\sum_{i=n}^{\infty} \delta_i < \sum_{i=n}^{\infty} 2^{-i+n}\delta_n = 2\delta_n < \varepsilon_n \text{ for every } n \in \omega.$$

Consider the following box neighborhood of $\vec{y}$ in S:

$$O(\vec{y}) = S \cap \square_{n\in\omega}(y_n - \delta_n, y_n + \delta_n).$$

We claim that for every $\vec{y}' = (y'_n)_{n\in\omega} \in O(\vec{y})$, the preimage $\vec{x}' = \partial^{-1}(\vec{y}')$ belongs to $O(\vec{x})$. Taking into account (6) and $\sum\vec{y} = \sum\vec{y}' = 0$, we get

$$\begin{aligned} |x_n - x'_n| &= \left|\sum_{i=0}^{n-1} y_i - \sum_{i=0}^{n-1} y'_i\right| = \left|-\sum_{i=n}^{\infty} y_i + \sum_{i=n}^{\infty} y'_i\right| \\ &\leq \sum_{i=n}^{\infty} |y_i - y'_i| < \sum_{i=n}^{\infty} \delta_i < \varepsilon_n \quad \text{for every } n \in \mathbb{N}, \end{aligned}$$

which means that $\vec{x}' \in O(\vec{x})$. $\square$

Next, in the box power $\square^{\mathbb{N}}(-1,\infty)$, consider the clopen subset Σ consisting of all sequences $(x_n)_{n\in\mathbb{N}} \in \square^{\mathbb{N}}(-1,\infty)$ such that the series $\sum_{n\in\mathbb{N}} x_n$ converges. Let also Σ_0 be the subspace of Σ consisting of all eventually zero sequences. For a sequence $\vec{x} = (x_n)_{n\in\mathbb{N}} \in \square^{\mathbb{N}}\mathbb{R}$, put $\sum\vec{x} = \sum_{n\in\mathbb{N}} x_n$ if the latter sum exists.

Lemma 2. *The pair (Σ, Σ_0) is homeomorphic to (S, S_0).*

Proof. Fix a homeomorphism $\varphi : (-1,\infty) \to \mathbb{R}$ such that

$$\varphi(x) = x \text{ for all } x \in [-1/2, \infty) \text{ and } \varphi(x) < x \text{ for all } x \in (-1, -1/2).$$

We shall define a homeomorphism $\Psi : \Sigma \to S$ such that $\Psi(\Sigma_0) = S_0$. Consider the following closed sets in Σ and S:

$$\Sigma' = \{\vec{x} \in \Sigma : \textstyle\sum\vec{x} \leq 1/2\}, \ \Sigma'' = \{\vec{x} \in \Sigma : \textstyle\sum\vec{x} \geq 1/2\},$$
$$S' = \{\vec{y} = (y_i)_{i\in\omega} \in S : y_0 \geq -1/2\}, \ S'' = \{\vec{y} = (y_i)_{i\in\omega} \in S : y_0 \leq -1/2\}.$$

Let $\Sigma'_0 = \Sigma' \cap \Sigma_0$, $\Sigma''_0 = \Sigma'' \cap \Sigma_0$, $S'_0 = S' \cap S_0$ and $S''_0 = S'' \cap S_0$. We can define a homeomorphism $\Psi' = \Psi : \Sigma' \to S'$ as follows:

$$\Psi'(\vec{x}) = (-\sum \vec{x}, x_1, x_2, \dots) \in S' \quad \text{for each } x \in \Sigma'.$$

Since $\sum_{i=1}^{n} y_i = -y_0 \le 1/2$ for each $\vec{y} = (y_i)_{i\in\omega} \in S'$, the inverse ${\Psi'}^{-1}$ can be defined by

$${\Psi'}^{-1}(\vec{y}) = (y_i)_{i\in\mathbb{N}} = (y_1, y_2, \dots).$$

The continuity of both Ψ' and ${\Psi'}^{-1}$ is clear and $\Psi'(\Sigma'_0) = S'_0$ by the definition. Thus, it suffices to construct a homeomorphism $\Psi'' : \Sigma'' \to S''$ such that

$$\Psi''(\Sigma''_0) = S''_0 \quad \text{and} \quad \Psi''|\Sigma' \cap \Sigma'' = \Psi'|\Sigma' \cap \Sigma''.$$

First, we define maps $s_n : \Sigma'' \to \mathbb{R}$, $f_n : \Sigma'' \to (-1, \infty)$, $n \in \omega$, as follows:

$$f_0(\vec{x}) = \varphi^{-1}(-\sum \vec{x}) \le -1/2, \tag{7}$$

$$f_n(\vec{x}) = \varphi^{-1}(\varphi(x_n) - \sum \vec{x}) \quad \text{for } n \in \mathbb{N}, \text{ and} \tag{8}$$

$$s_n(\vec{x}) = \sum_{i=0}^{n} f_i(\vec{x}) + \sum_{i=n+1}^{\infty} x_i \quad \text{for } n \in \omega, \tag{9}$$

where $\vec{x} = (x_i)_{i\in\mathbb{N}} \in \Sigma''$. Since $\varphi^{-1}(x) \ge x$ for every $x \in \mathbb{R}$, it follows that $s_0(\vec{x}) \ge 0$ for every $\vec{x} \in \Sigma''$. It should be noticed that $s_0(\vec{x}) = 0$ if and only if $\vec{x} \in \Sigma' \cap \Sigma''$. Since φ^{-1} is order-preserving (increasing), it follows that $f_n(\vec{x}) < x_n$ and hence $s_{n-1}(\vec{x}) > s_n(\vec{x})$ for every $n \in \mathbb{N}$. Moreover, $s_n(\vec{x}) \to -\infty$ $(n \to \infty)$. Indeed, because $\sum_{i=1}^{\infty} x_n$ is convergent, we can take $k \in \mathbb{N}$ so large that $|x_n| < 1/8$ for every $n \ge k$. For each $n \ge k$,

$$\begin{aligned} s_{n-1}(\vec{x}) - s_n(\vec{x}) &= x_n - f_n(\vec{x}) = x_n - \varphi^{-1}(\varphi(x_n) - \sum \vec{x}) \\ &= x_n - \varphi^{-1}(x_n - \sum \vec{x}) > -1/8 - \varphi^{-1}(-3/8)) = 1/4. \end{aligned}$$

We consider the following closed sets in Σ'':

$$\Delta_n = \{\vec{x} \in \Sigma'' : s_{n-1}(\vec{x}) \ge 0 \ge s_n(\vec{x})\}, \ n \in \omega, \tag{10}$$

where $s_{-1}(\vec{x}) = \infty$, so $\Delta_0 = \Sigma' \cap \Sigma'' \subset \Delta_1$. Since $s_0(\vec{x}) \ge 0$ and $s_n(\vec{x}) \to -\infty$ $(n \to \infty)$, each $\vec{x} \in \Sigma''$ is contained in some Δ_n, hence $\Sigma'' = \bigcup_{n\in\mathbb{N}} \Delta_n$. Moreover,

$$\Delta_{n-1} \cap \Delta_n = \{\vec{x} \in \Sigma'' : s_{n-1}(\vec{x}) = 0\} \quad \text{for every } n \in \mathbb{N}.$$

For each $n \in \mathbb{N}$, Δ_n has an open neighborhood

$$U_n = \{\vec{x} \in \Sigma'' : s_{n-2}(\vec{x}) > 0 > s_{n+1}(\vec{x})\}$$

such that $U_n \cap \Delta_i = \emptyset$ if $|n - i| > 1$, which means that $\{\Delta_n : n \in \omega\}$ is locally finite.

Next, we define maps $t_n : S'' \to \mathbb{R}$, $n \in \mathbb{N}$, and $g_n : S'' \to (-1, \infty)$, $n \in \omega$, as follows:

$$t_0(\vec{y}) = \sum_{i=1}^{\infty} y_i = -y_0 \ge 1/2, \tag{11}$$

$$g_n(\vec{y}) = \varphi^{-1}(\varphi(y_n) - \varphi(y_0)) \quad \text{and} \tag{12}$$

$$t_n(\vec{y}) = \sum_{i=1}^{n} g_i(\vec{y}) + \sum_{i=n+1}^{\infty} y_i \quad \text{for } n \in \mathbb{N}, \tag{13}$$

where $\vec{y} = (y_i)_{i\in\omega} \in S''$. Then, $t_0(\vec{y}) \le -\varphi(y_0)$ because $\varphi^{-1}(-t_0(\vec{y})) = \varphi^{-1}(y_0) \ge y_0$. Note that $t_0(\vec{y}) = -\varphi(y_0)$ if and only if $y_0 = \varphi(y_0)$ if and only if $y_0 = -1/2$.

Since $\varphi(y_0) \le y_0 \le -1/2$, it follows that $g_n(\vec{y}) > y_n$ and hence $t_{n-1}(\vec{y}) < t_n(\vec{y})$ for every $n \in \mathbb{N}$. Moreover, $t_n(\vec{y}) \to \infty$ $(n \to \infty)$. Indeed, take $k \in \mathbb{N}$ so large that $|y_n| < 1/8$ for every $n \ge k$, which implies that

$$\begin{aligned} t_n(\vec{y}) - t_{n-1}(\vec{y}) &= g_n(\vec{y}) - y_n = \varphi^{-1}(\varphi(y_n) - \varphi(y_0)) - y_n \\ &= \varphi^{-1}(y_n - \varphi(y_0)) - y_n > \varphi^{-1}(3/8) - 1/8 = 1/4. \end{aligned}$$

We also consider the following closed sets in S":

$$D_n = \{\vec{y} \in S'' : t_{n-1}(\vec{y}) \le -\varphi(y_0) \le t_n(\vec{y})\}, \; n \in \omega \tag{14}$$

where $t_{-1}(\vec{y}) = -\infty$, so $D_0 = S' \cap S'' \subset D_1$. Since $t_0(\vec{y}) \le -\varphi(y_0)$ and $t_n(\vec{y}) \to \infty$ $(n \to \infty)$, each $\vec{y} \in S''$ is conatined in some D_n, hence $S'' = \bigcup_{n\in\mathbb{N}} D_n$. Moreover,

$$D_{n-1} \cap D_n = \{\vec{y} \in S'' : t_{n-1}(\vec{y}) = -\varphi(y_0)\}.$$

For each $n \in \mathbb{N}$, D_n has an open neighborhood

$$V_n = \{\vec{y} \in S'' : t_{n-2}(\vec{y}) < -\varphi(y_0) < s_{n+1}(\vec{y})\}$$

such that $V_n \cap D_i = \emptyset$ if $|n-i| > 1$, which means that $\{D_n : n \in \omega\}$ is locally finite.

Now, it suffices to construct homeomorphisms $\Psi_n : \Delta_n \to D_n$, $n \in \mathbb{N}$, such that

$$\Psi_n(\Delta_n \cap \Sigma_0) = D_n \cap S_0 \;\text{ and }\; \Psi_n|\Delta_n \cap \Delta_{n-1} = \Psi_{n-1}|\Delta_n \cap \Delta_{n-1},$$

where $\Psi_0 = \Psi'$.

