CHARACTERIZING METRIC SPACES WHOSE HYPERSPACES ARE HOMEOMORPHIC TO ℓ_2

T.BANAKH AND R.VOYTSITSKYY

Abstract. It is shown that the hyperspace of nonempty (bounded) closed subsets $\mathrm{Cld}_H(X)$ ($\mathrm{Bdd}_H(X)$) of a metric space (X, d) is homeomorphic to ℓ_2 if and only if the completion $\overline{X}$ of X is connected, and locally connected, X is topologically complete and nowhere locally compact, and each (bounded) subset of X is totally bounded.

1. Introduction

In this paper we characterize metric spaces X whose hyperspaces $\mathrm{Cld}_H(X)$ and $\mathrm{Bdd}_H(X)$ of closed and closed bounded subsets are homeomorphic to the separable Hilbert space l_2. For a metric space (X, d) by $\mathrm{Cld}_H(X)$ we denote the space of non-empty closed subsets of X endowed with the topology generated by the Hausdorff "metric"

$$d_H(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}.$$

For an unbounded metric space (X, d) the "metric" d_H can accept the infinite value but still generates a topology on $\mathrm{Cld}_H(X)$ called the Hausdorff topology. More precisely, this topology is generated by the metric $\min\{1, d_H\}$. By $\mathrm{Bdd}_H(X)$ we denote the subspace of $\mathrm{Cld}_H(X)$ consisting of closed bounded subsets of the metric space (X, d). The hyperspace $\mathrm{Cld}_H(X)$ is a classical object of topology and has applications in Set-Valued Analysis, see e.g., [1]. For a compact metric space X the Hausdorff topology on $\mathrm{Cld}_H(X)$ coincides with the Vietoris topology, another classical topology on $\mathrm{Cld}(X)$, see [9, 2.7.20] (cf. [2]). More generally, the Vietoris topology coincides with the Hausdorff topology on the subspace $\mathrm{Comp}(X)$ of $\mathrm{Cld}(X)$ consisting of non-empty compact subsets of X, see [9, 8.5.16(c)].

One of the brightest results concerning the topology of hyperspaces is the famous Curtis-Shori Theorem [5] characterizing non-degenerate Peano continua as metric spaces X whose hyperspace $\mathrm{Cld}_H(X)$ is homeomorphic to the Hilbert cube $Q = [0, 1]^\omega$. The next step in this direction was made by D. Curtis who proved in [7] that the hyperspace $\mathrm{Comp}(X)$ of non-empty compact subsets of a metric space X is homeomorphic to $Q \times [0, 1)$ if and only if X is non-compact, locally compact, connected, and locally connected. Another result of D. Curtis [6] states that the hyperspace $\mathrm{Comp}(X)$ is homeomorphic to the separable Hilbert space l_2 if and only if X is connected, locally connected, topologically complete and nowhere locally compact. We recall that a space X is *topologically complete* if X is homeomorphic to a complete metric space.

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In this paper we characterize metric spaces X whose hyperspaces $\mathrm{Cld}_H(X)$ and $\mathrm{Bdd}_H(X)$ are homeomorphic to l_2. We call a metric space X *proper* if each closed bounded subset of X is compact.

Theorem 1. *The hyperspace* $\mathrm{Bdd}_H(X)$ *(resp.* $\mathrm{Cld}_H(X)$*) of a metric space* (X,d) *is homeomorphic to* ℓ_2 *if and only if* X *is a topologically complete nowhere locally compact space and the completion* $\overline{X}$ *of* X *is proper (resp. compact), connected, and locally connected.*

Applying this theorem to the metric spaces $\mathbb{R}\setminus\mathbb{Q}$ and $I\setminus\mathbb{Q}$ of irrational numbers on the real line and the interval $I=[0,1]$, we obtain the following result, second part of which is due to W.Kubiś and K.Sakai [10].

