CARDINAL CHARACTERISTICS OF
THE IDEAL OF HAAR NULL SETS

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ABSTRACT. We calculate the cardinal characteristics of the \( \mathcal{H}_N(G) \) of Haar null subsets of a Polish non-locally compact group \( G \) with invariant metric and show that

\[
\text{cov}(\mathcal{H}_N(G)) \leq b \leq \max\{d, \text{non}(N)\} < \text{non}(\mathcal{H}_N(G)) < \text{cof}(\mathcal{H}_N(G)) > \min\{d, \text{non}(N)\}.
\]

If \( G = \prod_{n \geq 0} G_n \), then the \( G \) group is the product of abelian locally compact groups \( G_n \), and hence \( \text{cof}(\mathcal{H}_N(G)) = \text{cof}(\mathcal{N}) \), where \( N \) is the ideal of Lebesgue null subsets on the real line. Martin Axiom implies that \( \text{cof}(\mathcal{H}_N(G)) > 2^{\aleph_0} \) and hence \( G \) contains a Haar null subset of \( G \) that cannot be enlarged to a Borel or projective Haar null subset of \( G \). This gives a negative (consistent) answer to a question of S. Solecki. To obtain these estimates we show that for a Polish non-locally compact group \( G \) with invariant metric the ideal \( \mathcal{H}_N(G) \) contains all \( \sigma \)-bounded subsets (equivalently, subsets with the small ball property) of \( G \).

A subset \( N \) of a topological group \( G \) is called \textit{Haar null} if it is contained in a universally measurable set \( B \subset G \) with \( \mu(Bh) = 0 \) for all \( g, h \in G \) (a subset \( B \) of a topological space \( X \) is \textit{universally measurable} if it is measurable with respect to any Borel \( \sigma \)-additive probability measure on \( X \)). The family \( \mathcal{H}_N(G) \) of Haar null subsets of a Polish group \( G \) is closed under translations, taking subsets and countable unions, see [THJ, 2.4.5]. The notion of Haar null sets is a natural extension of the notion of sets of Haar measure zero: if \( G \) happens to be locally compact, then Haar null sets are precisely the sets of Haar measure zero. Since the publication of Christensen paper [C] who introduced this new notion, Haar null sets have found many applications, see [BL], [PZ].

In this paper we estimate the principal cardinal characteristics of the \( \sigma \)-ideal \( \mathcal{H}_N(G) \) of Haar null subsets of a Polish group \( G \). There is nothing surprising about \( \mathcal{H}_N(G) \) if the group \( G \) is locally compact and non-discrete. In this case the ideal \( \mathcal{H}_N(G) \) is isomorphic to the \( \sigma \)-ideal \( N \) of Lebesgue null subsets of the real line \( \mathbb{R} \) in the sense that there is a Borel isomorphism \( h : G \to \mathbb{R} \) such that a subset \( A \subset G \) belongs to \( \mathcal{H}_N(G) \) if and only if \( h(A) \in N \) (this follow from the classical theorem on isomorphism of Borel measure spaces, see [Ke, 17.41]). Consequently, for a non-discrete locally compact Polish group \( G \) the \( \sigma \)-ideals \( \mathcal{H}_N(G) \) and \( N \) have the same cardinal characteristics. Let us remind their definitions, see [V].

Given a \( \sigma \)-ideal \( I \) of subsets of a set \( X \) let

\[
\begin{align*}
\text{add}(I) &= \min\{ |J| : J \subset I \text{ and } \bigcup J \notin I \}; \\
\text{cov}(I) &= \min\{ |J| : J \subset I \text{ and } \bigcup J = X \}; \\
\text{non}(I) &= \min\{ |A| : A \subset X \text{ and } A \notin I \}; \\
\text{cof}(I) &= \min\{ |J| : J \subset I \text{ and } I = \{ A \subset X : \exists E \in J \text{ with } A \subset E \} \}.
\end{align*}
\]

It is easy to see that these cardinals are related as follows:

\[
\aleph_1 \leq \text{add}(I) \leq \min\{\text{non}(I), \text{cov}(I)\} \leq \max\{\text{non}(I), \text{cov}(I)\} \leq \text{cof}(I).
\]

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It follows from the famous Cichoń diagram (see [V], [BS]) that \( \aleph_1 \leq \text{add}(\mathcal{N}) \leq b \leq \mathfrak{d} \leq \text{cof}(\mathcal{N}) \), where \( b \) and \( \mathfrak{d} \) are two well-known small cardinals introduced by E. van Douwen in his seminal paper [vD]. Since for any non-discrete locally compact Polish group \( G \) the cardinal characteristics of the ideals \( \mathcal{H}N(G) \) and \( \mathcal{N} \) coincide, we get

\[
\aleph_1 \leq \text{add}(\mathcal{H}N(G)) \leq b \leq \mathfrak{d} \leq \text{cof}(\mathcal{H}N(G)).
\]

In [S2, 3.4] S. Solecki proved that the same estimates hold also for any non-locally compact Polish group \( G \) with invariant metric. There is however one crucial difference between locally compact and non-locally compact cases: for a Polish non-locally compact group the cardinal \( \text{cof}(\mathcal{H}N(G)) \) always exceeds \( \aleph_1 \). Moreover, under Martin Axiom, it exceeds the size of continuum. Thus for a non-locally compact Polish group \( G \) the \( \sigma \)-ideal \( \mathcal{H}N(G) \) differs substantially from other classical ideals whose cardinal characteristics lie between \( \aleph_1 \) and \( \mathfrak{c} \) (and thus fall into the category of so-called small cardinals). Unlike to the cofinality \( \text{cof}(\mathcal{H}N(G)) \) the other cardinal characteristics of the \( \sigma \)-ideal \( \mathcal{H}N(G) \) behave not so wildly and for some special groups (like \( \mathbb{R}^\omega \) or \( \mathbb{Z}^\omega \)) can be expressed via known small cardinals \( b, \mathfrak{d}, \text{add}(\mathcal{N}), \text{cov}(\mathcal{N}) \), and \( \text{non}(\mathcal{N}) \).

To calculate the cardinal characteristics of the ideal \( \mathcal{H}N(G) \) for a non-locally compact Polish group \( G \) with invariant metric we shall prove that for such a group \( G \) the ideal \( \mathcal{H}N(G) \) contains the \( \sigma \)-ideal \( \mathcal{B}(G) \) of \( \sigma \)-bounded subsets of \( G \). Following O. Okunev and M. Tkachenko [Tk1, 3.9] we define a subset \( B \) of a topological group \( G \) to be \( \sigma \)-bounded if for any sequence \( (U_n)_{n \geq 0} \) of neighborhoods of the neutral element of \( G \) there is a sequence \( (F_n)_{n \geq 0} \) of finite subsets of \( G \) such that \( B \subseteq \bigcup_{n \geq 0} F_n U_n \) (this is equivalent to saying that there is a sequence \( (F_n)_{n \geq 0} \) of finite subsets of \( G \) with \( B \subseteq \bigcap_{k \geq 0} \bigcup_{n \geq k} F_n U_n \), see [HRT, 2.7]). Recently \( \sigma \)-bounded set attracted a lot of attention, see [Tk2], [HRT], [Her], [Ba1], [Ba2], [BNS], [Ts]. It should be mentioned that in Banach space theory they are known as sets with the small ball property, i.e., sets which can be covered by a sequence of small balls whose radii tend to zero, see [BK]. It is easy to see that the family \( \mathcal{B}(G) \) of all \( \sigma \)-bounded subsets of a topological group \( G \) forms a \( \sigma \)-ideal containing all compact subsets of \( G \).

Our main instrument in estimation of cardinal characteristics of the ideal \( \mathcal{H}N(G) \) is

**Theorem 1.** Let \( G \) be a non-locally compact Polish group.

1. If \( G \) admits an invariant metric, then \( \mathcal{B}(G) \subseteq \mathcal{H}N(G) \);
2. If \( G = \prod_{n \geq 0} G_n \) is the countable product of locally compact groups, then \( \mathcal{B}(G) \subseteq \mathcal{H}N(G) \);
3. For a continuous homomorphism \( h : G \to H \) onto a non-discrete (locally compact) Polish group \( H \), a subset \( A \subset H \) is Haar null (if and) only if its pre-image \( h^{-1}(A) \) is Haar null in \( G \), which implies that \( \text{cov}(\mathcal{H}N(G)) \leq \text{cov}(\mathcal{H}N(H)) \) and \( \text{non}(\mathcal{H}N(G)) \geq \text{non}(\mathcal{H}N(H)) \).

The first statement of Theorem 1 generalizes the result of Dougherty [D] who proved that for a Polish non-locally compact group with invariant metric the ideal \( \mathcal{H}N(G) \) contains all compact subsets of \( G \) (for abelian \( G \) this fact was proven by Chistensen [C]). Theorem 1 will help us to make the following estimations of the cardinal characteristics of the ideal \( \mathcal{H}N(G) \).
Theorem 2. Suppose $G$ is a non-discrete Polish group.

1. If $G$ is locally compact, then $\text{add}(\mathcal{H}_N(G)) = \text{add}(\mathcal{N})$, $\text{cov}(\mathcal{H}_N(G)) = \text{cov}(\mathcal{N})$, $\text{non}(\mathcal{H}_N(G)) = \text{non}(\mathcal{N})$, and $\text{cof}(\mathcal{H}_N(G)) = \text{cof}(\mathcal{N})$.