For each $n \in \mathbb{N}$, we can define a map $\Psi_n : \Delta_n \to D_n$ as follows:

$$\begin{aligned} \Psi_n(\vec{x}) &= (f_0(\vec{x}), \dots, f_{n-1}(\vec{x}), x_n - s_{n-1}(\vec{x}), x_{n+1}, x_{n+2}, \dots) \\ &= (f_0(\vec{x}), \dots, f_{n-1}(\vec{x}), f_n(\vec{x}) - s_n(\vec{x}), x_{n+1}, x_{n+2}, \dots). \end{aligned} \tag{15}$$

Indeed, let $\vec{x} = (x_i)_{i\in\mathbb{N}} \in \Delta_n$, that is, $s_{n-1}(\vec{x}) \ge 0 \ge s_n(\vec{x})$. Then, $f_n(\vec{x}) - s_n(\vec{x}) \ge f_n(\vec{x}) > -1$. By (15) and (9),

$$\sum \Psi_n(\vec{x}) = \sum_{i=0}^{n-1} f_i(\vec{x}) + x_n - s_{n-1}(\vec{x}) + \sum_{i=n+1}^{\infty} x_i = 0,$$

which means $\Psi_n(\vec{x}) \in S$. It follows from (7) that $\Psi_n(\vec{x}) \in S''$. By (12), (15) and (8), we have

$$g_i(\Psi_n(\vec{x})) = \varphi^{-1}(\varphi(f_i(\vec{x})) - \varphi(f_0(\vec{x}))) = x_i \;\text{ for } 0 < i < n. \tag{16}$$

Combining this with (15), (13) and (7), we have

$$\begin{aligned} t_{n-1}(\Psi_n(\vec{x})) &= \sum_{i=1}^{n-1} g_i(\Psi_n(\vec{x})) + x_n - s_{n-1}(\vec{x}) + \sum_{i=n+1}^{\infty} x_i \\ &= \sum_{i=1}^{n} x_i - s_{n-1}(\vec{x}) + \sum_{i=n+1}^{\infty} x_i \\ &= -\varphi(f_0(\vec{x})) - s_{n-1}(\vec{x}). \end{aligned} \tag{17}$$

Since $s_{n-1}(\vec{x}) \ge 0$, it follows that $t_{n-1}(\Psi_n(\vec{x})) \le -\varphi(f_0(\vec{x}))$. On the other hand, since $s_n(\vec{x}) \le 0$, we have $x_n - s_{n-1}(\vec{x}) = f_n(\vec{x}) - s_n(\vec{x}) \ge f_n(\vec{x})$, hence by (12), (15), (7) and (8),

$$\begin{aligned} g_n(\Psi_n(\vec{x})) &= \varphi^{-1}(\varphi(x_n - s_{n-1}(\vec{x})) - \varphi(f_0(\vec{x}))) \\ &\ge \varphi^{-1}(\varphi(f_n(\vec{x})) + \textstyle\sum \vec{x}) = \varphi^{-1}(\varphi(x_n)) = x_n. \end{aligned}$$

Combining this with (13), (16) and (7), we have

$$\begin{aligned} t_n(\Psi_n(\vec{x})) &= \sum_{i=1}^{n} g_i(\Psi_n(\vec{x})) + \sum_{i=n+1}^{\infty} x_i \\ &\geq \sum_{i=1}^{n} x_i + \sum_{i=n+1}^{\infty} x_i = -\varphi(f_0(\vec{x})). \end{aligned}$$

Thus, we conclude $\Psi_n(\vec{x}) \in D_n$. It is obvious that $\Psi_n(\Delta_n \cap \Sigma_0) \subset D_n \cap S_0$.

We can also define a map $\Psi_n^* : D_n \to \Delta_n$ as follows:

$$\begin{aligned} \Psi_n^*(\vec{y}) &= (g_1(\vec{y}), \dots, g_{n-1}(\vec{y}), y_n - t_{n-1}(\vec{y}) - \varphi(y_0), y_{n+1}, y_{n+2}, \dots) \\ &= (g_1(\vec{y}), \dots, g_{n-1}(\vec{y}), g_n(\vec{y}) - t_n(\vec{y}) - \varphi(y_0), y_{n+1}, y_{n+2}, \dots). \end{aligned} \tag{18}$$

Indeed, let $\vec{y} = (y_i)_{i\in\omega} \in D_n$. Then, $t_{n-1}(\vec{y}) \leq -\varphi(y_0) \leq t_n(\vec{y})$. Observe that

$$\begin{gathered} y_n - t_{n-1}(\vec{y}) - \varphi(y_0) \geq y_n > -1 \quad \text{and} \\ \sum \Psi_n^*(\vec{y}) = -\varphi(y_0) \geq -y_0 \geq 1/2, \end{gathered}$$

hence $\Psi_n^*(\vec{y}) \in \Sigma''$. Moreover,

$$f_0(\Psi_n^*(\vec{y})) = \varphi^{-1}(-\textstyle\sum \Psi_n^*(\vec{y})) = y_0 \quad \text{and} \tag{19}$$

$$f_i(\Psi_n^*(\vec{y})) = \varphi^{-1}(\varphi(g_i(\vec{y})) + \varphi(y_0)) = y_i \quad \text{for } 0 < i < n. \tag{20}$$

It follows from (9) and (18) that

$$\begin{aligned} s_{n-1}(\Psi_n^*(\vec{y})) &= \sum_{i=0}^{n-1} y_i + (y_n - t_{n-1}(\vec{y}) - \varphi(y_0)) + \sum_{i=n+1}^{\infty} y_i \\ &= \sum_{i=0}^{\infty} y_i - t_{n-1}(\vec{y}) - \varphi(y_0) = -t_{n-1}(\vec{y}) - \varphi(y_0). \end{aligned} \tag{21}$$

Since $t_{n-1}(\vec{y}) \leq -\varphi(y_0)$, it follows that $s_{n-1}(\Psi_n^*(\vec{y})) \geq 0$. Since $t_n(\vec{y}) \geq -\varphi(y_0)$, we have the following by (8) and (18):

$$\begin{aligned} f_n(\Psi_n^*(\vec{y})) &= \varphi^{-1}\big(\varphi(g_n(\vec{y}) - t_n(\vec{y}) - \varphi(y_0)) - \textstyle\sum \Psi^*(\vec{y})\big) \\ &\leq \varphi^{-1}(\varphi(g_n(\vec{y})) + \varphi(y_0)) = \varphi^{-1}(\varphi(y_n)) = y_n, \end{aligned}$$

which implies

$$s_n(\Psi_n^*(\vec{y})) \leq \sum_{i=0}^{n} y_i + \sum_{i=n+1}^{\infty} y_i = \sum_{i=0}^{\infty} y_i = 0.$$

Thus, $\Psi_n^*(\vec{y}) \in \Delta_n$. Obviously, $\Psi_n^*(D_n \cap S_0) \in \Delta_n \cap \Sigma_0$. Furthermore, we can apply (21) to obtain

$$\begin{aligned} \Psi_n(\Psi_n^*(\vec{y})) = (y_0, \dots, y_{n-1},& \\ & y_n - t_{n-1}(\vec{y}) - \varphi(y_0) - s_{n-1}(\Psi_n^*(\vec{y})), y_{n+1}, y_{n+2}, \dots) \\ = (y_0, \dots, y_{n-1}, y_n, y_{n+1}, \dots&) = \vec{y}. \end{aligned}$$

For each $\vec{x} = (x_i)_{i\in\mathbb{N}} \in \Delta_n$, it follows from (16) and (17) that

$$\begin{aligned} \Psi_n^*(\Psi_n(\vec{x})) = (x_1, \dots, x_{n-1},& \\ & x_n - s_{n-1}(\vec{x}) - t_{n-1}(\Psi_n(\vec{x})) - \varphi(f_0(\vec{x})), x_{n+1}, x_{n+2}, \dots) \\ = (x_1, \dots, x_{n-1}, x_n, x_{n+1}, \dots&) = \vec{x} \end{aligned}$$

Therefore, $\Psi_n : \Delta_n \to D_n$ is a homeomorphism with $\Psi_n^{-1} = \Psi_n^*$, which satisfies $\Psi_n(\Delta_n \cap \Sigma_0) = D_n \cap S_0$.

It should be noticed that if $\vec{x} \in \Delta_0$, that is, $\sum \vec{x} = 1/2$, then

$$\Psi_0(\vec{x}) = (f_0(\vec{x}), x_1, x_2, \dots) = (-\sum \vec{x}, x_1, x_2, \dots) = \Psi'(\vec{x}).$$

For each $\vec{x} \in \Delta_{n-1} \cap \Delta_n$, $n \in \mathbb{N}$, since $s_{n-1}(\vec{x}) = 0$, we have

$$\Psi_n(\vec{x}) = (f_0(\vec{x}), \dots, f_{n-1}(\vec{x}), x_n, x_{n+1}, x_{n+2}, \dots) = \Psi_{n-1}(\vec{x}).$$

This completes the proof. □

Lemma 3. *The pair* (Σ, Σ_0) *is homeomorphic to* $(\square^{\mathbb{N}}\mathbb{R}, \boxdot^{\mathbb{N}}\mathbb{R})$.

Proof. The homeomorphism $\varphi : (-1, \infty) \to \mathbb{R}$ from the proof of Lemma 2 induces a homeomorphism $\varphi^{\mathbb{N}} : \square^{\mathbb{N}}(-1, \infty) \to \square^{\mathbb{N}}\mathbb{R}$ mapping the set Σ_0 onto $\boxdot^{\mathbb{N}}\mathbb{R}$. Under this homeomorphism the set Σ maps onto the following subspace

$$\Sigma' = \Big\{ (y_n)_{n\in\mathbb{N}} \in \square^{\mathbb{N}}\mathbb{R} : \sum_{n\in\mathbb{N}} \varphi^{-1}(y_n) \text{ converges} \Big\}.$$

Thus, we obtain

$$(\Sigma, \Sigma_0) \approx (\Sigma', \boxdot^{\mathbb{N}}\mathbb{R}). \tag{22}$$

It remains to show that the latter pair is homeomorphic to $(\square^{\mathbb{N}}\mathbb{R}, \boxdot^{\mathbb{N}}\mathbb{R})$. To this end, consider the subgroup

$$l_1 = \Big\{ (x_n)_{n\in\mathbb{N}} \in \square^{\mathbb{N}}\mathbb{R} : \sum_{n\in\mathbb{N}} |x_n| < \infty \Big\},$$

which is clopen in $\square^{\mathbb{N}}\mathbb{R}$.

We claim that $\vec{x} + l_1 \subset \Sigma'$ for any $\vec{x} = (x_n)_{n\in\mathbb{N}} \in \Sigma'$. Indeed, take any vector $\vec{z} = (z_n)_{n\in\mathbb{N}} \in l_1$. The convergence of the series $\sum_{n\in\mathbb{N}} |z_n|$ and $\sum_{n\in\mathbb{N}} \varphi^{-1}(x_n)$ implies the existence of a number $k \in \mathbb{N}$ such that $\max\{|z_n|, |\varphi^{-1}(x_n)|\} < \frac{1}{4}$ for all $n \geq k$. Then, for every $n \geq k$, since $|\varphi^{-1}(x_n)| = |x_n| < \frac{1}{4}$, we get $|x_n + z_n| < \frac{1}{2}$ and thus $\varphi^{-1}(x_n + z_n) = x_n + z_n$. Note that the convergence of the series $\sum_{n=k}^{\infty} \varphi^{-1}(x_n)$ implies the convergence of the series $\sum_{n=k}^{\infty} x_n$. Since the series $\sum_{n=k}^{\infty} z_n$ is absolutely convergent, the series

$$\sum_{n=k} (x_n + z_n) = \sum_{n=k}^{\infty} \varphi^{-1}(x_n + z_n)$$

is convergent, which proves the inclusions $\vec{x} + \vec{z} \in \Sigma'$ and $\vec{x} + l_1 \subset \Sigma'$.

For any $\alpha \in (0, 1)$, consider the sequence

$$\vec{x}^{\alpha} = \big((-1)^n (n^{-\alpha} + (n-1)^{-\alpha})\big)_{n\in\mathbb{N}}.$$

Then, $\vec{x}^{\alpha} \in \Sigma'$ because $\sum \vec{x}^{\alpha} = \lim_{n\to\infty} (-1)^n n^{-\alpha} = 0$. However, $\vec{x}^{\alpha} - \vec{x}^{\beta} \notin l_1$ for $\alpha \neq \beta \in (0, 1)$. Indeed, if $0 < \alpha < \beta < 1$ then

$$\begin{aligned} &\big|(-1)^n (n^{-\alpha} + (n-1)^{-\alpha}) - (-1)^n (n^{-\beta} + (n-1)^{-\beta})\big| \\ &\qquad = n^{-\alpha} - n^{-\beta} + (n-1)^{-\alpha} - (n-1)^{-\beta} \\ &\qquad \geq n^{-\alpha} - n^{-\beta} = n^{-\beta}(n^{\beta-\alpha} - 1) \\ &\qquad \geq n^{-\beta}(2^{\beta-\alpha} - 1) \quad \text{for every } n \geq 2. \end{aligned}$$

Since $\sum_{n=2}^{\infty} n^{-\beta} = \infty$, it follows that $\vec{x}^{\alpha} - \vec{x}^{\beta} \notin l_1$.