Corollary 1. *The hyperspaces* $\mathrm{Cld}_H(I\setminus\mathbb{Q})$ *and* $\mathrm{Bdd}_H(\mathbb{R}\setminus\mathbb{Q})$ *are homeomorphic to* ℓ_2*.*

Let us remark that unlike the hyperspaces $\mathrm{Cld}_H(I\setminus\mathbb{Q})$ and $\mathrm{Bdd}_H(\mathbb{R}\setminus\mathbb{Q})$ the hyperspace $\mathrm{Cld}_H(\mathbb{R}\setminus\mathbb{Q})$ is not homeomorphic to ℓ_2 since it is neither connected nor separable.

As a by-product of the proof of Theorem 1 we obtain the following characterizations of metric spaces whose hyperspaces are separable absolute retracts.

Theorem 2. *The hyperspace* $\mathrm{Bdd}_H(X)$ *(resp.* $\mathrm{Cld}_H(X)$*) of a metric space* X *is a separable AR if and only if the completion* $\overline{X}$ *of* X *is proper (resp. compact), connected and locally connected.*

2. Homotopy dense subsets in the Hilbert cube

A subset Y of a topological space X is *homotopy dense* in X if there is a homotopy $(h_t)_{t\in I}:X\to X$ such that $h_0=id$ and $h_t(X)\subset Y$ for every $t>0$. The following lemma detecting topological copies of ℓ_2 in the Hilbert cube Q is due to D. Curtis [8] and is our main tool in the proof of Theorem 1.

Lemma 1. *A homotopy dense* G_δ*-subset* $X\subset Q$ *with homotopy dense complement in the Hilbert cube* Q *is homeomorphic to* ℓ_2*.*

3. Topology of Lawson semilattices

Theorem 2 will be derived from a more general result concerning Lawson semilattices. By a *topological semilattice* we understand a pair $(L,\vee)$ consisting of a topological space L and a continuous associative commutative idempotent operation $\vee:L\times L\to L$. A topological semilattice $(L,\vee)$ is a *Lawson semilattice* if open subsemilattices form a base of the topology of L. A typical example of a Lawson semilattice is the hyperspace $\mathrm{Cld}_H(X)$ endowed with the operation of union $\cup$, see [13, 5.4].

Each semilattice $(L,\vee)$ carries a natural partial order: $x\le y$ iff $x\vee y=y$. A semilattice $(L,\vee)$ is called *complete* if each subset $A\subset L$ has the smallest upper bound $\sup A\in L$. It is well-known (and can be easily proved) that each compact topological semilattice is complete.

Lemma 2. *If* L *is a locally compact Lawson semilattice, then each compact subset* $K\subset L$ *has the smallest upper bound* $\sup K\in L$*. Moreover, the map* $\sup:\mathrm{Comp}(L)\to L$, $\sup:K\mapsto\sup K$*, is a continuous semilattice homomorphism. Also for every subset* $A\subset L$ *with compact closure* $\overline{A}$ *we have* $\sup A=\sup\overline{A}$.

This lemma easily follows from its compact version proved by J. Lawson in [12].

In Lawson semilattices many geometric questions reduce to the one-dimensional level. The following fact illustrating this phenomenon is proved in [13].

Lemma 3. *Let X be a dense subsemilattice of a metrizable Lawson semilattice L. If X is relatively LC^0 in L (and X is path-connected), then X and L are ANRs (ARs) and X is homotopy dense in L.*

A subset $Y \subset X$ is defined to be *relatively* LC^0 in X if for every $x \in X$, each neighborhood U of x in X contains a smaller neighborhood V of x such that every two points of $V \cap Y$ can be joined by a path in $U \cap Y$.

Under a suitable completeness condition, the density of a subsemilattice is equivalent to the homotopical density.

A subsemilattice X of semilattice L is defined to be *relatively complete* in L if for any subset $A \subset X$ having the smallest upper bound $\sup A$ in L this bound belongs to X.