2. If $G$ is not locally compact and has an invariant metric, then $\text{cov}(\mathcal{H}_N(G)) \leq b$, $\text{non}(\mathcal{H}_N(G)) \geq \max\{d, \text{non}(\mathcal{N})\}$, and $\text{cof}(\mathcal{H}_N(G)) = \text{cof}(\mathcal{N})$.

3. If $G$ contains a closed normal subgroup $H$ such that either $H$ or $G/H$ is locally compact and not discrete, then $\text{add}(\mathcal{H}_N(G)) \leq \text{add}(\mathcal{N})$, $\text{cov}(\mathcal{H}_N(G)) \leq \text{cov}(\mathcal{N})$, $\text{non}(\mathcal{H}_N(G)) \geq \text{non}(\mathcal{N})$, and $\text{cof}(\mathcal{H}_N(G)) \geq \text{cof}(\mathcal{N})$.

4. If the center $Z = \{g \in G : \forall x \in G \quad gx = xg\}$ of $G$ is not locally compact, then $\text{cov}(\mathcal{H}_N(G)) \leq \text{cov}(\mathcal{H}_N(Z)) \leq b$, and $\text{non}(\mathcal{H}_N(G)) \geq \text{non}(\mathcal{H}_N(Z)) \geq \max\{d, \text{non}(\mathcal{N})\}$.

5. If $G$ admits a surjective continuous homomorphism onto a non-locally compact group $H$ with invariant metric, then $\text{cov}(\mathcal{H}_N(G)) \leq \text{cov}(\mathcal{H}_N(H)) \leq b$, and $\text{non}(\mathcal{H}_N(G)) \geq \text{non}(\mathcal{H}_N(H)) \geq \max\{d, \text{non}(\mathcal{N})\}$.

For linear complete metric spaces Theorem 2(2,3) implies

Corollary 1. If $X$ is an infinite-dimensional linear complete metric space, then

1. $\text{add}(\mathcal{H}_N(X)) \leq \text{add}(\mathcal{N})$;

2. $\text{cov}(\mathcal{H}_N(X)) \leq \min\{b, \text{cov}(\mathcal{N})\}$;

3. $\text{non}(\mathcal{H}_N(X)) \geq \max\{d, \text{non}(\mathcal{N})\}$;

4. $\text{cof}(\mathcal{N}) \leq \text{cof}(\mathcal{H}_N(X)) > \min\{d, \text{non}(\mathcal{N})\}$.

For groups $G$ which are countable products of locally compact amenable groups the first three inequalities of Corollary 1 can be reversed. Haar null subsets in such groups were characterized by S. Solecki [S_2]. We remind that a locally compact group $G$ is amenable if it admits a left invariant mean on the space $L^\infty(G)$ of all essentially bounded complex functions measurable with respect to the Haar measure. It is well-known [Pa, 4.10] that a locally compact group $G$ endowed with a left-invariant Haar measure $\mu$ is amenable if and only if it satisfies the Følner condition: for any $\varepsilon > 0$ and any compact subset $C \subset G$ there is a compact subset $K \subset G$ such that $\mu(xK \Delta K) < \varepsilon \mu(K)$ for all $x \in C$. The class of amenable locally compact groups contains all abelian (and even exponentially bounded) locally compact groups, see [Pa, Ch.6].

Another class containing all abelian groups is the class of groups admitting a finitely supported kaleidoscopic measure. A probability measure $\lambda$ on a topological group $G$ is called kaleidoscopic if there is a partition $G = A_1 \cup \cdots \cup A_n$ of $G$ into $n > 1$ $\lambda$-measurable pieces such that $\mu(xA_iy) = \frac{1}{n}$ for every $i \leq n$ and all $x, y \in G$. Groups admitting a kaleidoscopic finitely supported measure will be called kaleidoscopic, cf. [BP, §8]. We shall say that a group $G$ is almost kaleidoscopic if for any $\varepsilon > 0$ there is a finitely supported probability measure $\mu$ on $G$ and a partition $G = A_1 \cup \cdots \cup A_n$, $n > 1$, such that $|\mu(xA_iy) - \frac{1}{n}| < \frac{\varepsilon}{n}$ for all $x, y \in G$ and $i \leq n$.

The following result proved in [BP, §8] shows that the class of (almost) kaleidoscopic groups is quite large. We recall that a topological group $G$ is a SIN-group (abbreviated from “Small Invariant Neighborhoods”) if it has a neighborhood base $B$ at the unit such that $gUg^{-1} = U$ for any $U \in B$ and $g \in G$. It is well-known that each first countable SIN-group admits an invariant metric and that a topological group is a SIN-group if it is totally bounded in the sense that for any neighborhood $U \subset G$ of the unit there is a finite subset $F \subset G$ with $G = UF = FU$.

Proposition 1 ([BP, §8]).

1. A group admitting a homomorphism onto an (almost) kaleidoscopic group is (almost) kaleidoscopic.
(2) A group $G$ is kaleidoscopical provided $G$ admits a homomorphism onto a group containing a finite non-trivial normal subgroup.

(3) A group $G$ is almost kaleidoscopical provided $G$ admits a homomorphism onto a topological SIN-group containing a totally bounded non-trivial normal subgroup.

**Question 1.** Is every (amenable) group almost kaleidoscopical?

Now we can give some estimates of cardinal characteristics of the ideal $\mathcal{HN}(G)$ for Polish groups which are products of locally compact groups.

**Theorem 3.** Suppose that a Polish non-locally compact group $G = \prod_{n \geq 0} G_n$ is the countable product of locally compact groups $G_n$. Then

1. $\text{cov}(\mathcal{HN}(G)) \leq b$, $\text{non}(\mathcal{HN}(G)) \geq \max\{\mathfrak{d}, \text{non}(\mathcal{N})\}$, and $\text{cof}(\mathcal{HN}(G)) > \min\{\mathfrak{d}, \text{non}(\mathcal{N})\}$.

2. If all but finitely many groups $G_n$ are amenable, then $\text{add}(\mathcal{HN}(G)) \geq \text{add}(\mathcal{N})$, $\text{cov}(\mathcal{HN}(G)) \geq \min\{b, \text{cov}(\mathcal{N})\}$ and $\text{non}(\mathcal{HN}(G)) = \max\{\mathfrak{d}, \text{non}(\mathcal{N})\}$.

3. If some group $G_n$ is non-discrete or infinitely many of the groups $G_n$ are almost kaleidoscopical, then $\text{add}(\mathcal{HN}(G)) \leq \text{add}(\mathcal{N})$, $\text{cov}(\mathcal{HN}(G)) \leq \text{cov}(\mathcal{N})$, and $\text{cof}(\mathcal{HN}(G)) \leq \text{cof}(\mathcal{N})$.

4. If $G$ is abelian, then $\text{add}(\mathcal{HN}(G)) = \text{add}(\mathcal{N})$, $\text{cov}(\mathcal{HN}(G)) = \min\{b, \text{cov}(\mathcal{N})\}$, $\text{non}(\mathcal{HN}(G)) = \max\{\mathfrak{d}, \text{non}(\mathcal{N})\}$, and $\text{cof}(\mathcal{N}) \leq \text{cof}(\mathcal{HN}(G)) > \min\{\mathfrak{d}, \text{non}(\mathcal{N})\}$.

The strict inequality $\text{cof}(\mathcal{HN}(G)) > \min\{\mathfrak{d}, \text{non}(\mathcal{N})\}$ together with Martin Axiom has very strange consequences displaying a striking difference between properties of the $\sigma$-ideal $\mathcal{HN}(G)$ in the locally compact and non-locally compact cases.

It is well-known that any subset of zero Haar measure in a locally compact Polish group can be enlarged to a $\delta$-Zorn subset of a topological SIN-group containing a totally bounded non-trivial normal subgroup. The family of all $\mathfrak{c}$-algebra $\mathcal{HN}(X)$ containing $X$ and such that the image $f(A)$ of any set $A \in \sigma\mathcal{P}(X)$ under a continuous map $f : A \to X$ belongs to $\sigma\mathcal{P}(X)$. The $\sigma$-algebra $\sigma\mathcal{P}(X)$ contains all analytic and consequently all Borel subsets of $X$.

Following [Ke, 39.15] we call a subset $A$ of a Polish space $X$ $\sigma$-projective if it belongs to the smallest $\sigma$-algebra $\sigma\mathcal{P}(X)$ containing $X$ and such that the image $f(A)$ of any set $A \in \sigma\mathcal{P}(X)$ under a continuous map $f : A \to X$ belongs to $\sigma\mathcal{P}(X)$. The $\sigma$-algebra $\sigma\mathcal{P}(X)$ contains all analytic and consequently all Borel subsets of $X$.

Generalizing the notion of a Zorn set [PZ] let us call a subset $Z$ of a group $G$ a $\kappa$-Zorn set, where $\kappa$ is a cardinal, if $G \neq F \cdot Z$ for any subset $F \subset G$ of size $|F| \leq \kappa$. It is clear that each Haar null subset of a Polish group $G$ is $\kappa$-Zorn for any $\kappa < \text{cov}(\mathcal{HN}(G))$. The family of all $\kappa$-Zorn subsets of a topological $G$ group will be denoted by $\mathbb{Z}_\kappa(G)$. By $\mathcal{UN}(G)$ we denote the ideal of all universally null subsets of $G$ (a subset $N \subset G$ is universally null if it has zero measure with respect to any Borel non-atomic measure on $G$). Denote by $\text{cof}(\mathcal{UN}(G), \mathbb{Z}_2(G))$ the smallest size $|\mathcal{Z}|$ of a family $\mathcal{Z} \subset \mathbb{Z}_2(G)$ of 2-Zorn subsets of $G$ such that each universally null subset of $G$ lies in some set $Z \subset \mathcal{Z}$. Since $\mathcal{UN}(G) \subset \mathcal{HN}(G) \subset \mathbb{Z}_2(G)$ we get $\text{cof}(\mathcal{UN}(G), \mathbb{Z}_2(G)) \leq \text{cof}(\mathcal{HN}(G))$.