This implies that Σ' is the discrete union of continuum many shifts of the subgroup l_1 and hence

$$(\Sigma', \boxdot^{\mathbb{N}}\mathbb{R}) \approx (\mathfrak{c} \times l_1, 1 \times \boxdot^{\mathbb{N}}\mathbb{R}). \tag{23}$$

The same is true for the group $\square^{\mathbb{N}}\mathbb{R}$, that is, $\square^{\mathbb{N}}\mathbb{R}$ is a discrete union of continuum many shifts of l_1 and hence

$$(\square^{\mathbb{N}}\mathbb{R}, \boxdot^{\mathbb{N}}\mathbb{R}) \approx (\mathfrak{c} \times l_1, 1 \times \boxdot^{\mathbb{N}}\mathbb{R}). \tag{24}$$

Composing the homeomorphisms (22)–(24), we obtain the required homeomorphism $(\Sigma, \Sigma_0) \approx (\square^{\mathbb{N}}\mathbb{R}, \boxdot^{\mathbb{N}}\mathbb{R})$. $\square$

Unifying Lemmas 1–3, we get the promised

Corollary 7. *The pair (M, M_0) is homeomorphic to the pair $(\square^{\omega}\mathbb{R}, \boxdot^{\omega}\mathbb{R})$.*

9. The group $\mathcal{H}(\mathbb{R}_+)$ with the Whitney topology

In this section we shall recognize the topological structure of the triple

$$\big(\mathcal{H}(\mathbb{R}_+), \mathcal{H}^{\mathrm{PL}}(\mathbb{R}_+), \mathcal{H}_0(\mathbb{R}_+)\big) = \big(\mathcal{H}_+(\mathbb{R}_+), \mathcal{H}_+^{\mathrm{PL}}(\mathbb{R}_+), \mathcal{H}_0(\mathbb{R}_+)\big)$$

of the homeomorphism groups endowed with the Whitney topology. We shall show that the above triple is homeomorphic to the triple $(\square^{\omega} l_2, \square^{\omega} l_2^f, \boxdot^{\omega} l_2)$. The construction of a homeomorphism between those triples is made in Lemmas 4–7 below. All homeomorphism groups considered in this section are endowed with the Whitney topology.

Consider the following clopen subgroup of the group $\mathcal{H}(\mathbb{R}_+)$:

$$\mathcal{H}_u(\mathbb{R}_+) = \big\{h \in \mathcal{H}(\mathbb{R}_+) : \lim_{x\to\infty} |h(x) - x| = 0\big\},$$

which coincides with the closure of $\mathcal{H}_0(\mathbb{R}_+)$ in the topology of uniform convergence on $\mathcal{H}(\mathbb{R}_+)$. Let also $\mathcal{H}_u^{\mathrm{PL}}(\mathbb{R}_+) = \mathcal{H}_u(\mathbb{R}_+) \cap \mathcal{H}^{\mathrm{PL}}(\mathbb{R}_+)$.

Lemma 4. *If $\big(\mathcal{H}_u(\mathbb{R}_+), \mathcal{H}_u^{\mathrm{PL}}(\mathbb{R}_+), \mathcal{H}_0(\mathbb{R}_+)\big) \overset{W}{\approx} (\square^{\omega} l_2, \square^{\omega} l_2^f, \boxdot^{\omega} l_2)$, then*

$$\big(\mathcal{H}(\mathbb{R}_+), \mathcal{H}^{\mathrm{PL}}(\mathbb{R}_+), \mathcal{H}_0(\mathbb{R}_+)\big) \overset{W}{\approx} (\square^{\omega} l_2, \square^{\omega} l_2^f, \boxdot^{\omega} l_2).$$

Proof. Using the denseness of $\mathcal{H}^{\mathrm{PL}}(\mathbb{R}_+)$ in $\mathcal{H}(\mathbb{R}_+)$, choose a subset $D \subset \mathcal{H}^{\mathrm{PL}}(\mathbb{R}_+)$ that meets each left coset $h \circ \mathcal{H}_u(\mathbb{R}_+)$, $h \in \mathcal{H}(\mathbb{R}_+)$, at exactly one point. Moreover, we can select D so that $D \cap \mathcal{H}_u(\mathbb{R}_+) = \{\mathrm{id}\}$. Since $\mathcal{H}_u(\mathbb{R}_+)$ is a clopen subgroup of $\mathcal{H}(\mathbb{R}_+)$, the subspace D is closed and discrete in $\mathcal{H}(\mathbb{R}_+)$. Moreover, it is easy to check that D has cardinality of continuum. Observe that the map

$$\Psi : D \times \mathcal{H}_u(\mathbb{R}_+) \to \mathcal{H}(\mathbb{R}_+), \quad \Psi : (f, g) \mapsto f \circ g,$$

is a homeomorphism of the triple

$$\big(D \times \mathcal{H}_u(\mathbb{R}_+), D \times \mathcal{H}_u^{\mathrm{PL}}(\mathbb{R}_+), \{\mathrm{id}\} \times \mathcal{H}_0(\mathbb{R}_+)\big)$$

onto the triple $\big(\mathcal{H}(\mathbb{R}_+), \mathcal{H}^{\mathrm{PL}}(\mathbb{R}_+), \mathcal{H}_0(\mathbb{R}_+)\big)$.

Assuming that $\big(\mathcal{H}_u(\mathbb{R}_+), \mathcal{H}_u^{\mathrm{PL}}(\mathbb{R}_+), \mathcal{H}_0(\mathbb{R}_+)\big) \overset{W}{\approx} (\square^{\omega} l_2, \square^{\omega} l_2^f, \boxdot^{\omega} l_2)$, we conclude that the triple $\big(\mathcal{H}(\mathbb{R}_+), \mathcal{H}^{\mathrm{PL}}(\mathbb{R}_+), \mathcal{H}_0(\mathbb{R}_+)\big)$ is homeomorphic to the triple

$$(D \times \square^{\omega} l_2, D \times \square^{\omega} l_2^f, \{\mathrm{id}\} \times \boxdot^{\omega} l_2),$$

which is homeomorphic to $(\square^{\omega} l_2, \square^{\omega} l_2^f, \boxdot^{\omega} l_2)$ by Corollary 6. $\square$

Consider the following subsets of the group $\mathcal{H}_u(\mathbb{R}_+)$:

$$\mathcal{H}_u(\mathbb{R}_+,\mathbb{N}) = \{h \in \mathcal{H}_u(\mathbb{R}_+) : h|\mathbb{N} = \mathrm{id}\} \text{ and}$$
$$L = \{h \in \mathcal{H}_u(\mathbb{R}_+) : h \text{ is linear on each interval } [n, n+1],\ n \in \omega\}.$$

It is clear that $L \subset \mathcal{H}_u^{\mathrm{PL}}(\mathbb{R}_+)$ and $L_0 = L \cap \mathcal{H}_0(\mathbb{R}_+) \subset \mathcal{H}_0^{\mathrm{PL}}(\mathbb{R}_+)$.

Lemma 5. *The triple* $\big(\mathcal{H}_u(\mathbb{R}_+), \mathcal{H}_u^{\mathrm{PL}}(\mathbb{R}_+), \mathcal{H}_0(\mathbb{R}_+)\big)$ *is homeomorphic to the triple*

$$\big(L \times \mathcal{H}_u(\mathbb{R}_+,\mathbb{N}), L \times \mathcal{H}_u^{\mathrm{PL}}(\mathbb{R}_+,\mathbb{N}), L_0 \times \mathcal{H}_0(\mathbb{R}_+,\mathbb{N})\big).$$

Proof. The following is a required homeomorphism between the triples:

$$\Psi : L \times \mathcal{H}_u(\mathbb{R}_+,\mathbb{N}) \to \mathcal{H}_u(\mathbb{R}_+),\ \ \Psi : (f,g) \mapsto f \circ g.$$

Indeed, for each $h \in \mathcal{H}_u(\mathbb{R}_+)$, the PL homeomorphism $\phi(h) : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by $\phi(h)|\omega = h|\omega$ and $\phi(h)$ is linear on $[n, n+1]$ for each $n \in \omega$. The correspondence $h \mapsto (\phi(h), \phi(h)^{-1}h)$ gives the inverse of Ψ^{-1}. □

Thus, to recognize the topology of the triple $\big(\mathcal{H}(\mathbb{R}_+), \mathcal{H}^{\mathrm{PL}}(\mathbb{R}_+), \mathcal{H}_0(\mathbb{R}_+)\big)$, it suffices to know the topological types of the pair (L, L_0) and the triple

$$\big(\mathcal{H}_u(\mathbb{R}_+,\mathbb{N}), \mathcal{H}_u^{\mathrm{PL}}(\mathbb{R}_+,\mathbb{N}), \mathcal{H}_0(\mathbb{R}_+,\mathbb{N})\big). \tag{25}$$

Lemma 6. *The pair* (L, L_c) *is homeomorphic to* $(\square^\omega\mathbb{R}, \boxdot^\omega\mathbb{R})$.

Proof. Observe that the map

$$\Psi : L \to M,\ \ \Psi : h \mapsto (h(n))_{n\in\omega},$$

is a homeomorphism from the pair (L, L_c) onto the pair (M, M_0) considered in Section 8. Since $(M, M_0) \approx (\square^\omega\mathbb{R}, \boxdot^\omega\mathbb{R})$ by Corollary 7, we have the result. □

Lemma 7. *The triple* (25) *is homeomorphic to* $(\square^\omega l_2, \square^\omega l_2^f, \boxdot^\omega l_2)$.

Proof. The box power $\square^\omega\mathcal{H}_+(\mathbb{I})$ of the group $\mathcal{H}_+(\mathbb{I})$ of orientation-preserving homeomorphisms of the closed interval $\mathbb{I} = [0,1]$ is the group with the group operation:

$$\vec{f} \circ \vec{g} = (f_n \circ g_n)_{n\in\omega} \ \text{ for each } \vec{f} = (f_n)_{n\in\omega}, \vec{g} = (g_n)_{n\in\omega}$$

Consider the following open subgroup of $\square^\omega\mathcal{H}_+(\mathbb{I})$:

$$\square_u^\omega\mathcal{H}_+(\mathbb{I}) = \{(h_n)_{n\in\omega} : \lim_{n\to\infty} \|h_n - \mathrm{id}\| = 0\},$$

where $\|f - g\| = \sup_{t\in\mathbb{I}} |f(t) - g(t)|$.

Observe that the triple (25) is homeomorphic to the following triple:

$$\big(\square_u^\omega\mathcal{H}_+(\mathbb{I}), \square_u^\omega\mathcal{H}_+^{\mathrm{PL}}(\mathbb{I}), \boxdot^\omega\mathcal{H}_+(\mathbb{I})\big) \tag{26}$$

by the homeomorphism $G : \mathcal{H}_u(\mathbb{R}_+,\mathbb{N}) \to \square_u^\omega\mathcal{H}_+(\mathbb{I})$ assigning to each $h \in \mathcal{H}_u(\mathbb{R}_+,\mathbb{N})$ the sequence of homeomorphisms $(h_n)_{n\in\omega} \in \square_u^\omega\mathcal{H}_+(\mathbb{I})$, where each $h_n \in \mathcal{H}_+(\mathbb{I})$ is defined as follows:

$$h_n(t) = h(n+t) - n \ \text{ for } t \in \mathbb{I}.$$

For the open subgroup

$$\square_{l_1}^\omega\mathcal{H}_+(\mathbb{I}) = \{(h_n)_{n\in\omega} : \textstyle\sum_{n=0}^\infty \|h_n - \mathrm{id}\| < \infty\}$$

of $\square^\omega\mathcal{H}_+(\mathbb{I})$, we get $\boxdot^\omega\mathcal{H}_+(\mathbb{I}) \subset \square_{l_1}^\omega\mathcal{H}_+(\mathbb{I}) \subset \square_u^\omega\mathcal{H}_+(\mathbb{I})$. Since $\square^\omega\mathcal{H}_+^{\mathrm{PL}}(\mathbb{I})$ is dense in $\square^\omega\mathcal{H}_+(\mathbb{I})$, we can choose $D \subset \square^\omega\mathcal{H}_+^{\mathrm{PL}}(\mathbb{I})$ which meets each left coset $\vec{g} \circ \square_{l_1}^\omega\mathcal{H}_+(\mathbb{I})$, $\vec{g} \in \square_u^\omega\mathcal{H}_+(\mathbb{I})$, at exactly one point. Moreover, $D \cap \square_{l_1}^\omega\mathcal{H}_+(\mathbb{I})$ is the neutral element.