It is well-known that Martin Axiom implies $b = \mathfrak{d} = \text{add}(\mathcal{N}) = \mathfrak{c}$, where $\mathfrak{c}$ is the size of continuum.

**Theorem 4.** Let $G$ be a Polish non-locally compact group.

1. If $\text{non}(\mathcal{N}) \geq \mathfrak{d}$, then $\text{cof}(\mathcal{HN}(G)) \geq \text{cof}(\mathcal{UN}(G), \mathbb{Z}_2(G)) > \mathfrak{d}$;

2. If $\text{non}(\mathcal{N}) = \mathfrak{d} = \mathfrak{c}$ (which holds under Martin Axiom), then the group $G$ contains a universally null (and thus Haar null) subset that cannot be enlarged to a $\sigma$-projective Haar null (more generally, 2-Zorn) subset of $G$. 

It should be mentioned that the strict inequality $c < \text{cof}(\mathcal{UN}(G), \mathbb{Z}_2(G))$ from Theorem 4 cannot be proven in ZFC. According to [La] there is a model of ZFC in which $2^{\aleph_1} = c$ and each universally null set has size $\leq \aleph_1$. In this model $\text{cof}(\mathcal{UN}(G), \mathbb{Z}_2(G)) \leq \text{cof}(\mathcal{UN}(G)) \leq \aleph_1 = c$.

**Problem 1.** Is the inequality $\text{cof}(\mathcal{HN}(G)) \leq c$ consistent with ZFC for some Polish non-locally compact group $G$?

Assuming Martin Axiom we get $\text{add}(\mathcal{N}) = b = \text{non}(\mathcal{N}) = c$ and thus $\text{non}(\mathcal{HN}(G)) = c < \text{cof}(\mathcal{HN}(G))$ for any Polish non-locally compact group $G$.

**Problem 2.** Let $G$ be a nondiscrete Polish group (with invariant metric). Is $\text{add}(\mathcal{HN}(G)) = \text{cov}(\mathcal{HN}(G)) = c$ under MA or PFA?

Two topological groups $G, H$ are called **Haar null isomorphic** if there is a bijection $h : G \to H$ such that a subset $N \subset G$ is Haar null in $G$ if and only if $h(N)$ is Haar null in $H$. It follows from Isomorphism Theorem for non-atomic measure spaces [Ke, 17.41] that any two non-discrete locally compact Polish groups are Haar null isomorphic. On the other hand, the failure of the countable chain condition for the ideal $\mathcal{HN}(G)$ in the non-locally compact case [Si] implies that a Polish locally compact group cannot be Haar null isomorphic to a Polish non-locally compact group with invariant metric.

**Problem 3.** Are there two Polish non-locally compact groups (with invariant metric) that fail to be Haar null isomorphic? In particular, is the Hilbert space $\ell^2$ Haar null isomorphic to $\mathbb{R}^\omega$ or $\mathbb{Z}^\omega$? Have the ideals $\mathcal{HN}(\ell^2)$ and $\mathcal{HN}(\mathbb{R}^\omega)$ the same cardinal characteristics?

### Cardinal characteristics of the ideal $\mathcal{OB}(G)$

In this section we shall estimate the cardinal characteristics of the ideal $\mathcal{OB}(G)$ of $\omega$-bounded sets in a Polish non-locally compact group $G$ with invariant metric. First we remind the definition of the small cardinals $\mathfrak{b}$ and $\mathfrak{d}$. For two functions $f, g \in \mathbb{N}^\omega$ we write $f \leq^* g$ if $f(n) \leq g(n)$ for all sufficiently large $n$. A subset $B \subset \mathbb{N}^\omega$ is called

- **bounded** in $\mathbb{N}^\omega$ if there is $f \in \mathbb{N}^\omega$ such that $g \leq^* f$ for all $g \in B$;
- **dominating** if for any $f \in \mathbb{N}^\omega$ there is $g \in B$ with $f \leq^* g$.

By definition, $\mathfrak{b}$ is the smallest size of an unbounded subset of $\mathbb{N}^\omega$ while $\mathfrak{d}$ is the smallest size of a dominating subset of $\mathbb{N}^\omega$, see [vdD] or [V].

It is well-known (and easily seen) that the family $\mathcal{B}$ (resp. $\mathcal{N}\mathcal{D}$) of bounded (resp. non-dominating) subsets of $\mathbb{N}^\omega$ forms a $\sigma$-ideal. As we shall see, the ideal $\mathcal{N}\mathcal{D}$ is closely related to the ideal $\mathcal{OB}(G)$ while $\mathcal{B}$ is related to the $\sigma$-ideal $\mathcal{B}(G)$ generated by compact subsets of $G$.

**Lemma 1.** If $G$ is a Polish non-locally compact group, then $\text{cov}(\mathcal{OB}(G)) \leq \text{cov}(\mathcal{N}\mathcal{D}) = \mathfrak{b} \leq \mathfrak{d} = \text{non}(\mathcal{N}\mathcal{D}) \leq \text{non}(\mathcal{OB}(G))$.

**Proof.** To prove the lemma we shall construct a function $\psi : G \to \mathbb{N}^\omega$ such that for any non-dominating subset $D \subset \mathbb{N}^\omega$ the set $\psi^{-1}(D)$ is $\omega$-bounded in $G$. Fix a decreasing neighborhood base $(U_n)_{n \geq 0}$ at the unit of the group $G$ and a countable dense subset $\{a_k\}_{k \in \omega}$ of $G$. Define a function $\psi : G \to \mathbb{N}^\omega$ assigning to each $x \in G$ the function $y \in \mathbb{N}^\omega$ such that $y(n)$ is the smallest number with $x \in a_{y(n)}U_n$. We claim that the map $\psi : G \to \mathbb{N}^\omega$ satisfies our requirements.

Fix any non-dominating subset $D \subset \mathbb{N}^\omega$ and consider the preimage $\psi^{-1}(D) \subset G$. To show that $\psi^{-1}(D)$ is $\omega$-bounded in $G$, fix any sequence $(W_n)_{n \geq 0}$ of neighborhoods of the origin of $G$. By induction construct an increasing function $f : \omega \to \omega$ such that $U_{f(n)} \subset W_n$ for all $n \in \omega$. Since $D$ is not dominating, there is an increasing function $y \in \mathbb{N}^\omega$ such
that \( y \not\leq^* z \) for all \( z \in D \). Take any function \( g \in \mathbb{N}^\omega \) such that \( \min\{g(i) : f(k) \leq i < f(k+1)\} \geq y(f(k+1)) \) for every \( k \geq 0 \).

For every \( n \geq 0 \) let \( F_n = \{a_k : k \leq g(n)\} \). We claim that \( \psi^{-1}(D) \subset \bigcup_{n \geq 0} F_n U_n \).

Assuming the converse find a point \( x \in \psi^{-1}(D) \setminus \bigcup_{n \geq 0} F_n U_n \). Consider the function \( z = \psi(x) \in D \). It follows from the definition of \( \psi \) that \( z(i) > g(i) \) for all \( i \geq 0 \). Let us show that \( z(i) \geq y(i) \) for all \( i \geq f(0) \). Indeed, given such an \( i \), find \( k \geq 0 \) with \( f(k) \leq i < f(k+1) \) and observe that \( z(i) \geq g(i) \geq y(f(k+1)) \geq y(i) \). Thus \( y \not\leq^* z \in D \) which contradicts the choice of \( y \).

It follows from the property of the function \( \psi \) that \( \text{cov}(oB(G)) \leq \text{cov}(N^D) \) and \( \text{non}(oB(G)) \geq \text{non}(N^D) \). To complete the proof it rests to note that \( \text{cov}(N^D) = b \) and \( \text{non}(N^D) = d \). To establish these equalities observe that a subset \( D \subset \mathbb{N}^\omega \) is not dominating if and only if \( D \subset \{x \in \mathbb{N}^\omega : f \not\leq^* x\} \) for some \( f \in \mathbb{N}^\omega \).

In the sequel we shall also need some information concerning cardinal characteristics of the \( \sigma \)-ideal \( B(G) \) generated by compact subsets of a topological group \( G \).

**Lemma 2.** Suppose \( G \) is a Polish non-locally compact group. Then \( \text{add}(B(G)) = \text{add}(B) = b = \text{non}(B) = \text{non}(B(G)) \) and \( \text{cov}(B(G)) = \text{cov}(B) = d = \text{cof}(B) = \text{cof}(B(G)) \).

**Proof.** Let \( \hat{G} \) be any metrizable compactification of \( G \) and \( f : K \to \hat{G} \) be a continuous surjective map from a zero-dimensional compact space. Consider the preimage \( f^{-1}(G) \) and by Zorn Lemma find a minimal closed subset \( Z \subset f^{-1}(G) \) with \( f(Z) = G \). Then \( Z \), being Polish, zero-dimensional and nowhere locally compact, is homeomorphic to \( \mathbb{N}^\omega \) according to the Aleksandrov-Urysohn Theorem [Ke, 7.7]. Since the map \( f|Z \) is proper (that is the preimages of compact subsets are compact) we get that the space \( G \) is the image of the space \( \mathbb{N}^\omega \) under a continuous proper map \( \pi : \mathbb{N}^\omega \to G \).