Then, the map

$$\Psi : D \times \square^{\omega}_{l_1}\mathcal{H}_+(\mathbb{I}) \to \square^{\omega}_{u}\mathcal{H}_+(\mathbb{I}), \quad \Psi : (\vec{g}, \vec{h}) \mapsto \vec{g} \circ \vec{h},$$

is a homeomorphism mapping the triple

$$\big(D \times \square^{\omega}_{l_1}\mathcal{H}_+(\mathbb{I}), D \times \square^{\omega}_{l_1}\mathcal{H}^{\mathrm{PL}}_+(\mathbb{I}), \{0\} \times \boxdot^{\omega}\mathcal{H}_+(\mathbb{I})\big) \tag{27}$$

onto the triple (26).

By the same reason, the triple

$$(\square^{\omega}\mathcal{H}_+(\mathbb{I}), \square^{\omega}\mathcal{H}^{\mathrm{PL}}_+(\mathbb{I}), \boxdot^{\omega}\mathcal{H}_+(\mathbb{I})) \tag{28}$$

is homeomorphic to (27). On the other hand, $(\mathcal{H}_+(\mathbb{I}), \mathcal{H}^{\mathrm{PL}}_+) \approx (l_2, l_2^f)$ by Proposition 5, hence the above triple (28) is homeomorphic to $(\square^{\omega} l_2, \square^{\omega} l_2^f, \boxdot^{\omega} l_2)$. Therefore, the triples (27) and (25) are also homeomorphic to $(\square^{\omega} l_2, \square^{\omega} l_2^f, \boxdot^{\omega} l_2)$. □

With Lemmas 4–7 at hands we are able to prove the promised

Proposition 10. $(\mathcal{H}(\mathbb{R}_+), \mathcal{H}^{\mathrm{PL}}(\mathbb{R}_+), \mathcal{H}_0(\mathbb{R}_+)) \overset{W}{\approx} (\square^{\omega} l_2, \square^{\omega} l_2^f, \boxdot^{\omega} l_2)$.

Proof. By Lemmas 5–7, we have the following homeomorphisms of the triples:

$$\begin{aligned}
&\big(\mathcal{H}_u(\mathbb{R}_+), \mathcal{H}^{\mathrm{PL}}_u(\mathbb{R}_+), \mathcal{H}_0(\mathbb{R}_+)\big)\\
&\qquad \overset{W}{\approx} \big(L \times \mathcal{H}_u(\mathbb{R}_+, \mathbb{N}), L \times \mathcal{H}^{\mathrm{PL}}_u(\mathbb{R}_+, \mathbb{N}), L_0 \times \mathcal{H}_0(\mathbb{R}_+)\big)\\
&\qquad \overset{W}{\approx} \big(\square^{\omega}\mathbb{R} \times \square^{\omega} l_2, \square^{\omega}\mathbb{R} \times \square^{\omega} l_2^f, \boxdot^{\omega}\mathbb{R} \times \boxdot^{\omega} l_2\big)\\
&\qquad \approx \big(\square^{\omega}(\mathbb{R} \times l_2), \square^{\omega}(\mathbb{R} \times l_2^f), \boxdot^{\omega}(\mathbb{R} \times l_2)\big)\\
&\qquad \approx \big(\square^{\omega} l_2, \square^{\omega} l_2^f, \boxdot^{\omega} l_2\big).
\end{aligned}$$

Lemma 4 completes the proof. □

10. The group $\mathcal{H}_+(\mathbb{R})$ with the Whitney topology

The principal result of this section is the following:

Proposition 11. $\big(\mathcal{H}_+(\mathbb{R}), \mathcal{H}^{\mathrm{PL}}_+(\mathbb{R}), \mathcal{H}_0(\mathbb{R})\big) \overset{W}{\approx} (\square^{\omega} l_2, \square^{\omega} l_2^f, \boxdot^{\omega} l_2)$.

Proof. For a real number a, consider the homeomorphism $l_a \in \mathcal{H}(\mathbb{R})$ defined by the three conditions:

- $l_a(0) = a$,
- $l_a(x) = x$ if $|x| \geq 2|a|$;
- l_a is linear on the intervals $[-2|a|, 0]$ and $[0, 2|a|]$.

It is easy to see that the correspondence $a \mapsto l_a$ determines a homeomorphic embedding of the real line $\mathbb{R}$ into the group $\mathcal{H}^{\mathrm{PL}}_c(\mathbb{R})$.

In the group $\mathcal{H}_+(\mathbb{R})$ consider the closed subgroups

$$\mathcal{H}(\mathbb{R}_+) = \{h \in \mathcal{H}_+(\mathbb{R}) : h|(-\infty, 0] = \mathrm{id}\},$$
$$\mathcal{H}(\mathbb{R}_-) = \{h \in \mathcal{H}_+(\mathbb{R}) : h|[0, +\infty) = \mathrm{id}\},$$

and let $\mathcal{H}_0(\mathbb{R}_+) = \mathcal{H}(\mathbb{R}_+) \cap \mathcal{H}_0(\mathbb{R})$, $\mathcal{H}_0(\mathbb{R}_-) = \mathcal{H}(\mathbb{R}_-) \cap \mathcal{H}_0(\mathbb{R})$.

Observe that the map

$$\Psi : \mathbb{R} \times \mathcal{H}(\mathbb{R}_+) \times \mathcal{H}(\mathbb{R}_-) \to \mathcal{H}_+(\mathbb{R}), \ \Psi : (a, f, g) \mapsto l_a \circ f \circ g,$$

is a homeomorphism mapping the triple

$$\big(\mathbb{R}\times\mathcal{H}(\mathbb{R}_+)\times\mathcal{H}(\mathbb{R}_-),\ \mathbb{R}\times\mathcal{H}^{\mathrm{PL}}(\mathbb{R}_+)\times\mathcal{H}^{\mathrm{PL}}(\mathbb{R}_-),\mathbb{R}\times\mathcal{H}_0(\mathbb{R}_+)\times\mathcal{H}_0(\mathbb{R}_-)\big)$$

onto the triple

$$\big(\mathcal{H}_+(\mathbb{R}),\mathcal{H}^{\mathrm{PL}}_+(\mathbb{R}),\mathcal{H}_0(\mathbb{R})\big). \tag{29}$$

By Proposition 10, the triples

$$\big(\mathcal{H}(\mathbb{R}_+),\mathcal{H}^{\mathrm{PL}}(\mathbb{R}_+),\mathcal{H}_0(\mathbb{R}_+)\big) \text{ and } \big(\mathcal{H}(\mathbb{R}_-),\mathcal{H}^{\mathrm{PL}}(\mathbb{R}_-),\mathcal{H}_0(\mathbb{R}_-)\big)$$

are homeomorphic to $(\square^\omega l_2,\square^\omega l_2^f,\boxdot^\omega l_2)$. Consequently, the triple (29) is homeomorphic to the triple

$$(\mathbb{R}\times\square^\omega l_2\times\square^\omega l_2,\mathbb{R}\times\square^\omega l_2^f\times\square^\omega l_2^f,\mathbb{R}\times\boxdot^\omega l_2\times\boxdot^\omega l_2),$$

which is homeomorphic to $(\square^\omega l_2,\square^\omega l_2^f,\boxdot^\omega l_2)$ because of the homeomorphism

$$\big(\mathbb{R}\times l_2\times l_2,\mathbb{R}\times l_2^f\times l_2^f\big)\approx(l_2,l_2^f). \qquad \square$$

11. Groups of orientation preserving homeomorphisms of a graph

In this section, for any graph Γ we shall recognize the topological type of the triple $\big(\mathcal{H}_+(\Gamma),\mathcal{H}^{\mathrm{PL}}_+(\Gamma),\mathcal{H}_0(\Gamma)\big)$ endowed with the compact-open or Whitney topology.

We recall that o_Γ stands for the number of isolated circles of Γ and

$$e_\Gamma=\kappa_\Gamma+\nu_\Gamma\cdot\aleph_0,$$

where κ_Γ (resp. ν_Γ) is the number of connected components of $\Gamma\setminus\Gamma^{(0)}$ having (non)-compact closure in Γ.

Theorem 8. *For any graph Γ the triple $\big(\mathcal{H}_+(\Gamma),\mathcal{H}^{\mathrm{PL}}_+(\Gamma),\mathcal{H}_0(\Gamma)\big)$ of the homeomorphism groups endowed with the Whitney topology is homeomorphic to*

$$(\square^{o_\Gamma}\mathbb{T}\times\square^{e_\Gamma}l_2,\square^{o_\Gamma}\mathbb{T}\times\square^{e_\Gamma}l_2^f,\boxdot^{o_\Gamma}\mathbb{T}\times\boxdot^{e_\Gamma}l_2).$$

Proof. Write the complement $\Gamma\setminus\Gamma^{(0)}$ as the disjoint union $\bigcup_{\alpha\in A}E_\alpha$ of connected components. We recall that $\Gamma^{(0)}$ is the (closed discrete) set of topological vertices of Γ. For every α let $\overline{E}_\alpha$ and ∂E_α be the closure and the boundary of E_α in Γ. It is clear that $\overline{E}_\alpha$ is homeomorphic to one of the spaces: $\mathbb{I}$, $\mathbb{T}$, $\mathbb{R}$ or $\mathbb{R}_+$.

Let $\mathcal{H}_+(\overline{E}_\alpha,\partial E_\alpha)$ denote the group of orientation-preserving homeomorphisms of $\overline{E}_\alpha$ that do not move the points of the boundary ∂E_α. We shall identify the group $\mathcal{H}_+(\overline{E}_\alpha,\partial E_\alpha)$ with the subgroup $\{h\in\mathcal{H}_+(\Gamma):h|\Gamma\setminus E_\alpha=\mathrm{id}\}$ of $\mathcal{H}_+(\Gamma)$.

Let also

$$\mathcal{H}^{\mathrm{PL}}_+(\overline{E}_\alpha,\partial E_\alpha)=\mathcal{H}_+(\overline{E}_\alpha,\partial)\cap\mathcal{H}^{\mathrm{PL}}(\Gamma) \ \text{ and } \ \mathcal{H}_0(\overline{E}_\alpha,\partial E_\alpha)=\mathcal{H}_+(\overline{E}_\alpha,\partial E_\alpha)\cap\mathcal{H}_0(\Gamma).$$

It is easy to see that the group isomorphism

$$\Xi:\mathcal{H}_+(\Gamma)\to\square_{\alpha\in A}\mathcal{H}_+(\overline{E}_\alpha,\partial E_\alpha),\ \ \Xi:h\mapsto(h|\overline{E}_\alpha)_{\alpha\in A},$$

establishes the topological equivalence of the triples $(\mathcal{H}_+(\Gamma),\mathcal{H}^{\mathrm{PL}}_+(\Gamma),\mathcal{H}_0(\Gamma))$ and

$$(\square_{\alpha\in A}\mathcal{H}_+(\overline{E}_\alpha,\partial E_\alpha),\square_{\alpha\in A}\mathcal{H}^{\mathrm{PL}}_+(\overline{E}_\alpha,\partial E_\alpha),\boxdot_{\alpha\in A}\mathcal{H}_0(\overline{E}_\alpha,\partial E_\alpha)).$$

In the index set A consider the following three subsets:

$$\begin{aligned} S&=\{\alpha\in A:\bar{E}_\alpha \ \text{ is an isolated circle}\},\\ K&=\{\alpha\in A:\bar{E}_\alpha \ \text{ is compact}\},\\ N&=\{\alpha\in A:\bar{E}_\alpha \ \text{ is not compact}\}.\end{aligned}$$

and observe that $|S| = o_\Gamma$ and $|K| + |N| \cdot \aleph_0 = \kappa_\Gamma + \nu_\Gamma \cdot \aleph_0 = e_\Gamma$.

Propositions 5, 6, 10, 11 imply that for every $\alpha \in A$ the triple

$$\big(\mathcal{H}_+(\overline{E}_\alpha, \partial E_\alpha), \mathcal{H}_+^{\mathrm{PL}}(\overline{E}_\alpha, \partial E_\alpha), \mathcal{H}_0(\overline{E}_\alpha, \partial E_\alpha)\big)$$

is homeomorphic to

- $(\mathbb{T} \times l_2, \mathbb{T} \times l_2^f, \mathbb{T} \times l_2)$ if $\alpha \in S$,
- (l_2, l_2^f, l_2) if $\alpha \in K \setminus S$,
- $(\square^\omega l_2, \square^\omega l_2^f, \boxdot^\omega l_2)$ if $\alpha \in N$.