Call a subset of \( G \) \( \sigma \)-bounded if it lies in a \( \sigma \)-compact subset of \( G \). Observe that a subset \( B \subset \mathbb{N}^\omega \) lies in a \( \sigma \)-compact subset of \( \mathbb{N}^\omega \) if and only if it is bounded in the sense of the pre-order \( \leq^* \). Consequently, for any bounded subset \( A \) of \( (\mathbb{N}^\omega, \leq^*) \) the image \( \pi(A) \) is \( \sigma \)-bounded in \( G \) and for any \( \sigma \)-bounded subset \( B \subset G \) the pre-image \( \pi^{-1}(B) \) is bounded in \( (\mathbb{N}^\omega, \leq^*) \). This observation together with known equalities \( \text{add}(B) = b = \text{non}(B) \) and \( \text{cov}(B) = d = \text{cof}(B) \) allow us to conclude that \( \text{add}(B(G)) = \text{add}(B) = b = \text{non}(B) = \text{non}(B(G)) \) and \( \text{cov}(B(G)) = \text{cov}(B) = d = \text{cof}(B) = \text{cof}(B(G)) \).

Finally let us prove another useful lemma which probably belongs to the mathematical folklore.

**Lemma 3.** Let \( \mathcal{F} \) be a family of universally measurable subsets of a Polish space \( X \). If \( |\mathcal{F}| < \text{add}(\mathcal{N}) \), then the union \( \bigcup \mathcal{F} \) is universally measurable in \( X \).

**Proof.** Fix any finite Borel measure \( \mu \). We have to show that the union \( \bigcup \mathcal{F} \) is \( \mu \)-measurable. Let \( C = \{x \in X : \mu(\{x\}) > 0\} \). It is clear that the set \( C \) is at most countable and thus Borel. Consider the discrete measure \( \nu = \sum_{x \in C} \mu(\{x\}) \delta_x \) where \( \delta_x \) is the Dirac measure concentrated at \( x \). Then \( \eta = \mu - \nu \) is a non-atomic measure. Since each subset of \( X \) is \( \nu \)-measurable, it suffices to show that the set \( \bigcup \mathcal{F} \) is \( \eta \)-measurable. That is so if \( \eta = 0 \). So we consider the case of non-trivial measure \( \eta \). Multiplying \( \eta \) by a suitable constant we may assume that \( \eta \) is a probability measure. Then by Isomorphism Theorem for non-atomic probability measures [Ke, 17.41] the measure \( \eta \) is equivalent to the Lebesgue measure \( \lambda \) on \([0, 1]\). Hence we may assume that \( X = [0, 1] \) and \( \eta = \lambda \). Let \( \lambda_*(\bigcup \mathcal{F}) = \sup \{\lambda(S) : S \subset \bigcup \mathcal{F} \text{ is } \sigma \text{-compact}\} \) and find a \( \sigma \)-compact subset \( S \subset \bigcup \mathcal{F} \) with \( \lambda(S) = \lambda_*(\bigcup \mathcal{F}) \). Then \( \lambda(B) = 0 \) for any measurable subset \( B \subset \bigcup \mathcal{F} \setminus S \). It follows that \( \lambda(F \setminus S) = 0 \) for each \( F \in \mathcal{F} \). Since \( |\mathcal{F}| < \text{add}(\mathcal{N}) \) we conclude \( \lambda(\bigcup \mathcal{F} \setminus S) = 0 \) which implies that \( \bigcup \mathcal{F} = S \cup (\bigcup \mathcal{F} \setminus S) \) is \( \lambda \)-measurable.
We divide the proof of Theorem 1 into three lemmas.

Lemma 4. If \( G \) is a Polish non-locally compact group with invariant metric, then \( oB(G) \subset \mathcal{HN}(G) \).

Proof. Fix any complete invariant metric \( d \) on the group \( G \). Since \( G \) is not locally compact, no non-empty open subset of \( G \) is totally bounded. Using this observation we can inductively construct a sequence \( (\varepsilon_n)_{n \geq 0} \subset (0, 1] \) of positive reals such that for every \( n \geq 0 \) the \( \varepsilon_n \)-ball \( B(\varepsilon_n) = \{ x \in G : d(x, 0) < \varepsilon_n \} \) around the origin of \( G \) fails to have a finite \( 6\varepsilon_{n+1} \)-net. Then \( \varepsilon_{n+1} \leq \frac{1}{2} \varepsilon_n \leq \frac{1}{2^n} \) for all \( n \). By the invariance of the metric \( d \) we get \( F \cdot B(\varepsilon_n) = B(\varepsilon_n) \cdot F \) for any finite subset \( F \subset G \).

To show that \( oB(G) \subset \mathcal{HN}(G) \), fix any \( o \)-bounded subset \( B \subset G \) and find a sequence \( (F_n)_{n \geq 0} \) of finite subsets of \( G \) such that \( B \subset \bigcap_{k \geq 0} \bigcup_{n \geq k} F_n B(\varepsilon_n) \). Observe that the set \( M = \bigcap_{k \geq 0} \bigcup_{n \geq k} F_n B(\varepsilon_n) \) is Borel in \( G \).

Using the fact that the \( \varepsilon_n \)-ball \( B(\varepsilon_n) \) admits no finite \( 6\varepsilon_{n+1} \)-net, for every \( n \geq 0 \) fix a finite subset \( D_n \subset B(\varepsilon_n) \) of size \( |D_n| = 2^{n+1} |F_{n+2}| \) which is \( 6\varepsilon_{n+1} \)-separated in the sense that \( d(x, y) \geq 6\varepsilon_{n+1} \) for any distinct points \( x, y \in D_n \).

Let \( D = \bigcup_{n \geq 0} \prod_{k \leq n} D_k \) and let \( \hat{D} = \prod_{k \geq 0} D_k \) be the infinite product endowed with the Tychonov product topology. Consider the map \( \psi : D \rightarrow G \) assigning to each finite sequence \( (x_0, \ldots, x_n) \in D \) the product \( x_0 \cdots x_n \) in \( G \). Also let \( \varphi : \hat{D} \rightarrow G \) be the continuous map assigning to each infinite sequence \( (x_n)_{n \geq 0} \) the limit \( \lim_{n \rightarrow \infty} x_0 \cdots x_n \).

Also let \( \varphi^{-1}(g) = \{ (x_i)_{i \geq 0} : \varphi(x_i) = g \} \). This implies that for any \( g, h \in G \) and any \( k \geq 1 \) the preimage \( \varphi^{-1}(gB(\varepsilon_{k+1}^+)) \) is small in the sense that there is a finite sequence \( (x_0, \ldots, x_{k-1}) \in D \) such that for any \( y \in \varphi^{-1}(gB(\varepsilon_{k+1}^+)) \) we get \( y_i = x_i \) for all \( i < k \).

Let \( \lambda = \otimes_{n \geq 0} \lambda_n \) be the tensor product of probability counting measures \( \lambda_n \) on \( D_n \) (i.e., \( \lambda_n(A) = |A|/|D_n| \) for \( A \subset D_n \)) and \( \mu \) be the image of the measure \( \lambda \) under the map \( \varphi \) (i.e., \( \mu(A) = \lambda(\varphi^{-1}(A)) \) for a Borel subset \( A \subset G \)).

We claim that \( \mu(gMh) = 0 \) for each \( g, h \in G \). For this we note that for any \( g, h \in G \) and \( k \geq 1 \) we get \( \mu(gB(\varepsilon_{k+1}^+)) = \lambda(\varphi^{-1}(gB(\varepsilon_{k+1}^+))) \leq (\prod_{i < k} |D_i|)^{-1} \). Consequently, \( \mu(gF_{k+1}^+B(\varepsilon_{k+1}^+)) \leq |F_{k+1}^+| (\prod_{i < k} |D_i|)^{-1} \leq |F_{k+1}^+|/|D_{k-1}^+| = \frac{1}{2^k} \) and

\[
\mu(gMh) \leq \mu\left( \bigcup_{i > k} gF_i B(\varepsilon_i) \right) \leq \sum_{i > k} \frac{1}{2^i} = \frac{1}{2^k - 1}.
\]

Sending \( k \) to \( \infty \) we get \( \mu(gMh) = 0 \), which means that \( B \) lies in the Haar null \( G_\delta \)-subset \( M \) of \( G \).

Lemma 5. If \( \pi : G \rightarrow H \) is a continuous surjective homomorphism from a Polish group \( G \) onto a non-discrete (locally compact) Polish group \( H \), then a subset \( A \subset H \) is Haar null (if and only if) its preimage \( \pi^{-1}(A) \) is Haar null in \( G \), which implies \( \text{cov}(\mathcal{HN}(G)) \leq \text{cov}(\mathcal{HN}(H)) \) and \( \text{non}(\mathcal{HN}(G)) \geq \text{non}(\mathcal{HN}(H)) \).

Proof. To prove the “only if” part assume that a subset \( A \) is Haar null in \( H \). Without loss of generality, \( A \) is universally measurable in \( H \). Then its preimage \( \pi^{-1}(A) \) is universally measurable in \( G \). Fix any probability measure \( \mu \) on \( H \) with \( \mu(xA) = 0 \) for all \( x, y \in H \) and find any probability measure \( \eta \) on \( G \) that maps onto \( \mu \) by the homomorphism \( \pi \) (the existence of such a measure \( \eta \) follows from the Jankov, von Neumann Uniformization Theorem [Ke, 18.1]). Then for any \( x, y \in G \) we get \( \eta(x \pi^{-1}(A)y) = \eta(\pi^{-1}(\pi(x)A \pi(y))) = \mu(\pi(x) A \pi(y)) = 0 \), which means that \( \pi^{-1}(A) \) is Haar null.
To prove that $\text{cov}(\mathcal{H}^N(G)) \leq \text{cov}(\mathcal{H}^N(H))$ take any cover $C$ of $H$ by Haar null sets with $|C| = \text{cov}(\mathcal{H}^N(H))$ and observe that $\pi^{-1}(C) = \{\pi^{-1}(C) : C \in C\}$ is a cover of $G$ by Haar null sets, which yields $\text{cov}(\mathcal{H}^N(G)) \leq |\pi^{-1}(C)| \leq \text{cov}(\mathcal{H}^N(H))$.

To prove that $\text{non}(\mathcal{H}^N(G)) \geq \text{non}(\mathcal{H}^N(H))$ take any subset $A \subset G$ of size $|A| < \text{non}(\mathcal{H}^N(H))$. Then $|\pi(A)| < \text{non}(\mathcal{H}^N(H))$ and hence $\pi(A)$ is Haar null in $H$ while $\pi^{-1}(\pi(A)) \supset A$ is Haar null in $G$. Thus $\text{non}(\mathcal{H}^N(G)) \geq \text{non}(\mathcal{H}^N(H))$.

To prove the “if” part, suppose that the group $H$ is locally compact and $A \subset H$ is such that $\pi^{-1}(A)$ is Haar null in $G$. Let $\lambda$ denote a left invariant Haar measure on $H$. We should show that the homomorphism $\pi$ maps the measure $\eta * \mu$ onto the Haar measure $\lambda$. Indeed, given a Borel subset $B \subset H$ denote by $\chi_B : H \to \{0,1\}$ the characteristic function of the set $B$ and applying the Fubini Theorem conclude that

$$\eta * \mu(\pi^{-1}(B)) = \int_\eta \int_\mu \chi_B \circ \pi(xy) dxy = \int_\mu \int_\eta \chi_B \circ \pi(xy) dydx =$$

$$= \int_\mu \eta(\pi^{-1}(\pi(x^{-1})B)) dx = \int_\mu \lambda(\pi(x^{-1})B) dx = \int_\mu \lambda(B) dx = \lambda(B).$$

Since $\eta * \mu(M) = 0$ there is a $\sigma$-compact set $S \subset G \setminus M$ such that $\eta * \mu(G \setminus S) = 0$. Then $\lambda(H \setminus \pi(S)) = 0$ and hence $\lambda(A) = 0$ since $A \cap \pi(S) = \emptyset$. □

**Lemma 6.** If a non-locally compact Polish group $G = \prod_{n \geq 0} G_n$ is the product of locally compact groups, then $\text{oB}(G) \subset \mathcal{H}^N(G)$.

**Proof.** We remind that the modular function on a locally compact group $H$ endowed with a left invariant Haar measure $\lambda$ is a unique homomorphism $\Delta : H \to \mathbb{R}_+$ into the multiplicative group of positive real numbers such that $\lambda(Bx) = \Delta(x)\lambda(B)$ for any $x \in H$ and a Borel subset $B \subset H$, see [He, §1.2] or [Za, §4]. A locally compact group $H$ is unimodular if its modular function is constant (this is equivalent to saying that any left invariant Haar measure on $H$ is right invariant).

To prove that $\text{oB}(G) \subset \mathcal{H}^N(G)$ fix any $\sigma$-bounded subset $B \subset G = \prod_{n \geq 0} G_n$. Without loss of generality, we can assume that all the groups $G_n$ are not compact.

If infinitely many groups $G_n$ fail to be unimodular, then the Polish abelian group $H = \prod_{n \geq 0} G_n/\text{Ker}(\Delta_n)$ is not locally compact and consequently, the group $G$ admits a continuous homomorphism $\pi : G \to H$ onto the Polish non-locally compact abelian group $H$. By [Tk1, 3.10] the set $\pi(B)$ is $\sigma$-bounded in $H$. Since the abelian group $H$ has invariant metric we may apply Lemma 4 to conclude that $\text{oB}(H) \subset \mathcal{H}^N(H)$ and thus the set $\pi(B)$ is Haar null in $H$. Applying Lemma 5 we get that the preimage $\pi^{-1}(\pi(B)) \supset B$ is Haar null in $G$.

Now consider the case when almost all the groups $G_n$ are unimodular. Without loss of generality, we can assume that the groups $G_n$ are unimodular for all $n \geq 1$. For every $n \geq 0$ fix a left invariant Haar measure $\lambda_n$ on the locally compact group $G_n$ and a neighborhood $W_n \subset G_n$ of the unit, having compact closure in $G_n$. Let $U_n = \{(x_i)_{i \geq 0} \in G : x_i \in W_i \text{ for } i \leq n\}$, $n \geq 1$. Using the $\sigma$-boundedness of the set $B$ find a sequence $(F_n)_{n \geq 1}$ of finite subsets of the group $G$ such that $B \subset \bigcap_{k \geq 1} \bigcup_{n \geq k} F_n U_n$. Note that the set $M = \bigcap_{k \geq 1} \bigcup_{n \geq k} F_n U_n$ is Borel and hence universally measurable. We claim that it is Haar null in $G$. 

To find a suitable measure \( \mu \) on \( G \), for every \( n \geq 0 \) fix a compact subset \( K_n \subset G_n \) with \( \lambda_n(K_n) \geq 2^n |F| \lambda_n(W_n) \) (such a set \( K_n \) exists since \( G_n \) is not compact and the measure \( \lambda_n \) is unbounded). Next, consider the probability measure \( \mu_n \) on \( G_n \) defined by \( \mu_n(B) = \frac{\lambda_n(B \cap K_n)}{\lambda_n(K_n)} \) for a Borel subset \( B \subset G_n \). Finally consider the tensor product \( \mu = \otimes_{n \geq 0} \mu_n \) of the measures \( \mu_n \).

We claim that \( \mu(xMy) = 0 \) for any \( x, y \in G \). For this notice that by the invariance of the measures \( \lambda_n \), for every \( n \geq 1 \) we get

\[
\mu(xF_n U_n y) \leq |F_n| \frac{\lambda_n(W_n)}{\lambda_n(K_n)} \leq \frac{1}{2^n}.
\]

Consequently, for every \( k \geq 0 \)

\[
\mu(xMy) \leq \mu( \bigcup_{n \geq k} xF_n U_n y ) \leq \sum_{n \geq k} \frac{1}{2^n} = \frac{1}{2^{k-1}}.
\]

Sending \( k \) to \( \infty \) we get \( \mu(xMy) = 0 \) which means that \( M \supset B \) is Haar null in \( G \). □

**Proof of Theorem 2**

Suppose that \( G \) is a non-discrete Polish group.

1. If \( G \) is locally compact, then a subset \( A \subset G \) is Haar null if and only if \( A \) has zero measure with respect to a left-invariant Haar measure \( \lambda \) on \( G \), see [THJ, p.374]. Replace the Haar measure \( \lambda \) by a Borel probability measure \( \mu \) equivalent to \( \lambda \) in the sense that \( \mu(B) = 0 \) for a Borel subset \( B \subset G \) if and only if \( \lambda(B) = 0 \). By Theorem [Ke, 17.41] on the isomorphism of measure spaces, there is a Borel isomorphism \( f: G \to [0,1] \) such that for any Borel subset \( B \subset G \) \( \mu(B) = \tau(f(B)) \) where \( \tau \) is the Lebesgue measure on \([0,1]\). This show that the ideal \( \mathcal{H}(G) \) is isomorphic to the ideal \( \mathcal{N} \) of Lebesgue null subsets of \([0,1]\) and consequently these ideals have the same cardinal characteristics, i.e., \( \text{add}(\mathcal{H}(G)) = \text{add}(\mathcal{N}), \text{cov}(\mathcal{H}(G)) = \text{cov}(\mathcal{N}), \text{non}(\mathcal{H}(G)) = \text{non}(\mathcal{N}), \) and \( \text{cof}(\mathcal{H}(G)) = \text{cof}(\mathcal{N}) \).

2. Suppose that \( G \) is not locally compact and admits an invariant metric. The inclusion \( \mathcal{O}(G) \subset \mathcal{N}(G) \) and estimates \( \text{cov}(\mathcal{O}(G)) \leq \mathfrak{b} \) \( \text{non}(\mathcal{O}(G)) \geq \mathfrak{v} \) proved in Lemmas 4 and 1 imply \( \text{cov}(\mathcal{H}(G)) \leq \text{cov}(\mathcal{O}(G)) \leq \mathfrak{b} \) and \( \text{non}(\mathcal{H}(G)) \geq \text{non}(\mathcal{O}(G)) \geq \mathfrak{v} \). The estimate \( \text{non}(\mathcal{N}) \leq \text{non}(\mathcal{H}(G)) \) follows from the inclusion \( \mathcal{U}(G) \subset \mathcal{H}(G) \) and the well-known equality \( \text{non}(\mathcal{U}(G)) = \text{non}(\mathcal{N}) \) (holding because of the Isomorphism Theorem for non-atomic measure spaces [Ke, 17.41]). Therefore \( \text{cov}(\mathcal{H}(G)) \leq \mathfrak{b} \leq \text{max}\{\mathfrak{d}, \text{non}(\mathcal{N})\} \leq \text{non}(\mathcal{H}(G)) \leq \text{cof}(\mathcal{H}(G)) \).