Then we get the following homeomorphisms

$$\begin{aligned}
&\big(\mathcal{H}_+(\Gamma), \mathcal{H}_+^{\mathrm{PL}}(\Gamma), \mathcal{H}_0(\Gamma)\big) \\
&\approx \big(\square_{\alpha\in A}\mathcal{H}_+(\overline{E}_\alpha, \partial E_\alpha), \square_{\alpha\in A}\mathcal{H}_+^{\mathrm{PL}}(\overline{E}_\alpha, \partial E_\alpha), \boxdot_{\alpha\in A}\mathcal{H}_0(\overline{E}_\alpha, \partial E_\alpha)\big) \\
&\approx (\square^S\mathbb{T} \times \square^K l_2 \times \square^N \square^\omega l_2, \square^S\mathbb{T} \times \square^K l_2^f \times \square^N \square^\omega l_2^f, \boxdot^S\mathbb{T} \times \boxdot^K l_2 \times \boxdot^N \boxdot^\omega l_2) \\
&\approx (\square^{o_\Gamma}\mathbb{T} \times \square^{e_\Gamma} l_2, \square^{o_\Gamma}\mathbb{T} \times \square^{e_\Gamma} l_2^f, \boxdot^{o_\Gamma}\mathbb{T} \times \boxdot^{e_\Gamma} l_2).
\end{aligned}$$

$\square$

By analogy we can prove the compact-open version of Theorem 8:

Theorem 9. *For any graph Γ the triple $\big(\mathcal{H}_+(\Gamma), \mathcal{H}_+^{\mathrm{PL}}(\Gamma), \mathcal{H}_0(\Gamma)\big)$ of the homeomorphism groups endowed with the compact-open topology is homeomorphic to*

$$(\textstyle\prod^{o_\Gamma}\mathbb{T} \times \prod^{e_\Gamma} l_2, \prod^{o_\Gamma}\mathbb{T} \times \prod^{e_\Gamma} l_2^f, \boxdot\!\!\prod^{o_\Gamma}\mathbb{T} \times \boxdot\!\!\prod^{e_\Gamma} l_2).$$

12. The automorphism groups of digraphs

In this section, we recall some infomation about automorphism groups $\mathrm{Aut}(\vec{\Gamma})$ of digraphs $\vec{\Gamma}$, which will be used in the next section to identify the quotient group $\mathcal{H}(\Gamma)/\mathcal{H}_+(\Gamma)$ with $\mathrm{Aut}(\vec{\Gamma})$ for certain digraph $\vec{\Gamma}$ associated to Γ.

By a *directed graph* (briefly, *digraph*), we understand a pair $\vec{\Gamma} = (V, E)$, where V is the set of vertices and $E \subset V \times V$ is the set of (directed) edges of $\vec{\Gamma}$. A digraph $\vec{\Gamma} = (V, E)$ is said to be *countable* if the set V is at most countable (hence so is $V \cup E$). By an *automorphism* of a digraph (V, E), we understand a bijective map $h : V \to V$ such that $(x, y) \in E$ if and only if $(h(x), h(y)) \in E$. Such automorphisms form a group $\mathrm{Aut}(\vec{\Gamma})$ called the *automorphism group* of the digraph $\vec{\Gamma} = (V, E)$.

The *geometric realization* $|\vec{\Gamma}|$ of a digraph $\vec{\Gamma} = (V, E)$ is the subspace $\bigcup_{(x,y)\in E}[x, y]$ of the Banach space

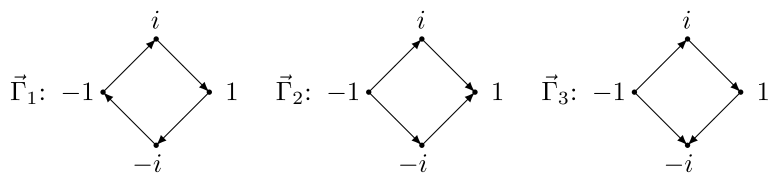
$$l_1(V) = \big\{\vec{x} = (x_v)_{v\in V} \in \mathbb{R}^V : \|\vec{x}\| = \textstyle\sum_{v\in V}|x_v| < \infty\big\},$$

where each vertex $x \in V$ is identified with the characteristic function $\chi_{\{x\}} : V \to \{0, 1\} \subset \mathbb{R}$ and $[x, y] = \{(1-t)x + ty : t \in [0, 1]\}$ is the closed interval in $l_1(V)$ connecting points $x, y \in l_1(V)$. Then, we also regard $|\vec{\Gamma}|$ as a graph (i.e., a simplicial complex) with the vertices $x \in V$ and the edges $[x, y]$, $(x, y) \in E$.

Each automorphism h of a digraph $\vec{\Gamma} = (V, E)$ determines the unique simplicial homeomorphism $\tilde{h} : |\vec{\Gamma}| \to |\vec{\Gamma}|$ such that $\tilde{h}(x) = h(x)$ and $\tilde{h}$ is linear on each interval $[x, y] \subset |\vec{\Gamma}|$ with $(x, y) \in V$. Identifying each $h \in \mathrm{Aut}(\vec{\Gamma})$ with $\tilde{h} \in \mathcal{H}(|\vec{\Gamma}|)$, the automorphism group $\mathrm{Aut}(\vec{\Gamma})$ of a digraph $\vec{\Gamma}$ can be embedded into the homomorpism group $\mathcal{H}(|\vec{\Gamma}|)$ of its geometric realization, where $\mathrm{Aut}(\vec{\Gamma}) \subset \mathcal{H}^{\mathrm{PL}}(|\vec{\Gamma}|)$.

One shoud note that $\mathrm{Aut}(\vec{\Gamma}) \not\cong \mathrm{Aut}(\vec{\Gamma}')$ even if $|\vec{\Gamma}| = |\vec{\Gamma}'|$ as simplicial complexes. As examples, consider three digraphs $\vec{\Gamma}_1 = (V, E_1)$, $\vec{\Gamma}_2 = (V, E_1)$, $\vec{\Gamma}_3 = (V, E_3)$ with the same set of vertices $V = \{1, -1, i, -i\} \subset \mathbb{C}$ and the sets of edges

$$\begin{aligned} E_1 &= \{(-1, i), (i, 1), (1, -i), (-i, -1)\}, \\ E_2 &= \{(-1, i), (-1, -i), (i, 1), (1, -i)\}, \\ E_3 &= \{(-1, i), (-1, -i), (i, 1), (-i, 1)\}. \end{aligned}$$



Then, $|\vec{\Gamma}_1| = |\vec{\Gamma}_2| = |\vec{\Gamma}_3|$ but

$$\mathrm{Aut}(\vec{\Gamma}_1) \cong \mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}, \ \mathrm{Aut}(\vec{\Gamma}_2) \cong \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}, \ \mathrm{Aut}(\vec{\Gamma}_3) = \{\mathrm{id}\}.$$

It is easy to see that the subgroup $\mathrm{Aut}(\vec{\Gamma})$ is discrete with respect to the Whitney topology inherited from $\mathcal{H}(|\vec{\Gamma}|)$. In case of the compact-open topology, $\mathrm{Aut}(\vec{\Gamma})$ can be embedded as a closed subspace of V^V, where V is the space of all vertices of $\vec{\Gamma}$ with discrete topology. Then, it is obvious that $\mathrm{Aut}(\vec{\Gamma}) \subset V^V$ is a topological group. Since the Cartesian product of 0-dimensional metrizable spaces is also 0-dimensional [20], we have the following proposition.

Proposition 12. *The automorphism group* $\mathrm{Aut}(\vec{\Gamma})$ *endowed with the compact-open (resp. WHitney) topology is a* 0*-dimensional (discrete) topological group.*

The following classification of separable completely metrizable 0-dimensional homogeneous spaces is well known, but for completeness, we shall give a proof.

Proposition 13. *Every separable completely metrizable* 0*-dimensional homogeneous space is homeomorphic to one of the spaces:* 2^ω*,* $2^\omega \times \mathbb{N}$*,* $\mathbb{N}^\omega$*,* $\mathbb{N}$*, or finite space.*

Proof. Let X be a separable completely metrizable 0-dimensional homogeneous space. If X has an isolated point, then every point of X is isolated by the homogeneity of X. Hence, X is discrete space. Since X is separable, X must be $\mathbb{N}$ or finite space.

Assume that X has no isolated point. In case X is compact, X is homemorphic to 2^ω by the characterization of the Cantor space, that is, every nonempty compact metrizable 0-dimensional perfect space is homeomorphic to 2^ω [13]. Finaly, we shall consider the non-compact case. If X is not locally compact, then every compact set in X has empty interior by the homogeneity of X. Since every separable completely metrizable nonempty 0-dimensional nowhere localy compact space is homeomorphic to the space of irrational numbers $\mathbb{N}^\omega$ [13], we have that X is homeomorphic to $\mathbb{N}^\omega$. On the other hand, if X is locally compact, then X has an open cover $\mathcal{U}$ so that each member of $\mathcal{U}$ has compact closure. Since X is 0-dimensional space, we can take open cover refinement $\mathcal{V}$ of $\mathcal{U}$ such that each two members of $\mathcal{V}$ are disjoint. Then, each member V of $\mathcal{V}$ is clopen, hence compact. Since X has no isolated point, also V is so. Thus, V is homeomorhic to 2^ω. Therefore, X is homeomrphic to $2^\omega \times \mathbb{N}$. $\square$

Thus, we have the following corollary:

Corollary 8. *For a countable digraph $\vec{\Gamma}$ the automorphism group* $\mathrm{Aut}(\vec{\Gamma})$ *endowed with the topology is homeomorphic to one of the spaces: 2^ω, $2^\omega \times \mathbb{N}$, $\mathbb{N}^\omega$, $\mathbb{N}$, or $n = \{0, \dots, n-1\} \in \mathbb{N}$.*

13. Describing the quotient group $\mathcal{H}(\Gamma)/\mathcal{H}_+(\Gamma)$

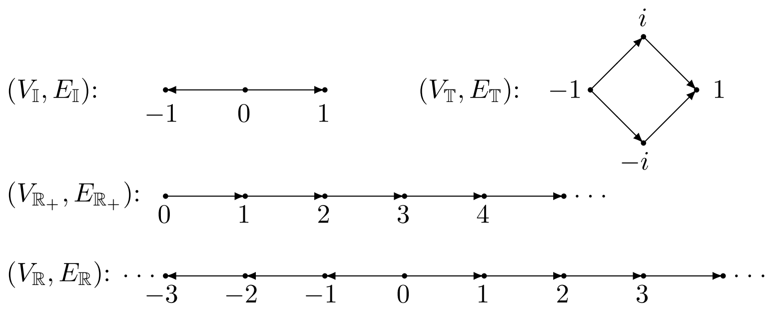
In this section, to each graph Γ we assign a digraph $\vec{\Gamma}$ whose automorphism group $\mathrm{Aut}(\vec{\Gamma})$ is naturally isomorphic to the quotient group $\mathcal{H}(\Gamma)/\mathcal{H}_+(\Gamma)$, and will show that the homeomorphism group $\mathcal{H}(\Gamma)$ is a semi-direct product of $\mathcal{H}_+(\Gamma)$ and $\mathcal{H}(\Gamma)/\mathcal{H}_+(\Gamma)$. We also give a proof of that the homeomorphism group $\mathcal{H}(\Gamma)$ endowed with the compact-open topology is a topological group. Probably, these are known result, but we could not find a corresponding reference.

We recall that a (topological) group $G = N \ltimes H$ is a *semi-direct product* of two (topological) groups N and H if N is a normal subgroup of G, H is a subgroup of G and the map $h : N \times H \to G$, $h : (x, y) \mapsto xy$, is a bijection (a homeomorphism).

We shall identify $\mathcal{H}(\Gamma)/\mathcal{H}_+(\Gamma)$ with the automorphism group $\mathrm{Aut}(\vec{\Gamma})$ of a certain digraph $\vec{\Gamma}$ whose geometric realization $|\vec{\Gamma}|$ is homeomoprhic to Γ.