To show that \( \text{cof}(\mathcal{H}(G)) \geq \min\{\mathfrak{d}, \text{non}(\mathcal{N})\} \), we first prove that \( \text{non}(\mathcal{N}) \geq \mathfrak{d} \) implies \( \text{cof}(\mathcal{H}(G)) \geq \text{cof}(\mathcal{U}(G), \mathcal{Z}(G)) \geq \mathfrak{d} \) (this will be used for the proof of Theorem 4).

Assuming that \( \text{non}(\mathcal{N}) \geq \mathfrak{d} \), and \( \text{cof}(\mathcal{U}(G), \mathcal{Z}(G)) \geq \mathfrak{d} \), fix a family \( \{Z_\alpha\}_{\alpha < \mathfrak{d}} \) of 2-Zorn subsets of \( G \) such that each universally null subsets of \( G \) lies in \( Z_\alpha \) for some ordinal \( \alpha < \mathfrak{d} \) (as usual, we identify cardinals with initial ordinals). Since \( \text{cof}(\mathcal{B}(G)) = \text{cof}(\mathcal{B}) = \mathfrak{d} \), we can also fix a family \( \{C_\alpha\}_{\alpha < \mathfrak{d}} \) of \( \sigma \)-compact subsets of \( G \) such that each \( \sigma \)-compact sets \( C \) lies in some \( C_\alpha \).

Let us show that for any ordinal \( \alpha < \mathfrak{d} \) we get \( G \neq Z_\alpha \cup (\bigcup_{\beta \leq \alpha} C_\beta) \). Let \( S_\alpha = \bigcup_{\beta \leq \alpha} C_\beta \) and consider the set \( S_\alpha \cdot S^{-1}_\alpha = \bigcup_{\beta, \gamma \leq \alpha} C_\beta \cdot C_\gamma \) which is the union of \( \mathfrak{d} \) compact subsets of \( G \). Since \( \text{cov}(\mathcal{B}(G)) = \text{cov}(\mathcal{B}) = \mathfrak{d} \), there is an element \( g \in G \setminus (S_\alpha \cdot S^{-1}_\alpha) \). For this element \( g \) we get \( S_\alpha \cap gS_\alpha = \emptyset \). Assuming that \( G = Z_\alpha \cup S_\alpha \), we would get \( gS_\alpha \subset Z_\alpha \) and \( S_\alpha \subset g^{-1}Z_\alpha \). Then \( G = Z_\alpha \cup g^{-1}Z_\alpha \) which is not possible as \( Z_\alpha \) is 2-Zorn. Consequently, \( G \neq Z_\alpha \cup S_\alpha \) and we can pick a point \( x_\alpha \in G \setminus (Z_\alpha \cup S_\alpha ) \).

We claim that the subset \( X = \{x_\alpha : \alpha < \mathfrak{d}\} \) is universally null. Fix any probability non-atomic measure \( \mu \) on \( G \) and find a \( \sigma \)-compact subset \( C \subset G \) with \( \mu(G \setminus C) = 0 \), see
By the choice of the family \( \{C_\alpha\} \), there is an ordinal \( \alpha < \vartheta \) with \( C \subset C_\alpha \).

It follows from the construction of \( X \) that \( X \cap C_\alpha \subset \{x_\beta : \beta \leq \alpha \} \) and \( |X \cap C_\alpha| < \vartheta \). Since \( \vartheta = \text{non}(\mathcal{N}) = \text{non}(\mathcal{U} \mathcal{N}(G)) \), the set \( X \cap C_\alpha \) is universally null. Consequently, \( \mu(X) \leq \mu(X \cap C_\alpha) + \mu(G \setminus C_\alpha) = 0 \), i.e., \( X \) is universally null and hence Haar null. By the choice of the family \( \{Z_\alpha\} \), there is an ordinal \( \alpha < \vartheta \) with \( X \subset Z_\alpha \). On the other hand \( X \setminus Z_\alpha \ni x_\alpha \), which is a contradiction.

Thus \( \text{cof}(\mathcal{H} \mathcal{N}(G)) \geq \text{cof}(\mathcal{U} \mathcal{N}(G), Z_\alpha(G)) > \vartheta \geq \min\{\vartheta, \text{non}(\mathcal{N})\} \) under \( \text{non}(\mathcal{N}) \geq \vartheta \).

If \( \text{non}(\mathcal{N}) < \vartheta \), then again \( \text{cof}(\mathcal{H} \mathcal{N}(G)) \geq \text{non}(\mathcal{H} \mathcal{N}(G)) \geq \vartheta > \min\{\vartheta, \text{non}(\mathcal{N})\} \).

3. Assume that \( H \) is a non-discrete locally compact group and either \( H \) is a closed normal subgroup of \( G \) or else \( H \) is a quotient of \( G \). In both cases we shall construct a map \( p : G \to H \) such that a subset \( N \subset H \) is Haar null in \( H \) if and only if \( p^{-1}(N) \) is Haar null in \( G \). If \( H \) is a quotient group of \( G \), then let \( p : G \to H \) be the quotient homomorphism and apply Lemma 5.

So now consider the case when \( H \) is a closed normal subgroup of \( G \). According to [Ke, 12.17] the quotient homomorphism \( \pi : G \to G/H \) admits a Borel section \( s : G/H \to G \).

The set \( T = s(G/H) \), being the image of the Polish space \( G/H \) under an injective Borel map, is Borel in \( G \), see [Ke, 15.1].

Consider the map \( p : G \to H \) assigning to a point \( x \in G \) the point \( p(x) = (s \circ \pi(x))^{-1}x \). We claim that \( p^{-1}(B) = TB \) for any subset \( B \subset H \). Indeed, for any \( t \in T \) and \( b \in B \)

\[
p(tb) = (s \circ \pi(tb))^{-1}tb = (s \circ \pi(t))^{-1}tb = t^{-1}tb = b \in B.
\]

On the other hand, if \( p(x) = b \in B \), then \( b = p(x) = (s \circ \pi(x))^{-1}x \) and thus \( x = (s \circ \pi(x))b \in TB \).

We claim that a subset \( N \subset H \) is Naar null in \( H \) if and only if \( TN \) is Haar null in \( G \). Fix a left-invariant Haar measure \( \lambda \) on \( H \).

Suppose that \( N \) is Haar null in \( H \). Then \( \lambda(N) = 0 \) and we can assume that \( N \) is Borel in \( H \). The product \( TN = p^{-1}(N) \), being the image of the Borel space \( T \times N \) under an injective continuous map, is Borel and thus universally measurable in \( G \), see [Ke, 15.1].

We claim that \( \lambda(xTNy) = 0 \) for all \( x, y \in G \). Given points \( x, y \in G \) let \( s = (x^{-1}y)^{-1} \) and observe that \( xTNy \cap H = xTNy \) and hence \( \lambda(xTNy) = \lambda(xTNy) = \lambda(Ny) = \lambda(N) = 0 \) which means that \( TN \) is Haar null in \( G \).

Now assume conversely, that \( TN \) is Haar null in \( G \). To show that \( N \) is Haar null in \( H \) it suffices to verify that \( \lambda(N) = 0 \). Fix a universally measurable subset \( M \supset TN \) of \( G \) and a probability measure \( \mu \) on \( G \) such that \( \mu(xMy) = 0 \) for all \( x, y \in G \). Consider the convolution \( \lambda \circ \mu \) assigning to a Borel function \( f : G \to \mathbb{R} \) the integral \( \int f \mu \). It is standard to show that \( \lambda \circ \mu(M) = 0 \), see [THJ, 2.4.4]. Denote by \( \chi_M : G \to \{0, 1\} \) the characteristic function of the set \( M \) and applying Fubini Theorem conclude that

\[
0 = \lambda \circ \mu(M) = \int \int \chi_M(xy)dxdy = \int \int \chi_M(xy)dydx = \int \mu \lambda(x^{-1}M)dx.
\]

Then \( \lambda(x^{-1}M) = 0 \) for some \( x \in G \). Since \( M \supset TN \), we get \( 0 = \lambda(x^{-1}TN) = \lambda(x^{-1}(s \circ \pi(x)))N) = \lambda(N) \).

Therefore a subset \( N \subset H \) is Haar zero if and only if \( p^{-1}(N) \) is Haar null in \( G \). Using this observation it is easy to show that \( \text{add}(\mathcal{H} \mathcal{N}(G)) \leq \text{add}(\mathcal{H} \mathcal{N}(H)) = \text{add}(\mathcal{N}) \), \( \text{cov}(\mathcal{H} \mathcal{N}(G)) \leq \text{cov}(\mathcal{H} \mathcal{N}(H)) = \text{cov}(\mathcal{N}) \), and \( \text{non}(\mathcal{H} \mathcal{N}(G)) \geq \text{non}(\mathcal{H} \mathcal{N}(H)) = \text{non}(\mathcal{N}) \).