First we do that for the graphs, PL homeomorphic to $\mathbb{I} = [0, 1]$, $\mathbb{R}$, $\mathbb{R}_+$ or the circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ endowed with the PL-structure such that the covering map $\mathbb{R} \to \mathbb{T}$, $t \mapsto e^{2\pi i t}$, is piecewise linear. The closed interval $\mathbb{I}$, the circle $\mathbb{T}$, the closed half-line $\mathbb{R}_+$ and the real line $\mathbb{R}$ are respectively PL homeomorphic to the geometric realizations of the digraphs $\vec{\mathbb{I}} = (V_\mathbb{I}, E_\mathbb{I})$, $\vec{\mathbb{T}} = (V_\mathbb{T}, E_\mathbb{T})$, $\vec{\mathbb{R}}_+ = (V_{\mathbb{R}_+}, E_{\mathbb{R}_+})$ and $\vec{\mathbb{R}} = (V_\mathbb{R}, E_\mathbb{R})$, where

$$V_\mathbb{I} = \{-1, 0, 1\},\ E_\mathbb{I} = \{(0, -1), (0, 1)\};$$
$$V_\mathbb{T} = \{1, -1, i, -i\},\ E_\mathbb{T} = \{(i, 1), (-i, 1), (-1, i), (-1, -i)\};$$
$$V_{\mathbb{R}_+} = \omega,\ E_{\mathbb{R}_+} = \{(n, n+1) : n \in \omega\} \text{ and}$$
$$V_\mathbb{R} = \mathbb{Z},\ E_\mathbb{R} = \{(n, n+1), (-n, -n-1) : n \in \omega\}.$$



Then, we have $\mathrm{Aut}(\vec{\mathbb{I}}) = \{\mathrm{id}_\mathbb{I}, \theta_\mathbb{I}\}$, $\mathrm{Aut}(\vec{\mathbb{T}}) = \{\mathrm{id}_\mathbb{T}, \theta_\mathbb{T}\}$ and $\mathrm{Aut}(\vec{\mathbb{R}}) = \{\mathrm{id}_\mathbb{R}, \theta_\mathbb{R}\}$ but $\mathrm{Aut}(\vec{\mathbb{R}}_+) = \{\mathrm{id}_{\mathbb{R}_+}\}$, which are naturally isomorphic to $\mathcal{H}(\mathbb{I})/\mathcal{H}_+(\mathbb{I})$, $\mathcal{H}(\mathbb{T})/\mathcal{H}_+(\mathbb{T})$, $\mathcal{H}(\mathbb{R})/\mathcal{H}_+(\mathbb{R})$ and $\mathcal{H}(\mathbb{R}_+)/\mathcal{H}_+(\mathbb{R}_+)$, respectively.

Write $\Gamma \setminus \Gamma^{(0)}$ as the disjoint union $\Gamma \setminus \Gamma^{(0)} = \bigcup_{\alpha \in A} E_\alpha$ of connected components and for every $\alpha \in A$ fix a PL homeomorphism $\varphi_\alpha : \overline{E}_\alpha \to M_\alpha$ onto a space

$M_\alpha \in \{\mathbb{I}, \mathbb{T}, \mathbb{R}, \mathbb{R}_+\}$. If $\overline{E}_\alpha$ is a non-isolated circle, we additionally assume that $\varphi_\alpha(\partial E_\alpha) \subset \{1\} \subset \mathbb{T}$.

Now, we define the digraph $\vec{\Gamma} = (V, E)$ by letting

$$V = \Gamma^{(0)} \cup \bigcup_{\alpha \in A} \varphi_\alpha^{-1}(V_{M_\alpha}) \text{ and}$$
$$E = \left\{(\varphi_\alpha^{-1}(x), \varphi_\alpha^{-1}(y)) : \alpha \in A,\ (x,y) \in E_{M_\alpha}\right\}.$$

The digraph $\vec{\Gamma}$ will be called the *associated digraph* to the graph Γ.

Since Γ is PL-homeomorphic to the geometric realization $|\vec{\Gamma}|$ of $\vec{\Gamma}$, we can identify Γ with $|\vec{\Gamma}|$ and $\mathcal{H}(\Gamma)$ with $\mathcal{H}(|\vec{\Gamma}|)$. Also we identify the automorphism group $\mathrm{Aut}(\vec{\Gamma})$ with the subgroup of $\mathcal{H}^{\mathrm{PL}}(\Gamma) = \mathcal{H}^{\mathrm{PL}}(|\vec{\Gamma}|)$ consisting of simplicial homeomorphisms of $|\vec{\Gamma}|$. Let $\mathrm{Aut}_c(\vec{\Gamma})$ denote the subgroup of $\mathrm{Aut}(\vec{\Gamma})$, consisting of the automorphisms of $\vec{\Gamma}$ with compact support.

To see $\mathcal{H}(\Gamma) = \mathcal{H}_+(\Gamma) \ltimes \mathrm{Aut}(\vec{\Gamma})$, we shall show the follwing lemmas at first. Recall that $\mathrm{Aut}(\vec{\Gamma})$ is closed (resp. discrete) subgroup of the homeomorphism group $\mathcal{H}(|\vec{\Gamma}|)$ endowed with the compact-open (resp. the Whitney) topology (see Section 12)

Lemma 8. *There exists a continuous section $s : \mathcal{H}(\Gamma)/\mathcal{H}_+(\Gamma) \to \mathrm{Aut}(\vec{\Gamma})$ of the quotient map $q : \mathcal{H}(\Gamma) \to \mathcal{H}(\Gamma)/\mathcal{H}_+(\Gamma)$, $q : h \mapsto \mathcal{H}_+(\Gamma)h$.*

Proof. For each $h \in \mathcal{H}(\Gamma)$, we have a bijection $\psi : A \to A$ such that $h(E_\alpha) = E_{\psi(\alpha)}$ for $\alpha \in A$. For each $\alpha \in A$, let $g_\alpha : \overline{E}_\alpha \to \overline{E}_{\psi(\alpha)}$ be the homeomorphism defined as follows:

$$g_\alpha = \begin{cases} \varphi_{\psi(\alpha)}^{-1}\varphi_\alpha & \text{if } \varphi_{\psi(\alpha)}h\varphi_\alpha^{-1} : M_\alpha \to M_\alpha \text{ is orientation-preserving,} \\ \varphi_{\psi(\alpha)}^{-1}\theta\varphi_\alpha & \text{if } \varphi_{\psi(\alpha)}h\varphi_\alpha^{-1} : M_\alpha \to M_\alpha \text{ is orientation-reversing,} \end{cases}$$

where $\theta(x) = -x$ if $M_\alpha = \mathbb{I}$ or $\mathbb{R}$; $\theta(z) = \bar{z}$ if $M_\alpha = \mathbb{T}$ (note $M_\alpha \neq \mathbb{R}_+$ in case $\varphi_{\psi(\alpha)}h\varphi_\alpha^{-1}$ is orientation-reversing). Note that if $\mathcal{H}_+(\Gamma)h = \mathcal{H}_+(\Gamma)h'$ then h and h' define the same homeomorphism g_α. Thus, we can define $s(\mathcal{H}_+(\Gamma)h) : \Gamma \to \Gamma$ by $s(\mathcal{H}_+(\Gamma)h)|\Gamma^{(0)} = h|\Gamma^{(0)}$ and $s(\mathcal{H}_+(\Gamma)h)|E_\alpha = g_\alpha$, $\alpha \in A$. Observe that $s(\mathcal{H}_+(\Gamma)h) \in \mathrm{Aut}(\vec{\Gamma})$ and $q \circ s = \mathrm{id}$. The continuity of s is obvious. □

In fact, the above section s is a group isomorphism. Hence, we have that $\mathrm{Aut}(\vec{\Gamma}) \equiv \mathcal{H}(\Gamma)/\mathcal{H}_+(\Gamma)$.

Lemma 9. *For any $h \in \mathcal{H}(\Gamma)$, the set $\mathcal{H}_+(\Gamma)h \cap \mathrm{Aut}(\vec{\Gamma})$ has exactly one point. Thus, we have $s \circ q|\mathrm{Aut}(\vec{\Gamma}) = \mathrm{id}$.*

Proof. By Lemma 8, each $\mathcal{H}_+(\Gamma)h \cap \mathrm{Aut}(\vec{\Gamma})$ is a non-empty set. Now, we shall show that $\mathcal{H}_+(\Gamma)h \cap \mathrm{Aut}(\vec{\Gamma})$ has at most one point. Suppose that $f, g \in \mathrm{Aut}(\vec{\Gamma})$ and $\mathcal{H}_+(\Gamma)f = \mathcal{H}_+(\Gamma)g$. Then, it follows that $f|\Gamma^{(0)} = g|\Gamma^{(0)}$ and $f(E_\alpha) = g(E_\alpha)$ for each $\alpha \in A$. Let $f(E_\alpha) = E_{\alpha'}$. Note that $f_\alpha = \varphi_{\alpha'}f\varphi_\alpha^{-1}$ and $g_\alpha = \varphi_{\alpha'}g\varphi_\alpha^{-1}$ belong to the same component of $\mathcal{H}(M_\alpha)$. Since f_α and g_α are linear, they are coincide with id_{M_α} or θ_{M_α} according as they are orientation-preserving or orientation reversing, respectively, where

$$\theta_{M_\alpha}(x) = -x \text{ if } M_\alpha = \mathbb{I} \text{ or } \mathbb{R}\ ;\ \theta_{M_\alpha}(z) = \bar{z} \text{ if } M_\alpha = \mathbb{T}$$

(note that f_α and g_α are id_{M_α} in case $M_\alpha = \mathbb{R}_+$). This means that $f_\alpha = g_\alpha$ and $f|E_\alpha = \varphi_{\alpha'}^{-1}f_\alpha\varphi_\alpha = \varphi_{\alpha'}^{-1}g_\alpha\varphi_\alpha = g|E_\alpha$ for each $\alpha \in A$, that is, $f = g$. □

Lemma 10. *For any graph* Γ*, the composition map* $\mathcal{H}(\Gamma) \times \mathcal{H}(\Gamma) \ni (f, g) \mapsto f \circ g \in \mathcal{H}(\Gamma)$ *is continuous with respect to the compact-open and the Whitney topologies.*

Proof. For the continuity with respect to the Whitney topology, refer Proposition 4.14 of [4]. Now we shall consider the compact-open topology. Let K be a compact set and U an open set in Γ. Suppose that $f \circ g \in [K, U]$, hence $g \in [K, f^{-1}(U)]$. Replacing U with smaller one, we may assume that $\mathrm{cl}_\Gamma(g(E_\alpha) \cap f^{-1}(U))$ is compact for all $\alpha \in A$. Since only finitely many E_α meets K, we can take a neighborhood $\mathcal{U}$ of g so that $g'(E_\alpha) = g(E_\alpha)$ if $E_\alpha \cap K \neq \emptyset$ and $g' \in \mathcal{U}$. Observe that

$$\begin{aligned} F &= \mathrm{cl}_\Gamma \bigcup \{\, g'(K) \mid g' \in [K, f^{-1}(U)] \cap \mathcal{U} \,\} \\ &\subset \mathrm{cl}_\Gamma \bigcup \{\, g(E_\alpha) \cap f^{-1}(U) \mid E_\alpha \cap K \neq \emptyset,\ \alpha \in A \,\}. \end{aligned}$$

Then, F is compact. Let Γ_1 be a compact subgraph of Γ which contains F. Since $g(K) \subset f^{-1}(U) \cap \Gamma_1$, we can take an open neighborhood W of $g(K)$ in Γ_1 such that $g(K) \subset W \subset \mathrm{cl}_\Gamma W \subset f^{-1}(U)$. Let $\tilde{W}$ be an open set in Γ such that $\tilde{W} \cap \Gamma_1 = W$. Now, we define a neighborhood $\mathcal{W}$ of (f, g) in $\mathcal{H}(\Gamma) \times \mathcal{H}(\Gamma)$ by

$$\mathcal{W} = [\mathrm{cl}_\Gamma W, U] \times \big([K, \tilde{W}] \cap [K, f^{-1}(U)] \cap \mathcal{U}\big).$$

Then, we have $f' \circ g' \in [K, U]$ for any $(f', g') \in \mathcal{W}$. Indeed, $g' \in [K, f^{-1}(U)] \cap \mathcal{U}$ means that $g'(K) \subset F \subset \Gamma_1$ and $g'(K) \subset \Gamma_1 \cap \tilde{W} = W \subset \mathrm{cl}_\Gamma W$. □

In Section 15, we shall show that $\mathcal{H}_+(\Gamma)$ is a normal subgroup of $\mathcal{H}(\Gamma)$. Then, the following proposition implies that $\mathcal{H}_+(\Gamma) \ltimes \mathrm{Aut}(\vec{\Gamma}) = \mathcal{H}(\Gamma)$.