To show that \( \text{cof}(\mathcal{H} \mathcal{N}(G)) \geq \text{cof}(\mathcal{H} \mathcal{N}(H)) = \text{cof}(\mathcal{N}) \) fix any family \( \mathcal{F} \subset \mathcal{H} \mathcal{N}(G) \) of size \( |\mathcal{F}| = \text{cof}(\mathcal{H} \mathcal{N}(G)) \) such that each Haar null subset of \( G \) lies in some \( F \in \mathcal{F} \). For each set \( F \in \mathcal{F} \) consider the subset \( F' = H \setminus p(G \setminus F) \) of \( H \) which is Haar null in \( H \) since \( p^{-1}(F') \subset F \). We claim that the family \( \{F' : F \in \mathcal{F}\} \) is cofinal in \( \mathcal{H} \mathcal{N}(H) \). Indeed, for any Haar null set \( N \subset H \) the set \( p^{-1}(N) \) is Haar null in \( G \). Then \( p^{-1}(N) \subset F \) for some
$F \in \mathcal{F}$ and hence $N \subset F'$. Therefore $\text{cof}(\mathcal{N}) = \text{cof}(\mathcal{HN}(H)) \leq |\{F' : F \in \mathcal{F}\}| \leq |\mathcal{F}| = \text{cof}(\mathcal{HN}(G))$.

4. Assume that the center $Z = \{g \in G : \forall x \in G \ xg = gx\}$ of $G$ is not locally compact. Then $\text{cov}(\mathcal{HN}(Z)) \leq b \leq \max\{\mathfrak{d}, \text{non}(\mathcal{N})\} \leq \text{non}(\mathcal{HN}(Z))$ by the second statement of this theorem. So it rests to verify that $\text{cov}(\mathcal{HN}(G)) \leq \text{cov}(\mathcal{HN}(Z))$ and $\text{non}(\mathcal{HN}(G)) \geq \text{non}(\mathcal{HN}(Z))$.

According to [Ke, 12.17] the quotient homomorphism $\pi : G \to G/Z$ admits a Borel section $s : G/Z \to G$. Let $T = s(G/Z)$ and consider the map $p : G \to Z$ defined by $p(x) = (s \circ \pi(x))^{-1}x$ for $x \in G$. In the preceding item we have shown that $p$ is a Borel map with $p^{-1}(N) = TN$ for any subset $N \subset H$.

We claim that for any universally measurable Haar null set $N \subset Z$ the set $TN$ is Haar null in $G$. First we note that the set $TN = p^{-1}(N)$, being the preimage of the universally measurable set $N$ under the Borel map $p$, is universally measurable.

Since the set $N$ is Haar null in $Z$, there is a Borel measure $\mu$ on $Z$ such that $\mu(xNy) = 0$ for all $x, y \in H$. We claim that $\mu(xTNy) = 0$ for all $x, y \in G$. Given any points $x, y \in G$ let $t = s \circ \pi(x^{-1}y^{-1})$ and note that $xty \in Z$ and $xTNy \cap Z = xTNy = xtyN$. Then $\mu(xTNy) = \mu(xtyN) = 0$ which means that $TN = p^{-1}(N)$ is a Haar null subset of $G$.

Therefore for any Haar null subset $N \subset Z$ the pre-image $p^{-1}(N) = TN$ is Haar null in $G$. Using this fact it is trivial to show that $\text{cov}(\mathcal{HN}(G)) \leq \text{cov}(\mathcal{HN}(Z))$ and $\text{non}(\mathcal{HN}(G)) \geq \text{non}(\mathcal{HN}(Z))$.

5. If $G$ admits a surjective continuous homomorphism onto a non-locally Polish compact group $H$ with invariant metric, then Lemma 5 and the second item of this theorem imply that $\text{cov}(\mathcal{HN}(G)) \leq \text{cov}(\mathcal{HN}(H)) \leq b$, and $\text{non}(\mathcal{HN}(G)) \geq \text{non}(\mathcal{HN}(H)) \geq \max\{\mathfrak{d}, \text{non}(\mathcal{N})\}$.

**Proof of Theorem 3**

Suppose that a non-locally compact Polish group $G = \prod_{n \geq 0} G_n$ is the product of locally compact Polish groups $G_n$. Without loss of generality we can assume that the groups $G_n$ are not trivial.

1. The estimates $\text{cov}(\mathcal{HN}(G)) \leq \text{cov}(\rho\mathcal{B}(G)) \leq b$ and $\max\{\text{non}(\mathcal{N}), \mathfrak{d}\} \leq \max\{\text{non}(\mathcal{HN}(G)), \text{non}(\rho\mathcal{B}(G))\} \leq \text{non}(\mathcal{HN}(G)) \leq \text{cof}(\mathcal{HN}(G))$ follow from the inclusion $\mathcal{U}(G) \cup \rho\mathcal{B}(G) \subset \mathcal{HN}(G)$ (see Theorem 1) and Lemma 1. To prove that $\text{cof}(\mathcal{HN}(G)) \geq \min\{\mathfrak{d}, \text{non}(\mathcal{N})\}$ we consider separately two cases. If $\text{non}(\mathcal{N}) < \mathfrak{d}$, then $\text{cof}(\mathcal{HN}(G)) \geq \mathfrak{d} > \min\{\mathfrak{d}, \text{non}(\mathcal{N})\}$. If $\text{non}(\mathcal{N}) \geq \mathfrak{d}$, then $\text{cof}(\mathcal{HN}(G)) \geq \mathfrak{d} \geq \min\{\mathfrak{d}, \text{non}(\mathcal{N})\}$ according to (the proof of) Theorem 2.(2).

2. Suppose that all but finitely many groups $G_n$ are amenable. Without loss of generality we can assume that the groups $G_n$ are amenable for $n \geq 1$. For every $n \geq 0$ fix a left-invariant Haar measure $\lambda_n$ on $G_n$. Each group $G_n$, $n \geq 1$, being amenable, satisfies the Følner condition. Using this condition, for every $n \geq 1$ we can construct an increasing sequence $(K_{n,m})_{m \geq 0}$ of compact subsets of the group $G_n$ such that $\bigcup_{m \geq 0} K_{n,m} = G_n$, each $K_{n,m}$ lies in the interior of $K_{n,m+1}$ and $\lambda_n(xK_{n,m+1} \Delta K_{n,m+1}) < 2^{-m} \lambda_n(K_{n,m+1})$ for any $x \in K_{n,m}$.

For every $n \geq 0$ fix a probability measure $\hat{\lambda}_n$ on $G_n$ equivalent to the Haar measure $\lambda_n$ (in the sense that they have the same null sets) and let $\lambda_{0,m} = \hat{\lambda}_n$ for all $m \in \mathbb{N}$. For every $n, m \in \mathbb{N}$ define a probability measure $\lambda_{n,m}$ on the group $G_n$ letting

$$\lambda_{n,m}(B) = \left(1 - \frac{1}{2^m}\right) \frac{\lambda_n(B \cap K_{n,m})}{\lambda_n(K_{n,m})} + \frac{1}{2^m} \hat{\lambda}_n(B)$$
for any Borel subset $B \subset G_n$. For any function $f \in \mathbb{N}^\omega$ by $\mu_f$ denote the tensor product
$\mu_f = \otimes_{n \in \mathbb{N}} \lambda_{n,f(n)}$ which is a probability measure on $G$. In (the proof of) Theorem 4.1 [S2] S. Solecki has shown that a universally measurable subset $N \subset G$ is Haar null in $G$ if and only if there is a function $f \in \mathbb{N}^\omega$ such that $\mu_f(xNy) = 0$ for any $x, y \in G$ if and only if there is a function $f \in \mathbb{N}^\omega$ such that $\mu_g(xNy) = 0$ for any $x, y \in G$ and any $g \in \mathbb{N}^\omega$ with $f \preceq g$.

To estimate the cardinals $\text{add}(\mathcal{HN}(G))$ and $\text{cov}(\mathcal{HN}(G))$ fix any family $S \subset \mathcal{HN}(G)$ of universally measurable Haar null subsets of $G$ with $|S| < b$. Using the mentioned result of S. Solecki, for any $S \subset S$ find a function $f_S \in \mathbb{N}^\omega$ such that $\mu_g(xSy) = 0$ for any $x, y \in G$ and any $g \in \mathbb{N}^\omega$ with $f_S \preceq g$. Since $|S| < b$, the set $\{f_S : S \in S\}$ is bounded in $(\mathbb{N}^\omega, \preceq)$. Consequently, there is a function $f \in \mathbb{N}^\omega$ such that $f_S \preceq f$ for all $S \in S$. For this function $f$ we get $\mu_f(xSy) = 0$ for all $x, y \in G$ and $S \subset S$. Now consider the union $\bigcup S$. If $|S| < \text{add}(N)$, then $\bigcup S$ is universally measurable by Lemma 3 and $\mu_f(x(\bigcup S)y) = 0$ for all $x, y \in G$. Applying S. Solecki Theorem 4.1 [S2] we conclude that the union $\bigcup S$ is Haar null in $G$ and hence $\text{add}(\mathcal{HN}(G)) = \min\{b, \text{add}(N)\} = \text{add}(N)$. If $|S| < \text{cov}(N)$, then $\bigcup S \not\subset G$ (being the union of $< \text{cov}(N)$ many $\mu_f$-zero sets) and thus $\text{cov}(\mathcal{HN}(G)) < \min\{b, \text{cov}(N)\}$.

To prove that $\text{non}(\mathcal{HN}(G)) \geq \max\{c, \text{non}(N)\}$ fix any dominating subset $D \subset \mathbb{N}^\omega$ of size $|D| = c$. For any $f \in D$ find a subset $N_f \subset G$ of size $|N_f| = \text{non}(N)$ such that $\mu_f(N_f) \neq 0$. Then the union $N = \bigcup_{f \in D} N_f$ has size $|N| \leq \max\{c, \text{non}(N)\}$ and is not Haar null. Otherwise, using the S. Solecki result we would find a function $f \in D$ such that $\mu_f(N) = 0$ which is not possible since $\mu_f(N_f) \neq 0$ and $N_f \subset N$.