Proposition 14. *Let* Γ *be a graph and* $\vec{\Gamma}$ *be the associated digraph. Then,*

(i) *the map* $\mu : \mathcal{H}_+(\Gamma) \times \mathrm{Aut}(\vec{\Gamma}) \to \mathcal{H}(\Gamma)$, $(f, g) \mapsto f \circ g$*, is a homeomorphism with respect to the compact-open and the Whitney topologies;*

(ii) $\mu(\mathcal{H}_+^{\mathrm{PL}}(\Gamma) \times \mathrm{Aut}(\vec{\Gamma})) = \mathcal{H}^{\mathrm{PL}}(\Gamma)$*;*

(iii) $\mu(\mathcal{H}_0(\Gamma) \times \mathrm{Aut}_c(\vec{\Gamma})) = \mathcal{H}_c(\Gamma)$*.*

Proof. By Lemma 10, μ is continuous with respect to the both topologies. Thus, it suffices to show that μ has the continuous inverse. Let s be the section of q in Lemma 8. Since $\mathrm{Aut}(\vec{\Gamma})$ is a topological group, we can define the map $\zeta : \mathcal{H}(\Gamma) \to \mathcal{H}_+(\Gamma) \times \mathrm{Aut}(\vec{\Gamma})$ by $\zeta(h) = (h \circ s(q(h))^{-1}, s(q(h)))$. Then, observe that $\mu \circ \zeta = \mathrm{id}$ and $\zeta \circ \mu = \mathrm{id}$, that is, $\zeta = \mu^{-1}$. Thus, μ is a homeomorphism. □

Finally, we have the following.

Proposition 15. *For any graph* Γ*, the homeomorphism group* $\mathcal{H}(\Gamma)$ *endowed with the compact-open topology is a topological group.*

Proof. By Lemma 10, the composition map is continuous. We shall show that the inverse operator is continuous. Since $\mathcal{H}(\Gamma) = \mathcal{H}_+(\Gamma) \ltimes \mathrm{Aut}(\vec{\Gamma})$ by Proposition 14, it is sufficient to show that both $\mathcal{H}_+(\Gamma)$ and $\mathrm{Aut}(\vec{\Gamma})$ are topological groups. Indeed, we can express the inverse operator on $\mathcal{H}(\Gamma)$ as the composition of the inverse operators on $\mathcal{H}_+(\Gamma)$ and $\mathrm{Aut}(\vec{\Gamma})$.

Recall that $\mathcal{H}_+(\Gamma)$ is expressed as the product of topological groups, that is, the following map Ξ is a group isomorphism (see the proofs of Theorems 8 and 9):

$$\Xi : \mathcal{H}_+(\Gamma) \to \prod_{\alpha \in A} \mathcal{H}_+(\overline{E}_\alpha, \partial E_\alpha), \quad \Xi : h \mapsto (h|\overline{E}_\alpha)_{\alpha \in A}.$$

Due to [7], the homeomorphism group $\mathcal{H}(X)$ endowed with the compact-open topology is a topological group if each point of X has a compact connected neighborhood. Hence, $\mathcal{H}_+(\overline{E}_\alpha, \partial E_\alpha)$ is a topological group for each $\alpha \in A$. Thus, $\mathcal{H}_+(\Gamma)$ is also so. We have already known that $\mathrm{Aut}(\vec{\Gamma})$ is a topological group (see Section 12). Therefore, $\mathcal{H}(\Gamma)$ is a topological group. □

14. Proof of Theorems 1–3

Combining Proposition 14 with Theorems 9 and 8, we obtain Theorems 1 and 2 announced in the Introduction.

To prove Theorem 3, assume that Γ is a non-compact graph and let $\vec{\Gamma} = (V, E)$ be the associated digraph. By the definition of $\vec{\Gamma}$, we have $|V| \le p_\Gamma + 4e_\Gamma$. Now consider two cases.

(1) $2^{p_\Gamma} \ge 2^{e_\Gamma}$. Let $V_0 \subset V$ denote the set of isolated point of the graph $|\vec{\Gamma}|$. It follows that the group $\mathrm{Aut}(\vec{\Gamma})$ contains the subgroup S of all bijections of the set V_0. Since $|V_0| = p_\Gamma \ge \aleph_0$, the group S has cardinality $|S| = 2^{p_\Gamma}$. Now it follows from

$$2^{p_\Gamma} = |S| \le |\mathrm{Aut}(\vec{\Gamma})| \le |V|^{|V|} \le (p_\Gamma + e_\Gamma)^{p_\Gamma + e_\Gamma} = 2^{p_\Gamma + e_\Gamma} = 2^{p_\Gamma}$$

that $|\mathrm{Aut}(\vec{\Gamma})| = 2^{p_\Gamma + e_\Gamma}$ and thus

$$\begin{aligned}
&(\mathcal{H}(\Gamma), \mathcal{H}^{\mathrm{PL}}(\Gamma), \mathcal{H}_0(\Gamma)) \\
&\quad \overset{W}{\approx} (\mathrm{Aut}(\vec{\Gamma}) \times \square^{o_\Gamma}\mathbb{T} \times \square^{e_\Gamma} l_2, \mathrm{Aut}(\vec{\Gamma}) \times \square^{o_\Gamma}\mathbb{T} \times \square^{e_\Gamma} l_2^f, \{\mathrm{id}\} \times \boxdot^{o_\Gamma}\mathbb{T} \times \boxdot^{e_\Gamma}) \\
&\quad \approx (2^{p_\Gamma + e_\Gamma} \times \square^{o_\Gamma}\mathbb{T} \times \square^{e_\Gamma} l_2, 2^{p_\Gamma + e_\Gamma} \times \square^{o_\Gamma}\mathbb{T} \times \square^{e_\Gamma} l_2^f, 1 \times \boxdot^{o_\Gamma}\mathbb{T} \times \boxdot^{e_\Gamma})).
\end{aligned}$$

(2) $2^{p_\Gamma} \le 2^{e_\Gamma}$. The non-compactness of Γ implies that $p_\Gamma + e_\Gamma \ge \aleph_0$ and hence e_Γ is infinite. Then $|V| \le p_\Gamma + 4e_\Gamma = p_\Gamma + e_\Gamma$ and thus

$$|\mathrm{Aut}(\vec{\Gamma})| \le |V|^{|V|} \le (p_\Gamma + e_\Gamma)^{p_\Gamma + e_\Gamma} \le 2^{p_\Gamma + e_\Gamma} = 2^{e_\Gamma}.$$

Applying Proposition 9, we conclude that

$$\begin{aligned}
&(\mathrm{Aut}(\vec{\Gamma}) \times \square^{e_\Gamma} l_2, \mathrm{Aut}(\vec{\Gamma}) \times \square^{e_\Gamma} l_2^f, \{\mathrm{id}\} \times \boxdot^{e_\Gamma} l_2) \\
&\qquad \approx (2^{p_\Gamma + e_\Gamma} \times \square^{e_\Gamma} l_2, 2^{p_\Gamma + e_\Gamma} \times \square^{e_\Gamma} l_2^f, 1 \times \boxdot^{e_\Gamma} l_2) \\
&\qquad \approx (\square^{e_\Gamma} l_2, \square^{e_\Gamma} l_2^f, \boxdot^{e_\Gamma} l_2)
\end{aligned}$$

where $\mathrm{Aut}(\vec{\Gamma})$ is endowed with the Whithey topology (and hence is discrete).

Combining this fact with Theorem 2, we get the desired homeomorphism:

$$\begin{aligned}
&(\mathcal{H}(\Gamma), \mathcal{H}^{\mathrm{PL}}(\Gamma), \mathcal{H}_0(\Gamma)) \\
&\quad \overset{W}{\approx} (\mathrm{Aut}(\vec{\Gamma}) \times \square^{o_\Gamma}\mathbb{T} \times \square^{e_\Gamma} l_2, \mathrm{Aut}(\vec{\Gamma}) \times \square^{o_\Gamma}\mathbb{T} \times \square^{e_\Gamma} l_2^f, \{\mathrm{id}\} \times \boxdot^{o_\Gamma}\mathbb{T} \times \boxdot^{e_\Gamma} l_2) \\
&\quad \approx (2^{p_\Gamma + e_\Gamma} \times \square^{o_\Gamma}\mathbb{T} \times \square^{e_\Gamma} l_2, 2^{p_\Gamma + e_\Gamma} \times \square^{o_\Gamma}\mathbb{T} \times \square^{e_\Gamma} l_2^f, 1 \times \boxdot^{o_\Gamma}\mathbb{T} \times \boxdot^{e_\Gamma} l_2) \\
&\quad \approx (\square^{o_\Gamma}\mathbb{T} \times \square^{e_\Gamma} l_2, \square^{o_\Gamma}\mathbb{T} \times \square^{e_\Gamma} l_2^f, \boxdot^{o_\Gamma}\mathbb{T} \times \boxdot^{e_\Gamma} l_2).
\end{aligned}$$

15. The identity components of $\mathcal{H}(\Gamma)$

In this section we shall prove Proposition 1 announced in the introduction. Taking into account that the group $\mathrm{Aut}(\vec{\Gamma})$ is zero-dimensional, we see that Theorem 1 implies the "compact-open" part of Proposition 1.

Lemma 11. *The group $\mathcal{H}_+(\Gamma)$ (resp. $\mathcal{H}_+^{\mathrm{PL}}(\Gamma)$) coincides with the idenity component of the homeomorphism group $\mathcal{H}(\Gamma)$ (resp. $\mathcal{H}^{\mathrm{PL}}(\Gamma)$) endowed with the compact-open topology.*

In fact the topological structure of the automorphism group $\mathrm{Aut}(\vec{\Gamma})$ can be recovered from the topological structure of the group $\mathcal{H}(\Gamma)$:

Proposition 16. *Let Γ be a graph and $\vec{\Gamma}$ be its associated digraph. The automorphism group $\mathrm{Aut}(\vec{\Gamma})$ endowed with the compact-open topology is homeomorphic to the space of quasi-components of the group $\mathcal{H}(\Gamma)$ endowed with the compact-open topology.*

The subgroup $\mathcal{H}_+(\Gamma)$, being the connected component of $\mathcal{H}(\Gamma)$, is normal and closed in $\mathcal{H}(\Gamma)$ with compact-open topology. Next, we show that $\mathcal{H}_+(\Gamma)$ is open in the Whitney topology.

Proposition 17. *$\mathcal{H}_+(\Gamma)$ is an open normal subgroup of the group $\mathcal{H}(\Gamma)$ endowed with the Whitney topology.*

Proof. For any $h \in \mathcal{H}_+(\Gamma)$ and any closed edge $\overline{E}_\alpha$ in Γ, $h|\overline{E}_\alpha$ is isotopic to $\mathrm{id}_{\overline{E}_\alpha}$ in $\overline{E}_\alpha$. Hence, $g^{-1}hg|\overline{E}_\alpha \simeq g^{-1}g|\overline{E}_\alpha = \mathrm{id}_{\overline{E}_\alpha}$ in $\overline{E}_\alpha$ for all $g \in \mathcal{H}(\Gamma)$. Thus, $g^{-1}hg|\overline{E}_\alpha$ is orientation-preserving in $\overline{E}_\alpha$, and we have $g^{-1}hg \in \mathcal{H}_+(\Gamma)$. This is the normality of $\mathcal{H}_+(\Gamma)$.

To see open, take two points $v_1^\alpha, v_2^\alpha \in E_\alpha$ for any $\alpha \in A$. It follows from the definition of the CW-topology on Γ that the set $\Lambda = \Gamma^{(0)} \cup \{v_1^\alpha, v_2^\alpha : \alpha \in A\}$ is discrete in Γ. Thus, for any $v \in \Lambda$, we can take a neighborhood U_v such that $U_v \cap U_w = \emptyset$ if $v, w \in \Lambda$ and $v \neq w$. Then, $\mathcal{U} = \{U_v : v \in \Lambda\} \cup \{\Gamma \setminus \Lambda\}$ is an open cover of Γ. For any $h \in \mathcal{H}_+(\Gamma)$, take any $g \in \mathcal{H}(\Gamma)$ with $g \in \mathcal{U}(h)$. From the definition of $\mathcal{U}$, $g(v) \in U_v$ for any $v \in \Lambda$ and $g(w) = w$ for any $w \in \Gamma^{(0)}$. Hence, $g|\overline{E}_\alpha$ must be orientation-preserving in $\overline{E}_\alpha$, for any $\alpha \in A$. Thus, $\mathcal{H}_+(\Gamma)$ is open in $\mathcal{H}(\Gamma)$. □

Corollary 9. *The quotient group $\mathrm{Aut}(\vec{\Gamma}) = \mathcal{H}(\Gamma)/\mathcal{H}_+(\Gamma)$ endowed with the quotient Whitney topology is discrete.*

Next we describe the identity component of the group $\mathcal{H}(\Gamma)$ endowed with the Whitney topology and hence prove the second part of Proposition 1.