3. If one of the groups $G_n$ is non-discrete, then we may apply Theorem 2(3) to conclude that $\text{add}(\mathcal{HN}(G)) \leq \text{add}(N)$, $\text{cov}(\mathcal{HN}(G)) \leq \text{cov}(N)$, and $\text{non}(\mathcal{HN}(G)) \geq \text{non}(N)$. So assume that all groups $G_n$ are discrete and infinitely many of them are almost kaleidoscopic. Without loss of generality we can assume that each group $G_{2n}$, $n \geq 0$, is almost kaleidoscopic.

Fix any sequence $(\varepsilon_n)_{n \geq 0}$ of positive real numbers such that

$$\frac{1}{2} < \prod_{n \geq 0} (1 - \varepsilon_n) < \prod_{n \geq 0} (1 + \varepsilon_n) < 2.$$ 

Since the groups $G_{2n}$ are almost kaleidoscopic, for every $n \geq 0$ we can find a finitely supported probability measure $\mu_{2n}$ on $G_{2n}$ and a nontrivial finite partition $\mathcal{P}_n$ of $G_{2n}$ such that $|\mu_{2n}(xPy) - \frac{1}{|\mathcal{P}_n|}| < \frac{\varepsilon_n}{|\mathcal{P}_n|}$ for each $x, y \in G$ and $P \in \mathcal{P}_n$. Endow each set $\mathcal{P}_n$ with the discrete topology and consider the map $p_n : G_{2n} \to \mathcal{P}_n$ assigning to a point $g \in G$ a unique element $P \in \mathcal{P}_n$ containing $g$. Now consider the continuous map $\pi : \prod_{n \geq 0} G_n \to \prod_{n \geq 0} \mathcal{P}_n$ assigning to a sequence $(x_n)_{n \geq 0} \in \prod_{n \geq 0} G_n$ the sequence $(p_n(x_{2n}))_{n \geq 0}$ in $\prod_{n \geq 0} \mathcal{P}_n$. For every $n \geq 0$ fix any probability measure $\mu_{2n+1}$ on the group $G_{2n+1}$ and endow the group $G = \prod_{n \geq 0} G_n$ with the measure $\mu$ equal to the tensor product $\otimes_{n \geq 0} \mu_n$ of the measures $\mu_n$.

On the product $\mathbb{P} = \prod_{n \geq 0} \mathcal{P}_n$ consider the measure $\lambda$ equal to the tensor product $\otimes_{n \geq 0} \lambda_n$ of uniformly distributed measures of $\mathcal{P}_n$’s. For every $m \geq 1$ let $\text{pr}_m : \prod_{n \geq 0} \mathcal{P}_n \to \prod_{0 \leq n < m} \mathcal{P}_n$ be the projection onto the first $m$ coordinates. Let us call a subset $C$ of $\mathbb{P}$ cylindrical if $C = \text{pr}_m^{-1}(A)$ for some $m \geq 1$ and some set $A \subset \prod_{0 \leq n < m} \mathcal{P}_n$. It follows from the choice of the measures $\mu_n$ that for any point $y \in \prod_{0 \leq n < m} \mathcal{P}_n$ the preimage $P = (\text{pr}_m \circ \pi)^{-1}(y)$ has $\mu$-measure satisfying

$$\prod_{0 \leq n < m} \frac{1 - \varepsilon_n}{|\mathcal{P}_n|} \leq \mu(P) \leq \prod_{0 \leq n < m} \frac{1 + \varepsilon_n}{|\mathcal{P}_n|}.$$
and the same estimate is true for any shift \( xPy \) of \( P \). Consequently, for any \( x, y \in G \) and any cylindrical set \( C = pr^{-1}(A) \) we have

\[
\lambda(C) = |A| \prod_{0 \leq n < m} \frac{1}{|P_n|}
\]

and for its preimage \( \pi^{-1}(C) \) we get

\[
|A| \prod_{0 \leq n < m} \frac{1 - \varepsilon_n}{|P_n|} \leq \mu(x\pi^{-1}(C)y) \leq |A| \prod_{0 \leq n < m} \frac{1 + \varepsilon_n}{|P_n|}.
\]

Dividing (2) by (1) we get

\[
\frac{1}{2} \leq \prod_{0 \leq n < m} (1 - \varepsilon_n) \leq \frac{\mu(x\pi^{-1}(C)y)}{\lambda(C)} \leq \prod_{0 \leq n < m} (1 + \varepsilon_n) \leq 2.
\]

Next, we show that the same estimate

\[
\frac{1}{2} \lambda(M) \leq \mu(x\pi^{-1}(M)y) \leq 2\lambda(M).
\]

holds for any universally measurable subset \( M \) of \( \mathbb{P} \). Assuming that \( \mu(x\pi^{-1}(M)y) > 2\lambda(M) \) for some universally measurable set \( M \subset \mathbb{P} \) and points \( x, y \in G \), find a compact subset \( K \subset \pi^{-1}(M) \) with \( \mu(xKy) > 2\lambda(M) \). Express the compact set \( \pi(K) \) as a countable intersection \( \pi(K) = \bigcap_{n \geq 0} C_n \) of a decreasing sequence of cylindrical subsets of \( \mathbb{P} \). Since \( \mu(x\pi^{-1}(\pi(K))y) \geq \mu(xKy) > 2\lambda(M) \geq 2\lambda(\pi(K)) \), the countable additivity of the measures \( \mu \) and \( \lambda \) imply that \( \mu(x\pi^{-1}(C_n)y) > 2\lambda(C_n) \) for a sufficiently large \( n \), which is not possible because of (3). By a similar argument we can show that \( \mu(x\pi^{-1}(M)y) \geq \frac{1}{2}\lambda(M) \) for any universally measurable set \( M \subset \mathbb{P} \) and any points \( x, y \in G \) and thus finish the proof of (4).

This estimate implies that for any universally measurable set \( M \subset \mathbb{P} \) with \( \lambda(M) = 0 \) we get \( \mu(x\pi^{-1}(M)y) = 0 \) for all \( x, y \in G \), which means that \( \pi^{-1}(M) \) is Haar null in \( G \).

Now we prove that the converse is also true, that is a subset \( N \subset \mathbb{P} \) has zero \( \lambda \)-measure if its pre-image \( \pi^{-1}(N) \) is Haar null in \( G \). Assuming that the set \( \pi^{-1}(N) \) is Haar null in \( G \), fix a universally measurable set \( M \supset \pi^{-1}(N) \) of \( G \) and a probability measure \( \nu \) on \( G \) such that \( \nu(xM) = 0 \) for all \( x, y \in G \). Now consider the convolution \( \mu * \nu \) assigning to a bounded continuous function \( f : G \to \mathbb{R} \) the integral \( \int f(xy)dxdy \). It follows from the Fubini Theorem that \( \mu * \nu(M) = 0 \). Consequently, there is a \( \sigma \)-compact set \( S \subset G \setminus M \) with \( \mu * \nu(S) = 1 \). Now consider the image \( \pi(S) \subset \mathbb{P} \) and note that it is a \( \sigma \)-compact set disjoint with \( N \). Let \( Q = \pi^{-1}(\mathbb{P} \setminus \pi(S)) \) and note that \( S \cap Q = \emptyset \) and hence \( \mu * \nu(Q) = 0 \). Denote by \( \chi_Q : G \to \{0, 1\} \) the characteristic function of the set \( Q \). Using the Fubini Theorem and the estimate (4) we get

\[
0 = \mu * \nu(Q) = \int_\mu \int_\nu \chi_Q(xy)dxdy = \int_\nu \int_\mu \chi_Q(xy)dydx = \\
= \int_\nu \mu(x^{-1}Q)dx \geq \frac{1}{2} \int_\nu \lambda(\mathbb{P} \setminus \pi(S))dx = \frac{1}{2} \lambda(\mathbb{P} \setminus \pi(S)).
\]

Hence \( \lambda(\mathbb{P} \setminus \pi(S)) = 0 \) and \( \lambda(N) = 0 \) since \( N \cap \pi(S) = \emptyset \). Therefore we have shown that a subset \( N \subset \mathbb{P} \) has measure \( \lambda(N) = 0 \) if and only if its preimage \( \pi^{-1}(N) \) is Haar null in \( G \).

Now the estimations \( \text{add}(\mathcal{H}\mathcal{N}(G)) \leq \text{add}(\mathcal{N}) \), \( \text{cov}(\mathcal{H}\mathcal{N}(G)) \leq \text{cov}(\mathcal{N}) \) and \( \text{cof}(\mathcal{N}) \leq \text{cof}(\mathcal{H}\mathcal{N}(G)) \) can be derived by analogy with the proof of Theorem 2(3).

4. Finally assume that the group \( G \) is abelian. Then every group \( G_n \), being abelian is amenable and kaleidoscopical, see Proposition 1. Applying Theorem 3(1,2,3) we get the required estimations.
Proof of Theorem 4. Let $G$ be a Polish non-locally compact group. The estimate $\text{cof}(\mathcal{H}(G)) \geq \text{cof}(\mathcal{U}(G), Z_2(G)) > d$ under $\text{non}(\mathcal{N}) \geq d$ was proven in (the proof of) Theorem 2(2). Then under $\text{non}(\mathcal{N}) = d = c$ we get $\text{cof}(\mathcal{U}(G), Z_2(G)) > c$. Since the $\sigma$-algebra $\sigma P$ has size $|\sigma P| = c$, we conclude that there is a universally null subset of $G$, contained in no $\sigma$-projective 2-Zorn subset of $G$.

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