Lemma 12. *The identity component of the group $\mathcal{H}(\Gamma)$ endowed with the Whitney topology coincides with the subgroup $\mathcal{H}_0(\Gamma) = \mathcal{H}_+(\Gamma) \cap \mathcal{H}_c(\Gamma)$.*

Proof. Since Γ is a paracompact space, we may apply Proposition 4.2 of [4] to conclude that the identity component C_0 of the homeomorphism group $\mathcal{H}(\Gamma)$ lies in the subgroup $\mathcal{H}_c(\Gamma)$. The subgroup $\mathcal{H}_+(\Gamma)$, being open in $\mathcal{H}(\Gamma)$, also contains C_0. Consequently, $C_0 \subset \mathcal{H}_+(\Gamma) \cap \mathcal{H}_c(\Gamma) = \mathcal{H}_0(\Gamma)$. On the other hand, by Theorem 2, the group $\mathcal{H}_0(\Gamma)$ is homeomorphic to $\boxdot^{\mathrm{or}}\mathbb{T} \times \boxdot^{\mathrm{er}} l_2$, which is connected. Thus, $\mathcal{H}_0(\Gamma) \subset C_0$. Unifying both the inclusions, we obtain the required equality $C_0 = \mathcal{H}_0(\Gamma)$. □

By the same argument, we can prove

Lemma 13. *The identity component of the group $\mathcal{H}^{\mathrm{PL}}(\Gamma)$ endowed with the Whitney topology coincides with the subgroup*

$$\mathcal{H}_0^{\mathrm{PL}}(\Gamma) = \mathcal{H}^{\mathrm{PL}}(\Gamma) \cap \mathcal{H}_0(\Gamma) = \mathcal{H}_+^{\mathrm{PL}}(\Gamma) \cap \mathcal{H}_c(\Gamma).$$

16. Proof of Proposition 3

Let κ be any cardinal.

(1) First we prove that $b_1(\boxdot^\kappa\mathbb{T})=\kappa$. We recall that $b_1(\boxdot^\kappa\mathbb{T})=\dim_{\mathbb{Q}} H_1(\boxdot^\kappa\mathbb{T};\mathbb{Q})$. Using the definition of the box topology on $\boxdot^\kappa\mathbb{T}$, it is easy to show that for each compact subset $K\subset\boxdot^\kappa\mathbb{T}$ there is a finite subset $F\subset\kappa$ with

$$K\subset\mathbb{T}^F=\{(x_\alpha)_{\alpha\in\kappa}\in\boxdot^\kappa\mathbb{T}:\forall\alpha\in\kappa\setminus F\ (x_\alpha=1)\}.$$

This property of $\boxdot^\kappa\mathbb{T}$, combined with the compact support axiom for the singular homologies (see [22, 4.§8.11]) implies that the homology group $H_1(\boxdot^\kappa\mathbb{T};\mathbb{Q})$ coincides with the direct limit of the spectrum $\{H_1(\mathbb{T}^F;\mathbb{Q}):F\subset\kappa,\ |F|<\infty\}$. Since for each finite subset $F\subset\kappa$ the homology group $H_1(\mathbb{T}^F;\mathbb{Q})$ is isomorphic to $\mathbb{Q}^F$, we get that $H^1(\boxdot^\kappa\mathbb{T};\mathbb{Q})$ is isomorphic to $\oplus^\kappa\mathbb{Q}$ and hence

$$b_1(\boxdot^\kappa\mathbb{T})=\dim_{\mathbb{Q}} H_1(\boxdot^\kappa\mathbb{T};\mathbb{Q})=\dim_{\mathbb{Q}}(\oplus^\kappa\mathbb{Q})=\kappa.$$

(2) Next, we prove that $b^1(\prod^\kappa\mathbb{T})=\kappa$. Since $\prod^\kappa\mathbb{T}$ coincides with the limit of the inverse spectrum $\{\mathbb{T}^F:F\subset\kappa,\ |F|<\infty\}$ the continuity of Čech cohomologies (see [10, VIII.6.18]) implies that $\check H^1(\prod^\kappa\mathbb{T};\mathbb{Q})$ is the direct limit of a spectrum $\{\check H^1(\mathbb{T}^F;\mathbb{Q}):F\subset\kappa,\ |F|<\infty\}$. Since each group $\check H^1(\mathbb{T}^F;\mathbb{Q})$ is isomorphic to $\mathbb{Q}^F$, we conclude that $\check H^1(\prod^\kappa\mathbb{T})$ is isomorphic to the direct sum $\oplus^\kappa\mathbb{Q}$ and consequently,

$$b^1(\prod^\kappa\mathbb{T})=\dim_{\mathbb{Q}}(\prod^\kappa\mathbb{T};\mathbb{Q})=\dim_{\mathbb{Q}}(\oplus^\kappa\mathbb{Q})=\kappa.$$

(3) Finally, we prove that $cb^1(\dot\prod^\kappa\mathbb{T})=\kappa$. Observe that each continuous function $f:K\to\dot\prod^\kappa\mathbb{T}$ from a compact Hausdorff space K can be written as $f=i\circ\bar f$ where $\bar f=f:K\to f(K)$ and $i:f(K)\to\dot\prod^\kappa\mathbb{T}$ is the inclusion map. Then the induced operator $f^*:\check H^1(\dot\prod^\kappa\mathbb{T})\to\check H^1(K)$ can be written as $f^*=\bar f^*\circ i^*$ where $i^*:\check H^1(\dot\prod^\kappa\mathbb{T})\to\check H^1(f(K);\mathbb{Q})$ and $\bar f^*:\check H^1(f(K);\mathbb{Q})\to\check H^1(K;\mathbb{Q})$ are linear operators induced by the maps i and $\bar f$, respectively.

Taking into account that the weight $w(f(K))$ of the compactum $f(K)$ does not exceed κ, we conclude that $\dim_{\mathbb{Q}}\check H^1(f(K);\mathbb{Q})\le w(f(K))\le\kappa$ by the continuity of the Čech cohomologies, see [10, VIII.6.18]. Consequently,

$$\dim_{\mathbb{Q}} f^*(\check H^1(\dot\prod^\kappa\mathbb{T}))\le\dim_{\mathbb{Q}}\bar f^*(\check H^1(\dot\prod^\kappa\mathbb{T}))\le\dim_{\mathbb{Q}}\check H^1(f(K);\mathbb{Q})\le\kappa,$$

which implies $cb^1(\dot\prod^\kappa\mathbb{T})\le\kappa$.

To prove the reverse inequality, for every $\alpha\in\kappa$ consider the projection $\mathrm{pr}_\alpha:\dot\prod^\kappa\mathbb{T}\to\mathbb{T}$ onto the αth coordinate circle and the embedding $e_\alpha:\mathbb{T}\to\dot\prod^\kappa\mathbb{T}$ assigning to each point $x\in\mathbb{T}$ the sequence $(x_\beta)_{\beta\in\kappa}$ such that $x_\alpha=x$ and $x_\beta=1$ for all $\beta\ne\alpha$.

It is clear that the union $K=\bigcup_{\alpha\in\kappa}e_\alpha(\mathbb{T})$ is a compact subset of $\dot\prod^\kappa\mathbb{T}$. Denote by $i:K\to\dot\prod^\kappa\mathbb{T}$ the inclusion map. The chain of the maps

$$\mathbb{T}\xrightarrow{e_\alpha}K\xrightarrow{i}\dot\prod^\kappa\mathbb{T}\xrightarrow{\mathrm{pr}_\alpha}\mathbb{T}$$

with $\mathrm{pr}_\alpha\circ i\circ e_\alpha=\mathrm{id}$ induces the chain of linear operators

$$\check H^1(\mathbb{T};\mathbb{Q})\xleftarrow{e_\alpha^*}\check H^1(K;\mathbb{Q})\xleftarrow{i^*}\check H^1(\dot\prod^\kappa\mathbb{T};\mathbb{Q})\xleftarrow{\mathrm{pr}_\alpha^*}\check H^1(\mathbb{T};\mathbb{Q})$$

with $e_\alpha^* \circ i^* \circ \mathrm{pr}_\alpha^* = \mathrm{id}$. Fix any non-zero element $g \in \check{H}^1(\mathbb{T};\mathbb{Q})$ and consider its image $g_\alpha = i^* \circ \mathrm{pr}_\alpha^*(g) \in i^*(\check{H}^1(\prod^\kappa \mathbb{T};\mathbb{Q})) \subset \check{H}^1(K,\mathbb{Q})$. It is non-zero because $e_\alpha^*(g_\alpha) = g \neq 0$.

We claim that the vectors g_α, $\alpha \in \kappa$, are linearly independent and consequently, $\dim_{\mathbb{Q}} i^*(\check{H}^1(K;\mathbb{Q})) \geq \kappa$. For every $\alpha \in \kappa$, consider the linear operator

$$P_\alpha = i^* \circ \mathrm{pr}_\alpha^* \circ e_\alpha^* : \check{H}^1(K;\mathbb{Q}) \to \check{H}^1(K;\mathbb{Q})$$

and observe that

$$\begin{aligned} P_\alpha(g_\alpha) &= i^* \circ \mathrm{pr}_\alpha^* \circ e_\alpha^*(g_\alpha) = i^* \circ \mathrm{pr}_\alpha^* \circ e_\alpha^* \circ i^* \circ \mathrm{pr}_\alpha^*(g) = \\ &= i^* \circ \mathrm{pr}_\alpha^* \circ (\mathrm{pr}_\alpha \circ i \circ e_\alpha)^*(g) = i^* \circ \mathrm{pr}_\alpha^* \circ \mathrm{id}^*(g) = i^* \circ r_\alpha^*(g) = g_\alpha \end{aligned}$$

while for every $\beta \neq \alpha$ in κ we get

$$\begin{aligned} P_\alpha(g_\beta) &= i^* \circ \mathrm{pr}_\alpha^* \circ e_\alpha^*(g_\beta) = i^* \circ \mathrm{pr}_\alpha^* \circ e_\alpha^* \circ i^* \circ \mathrm{pr}_\beta^*(g) = \\ &= i^* \circ \mathrm{pr}_\alpha^* \circ (\mathrm{pr}_\beta \circ i \circ e_\alpha)^*(g) = i^* \circ \mathrm{pr}_\alpha^* \circ 0^*(g) = 0 \end{aligned}$$

because $\mathrm{pr}_\beta \circ i \circ e_\alpha : \mathbb{T} \to \{1\} \subset \mathbb{T}$ is a constant map.

To see that the vectors g_α, $\alpha \in \kappa$, are linearly dependent assume that

$$0 = \sum_{\alpha\in\kappa} \lambda(\alpha) \cdot g_\alpha$$

for some function $\lambda : \kappa \to \mathbb{Q}$ such that the set $\{\alpha \in \kappa : \lambda(\alpha) \neq 0\}$ is finite. Applying the operators P_β, $\beta \in \kappa$, to the above equality, we conclude that

$$0 = P_\beta(0) = P_\beta\big(\sum_{\alpha\in\kappa} \lambda(\alpha) \cdot g_\alpha\big) = \sum_{\alpha\in\kappa} \lambda(\alpha) \cdot P_\beta(g_\alpha) = \lambda(\beta) \cdot g_\beta$$

and thus $\lambda(\beta) = 0$. This means that $\lambda \equiv 0$ and the vectors g_α, $\alpha \in \kappa$, are linearly indepenedent. □

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(T. Banakh) Instytut Matematyki, Uniwersytet Humanistyczno-Przyrodniczy im. Jana Kochanowskiego w Kielcach, Poland, and Department of Mathematics, Lviv National University, Lviv, 79000, Ukraine
E-mail address: tbanakh@yahoo.com

(K. Mine) Institute of Mathematics, University of Tsukuba, Tsukuba, 305-8571, Japan
E-mail address: pen@math.tsukuba.ac.jp

(K. Sakai) Institute of Mathematics, University of Tsukuba, Tsukuba, 305-8571, Japan
E-mail address: sakaiktr@sakura.cc.tsukuba.ac.jp