Proposition 1. *Let L be a metrizable locally compact locally connected Lawson semilattice. Each dense relatively complete subsemilattice $X \subset L$ is homotopy dense in L.*

Proof. According to Lemma 3 it suffices to check that X is relatively LC^0 in L. Given a point $x_0 \in L$ and a neighborhood $U \subset L$ of x_0, consider the canonical retraction $\sup : \mathrm{Comp}(L) \to L$. The space L, being locally compact and locally connected, is locally path-connected, see [11, §50.II]. By Lemma 3, the Lawson semilattice L is an ANR. Using the continuity of sup, find a path-connected neighborhood $V \subset L$ of x_0 such that $\sup(\mathrm{Comp}(\overline{V})) \subset U$. We claim that any two points $x, y \in X \cap V$ can be connected by a path in $X \cap U$. First we construct a path $\gamma : [0,1] \to \overline{V}$ such that $\gamma(0) = x$, $\gamma(1) = y$ and $\gamma^{-1}(X)$ is dense in $[0,1]$. Let $\{q_n : n \in \omega\}$ be a countable dense subset in $[0,1]$ with $q_0 = 0$ and $q_1 = 1$. The space L, being locally compact, admits a complete metric ρ. The path-connectedness of V implies the existence of a continuous map $\gamma_0 : [0,1] \to V$ such that $\gamma_0(0) = x$ and $\gamma_0(1) = y$. Using the local path-connectedness of L we can construct inductively a sequence of functions $\gamma_n : [0,1] \to V$ such that

- $\gamma_n(q_k) = \gamma_{n-1}(q_k)$ for all $k \le n$;
- $\gamma_n(q_{n+1}) \in X$;
- $\sup_{t\in[0,1]} \rho(\gamma_n(t), \gamma_{n-1}(t)) < 2^{-n}$.

Then the map $\gamma = \lim_{n\to\infty} \gamma_n : [0,1] \to \overline{V}$ is continuous and has the desired properties: $\gamma(0) = x$, $\gamma(1) = y$ and $\gamma(q_n) \in X$ for all $n \in \omega$.

For every $t \in [0,1]$ consider the set $\Gamma(t) = \{\gamma(s) : |t - s| \le \mathrm{dist}(t, \{0,1\})\}$. Is is clear that the map $\Gamma : [0,1] \to \mathrm{Comp}(L)$ is continuous and so is the composition $\sup \circ \Gamma : [0,1] \to L$. Observe that $\sup \circ \Gamma(0) = \sup\{\gamma(0)\} = \gamma(0) = x$, $\sup \circ \Gamma(1) = y$, and $\sup \circ \Gamma([0,1]) \subset \sup(\mathrm{Comp}(\overline{V})) \subset U$. Since for every $t \in (0,1)$ the set $\Gamma(t) = \overline{\Gamma(t) \cap X}$, we get $\sup \Gamma(t) = \sup(\Gamma(t) \cap X) \in X$ by the relative completeness of X in L. Thus $\sup \circ \Gamma : [0,1] \to U \cap X$ is a path connecting x and y in U. □

4. Some topological properties of hyperspaces

In this section we collect some easy (and known) lemmas that will be used in the subsequent proofs.

Lemma 4. *For a metric space X the following conditions are equivalent:*

(1) *X is topologically complete;*
(2) *$\mathrm{Cld}_H(X)$ is topologically complete;*
(3) *$\mathrm{Bdd}_H(X)$ is topologically complete.*

Lemma 5. *For a metric space X the following conditions are equivalent:*

(1) *X is nowhere locally compact;*
(2) *$\mathrm{Cld}_H(X)$ is nowhere locally compact;*
(3) *$\mathrm{Bdd}_H(X)$ is nowhere locally compact.*

Lemma 6. *Let X be a metric space. The hyperspace $\mathrm{Cld}_H(X)$ (resp. $\mathrm{Bdd}_H(X)$) is separable if and only if each (bounded) subset of X is totally bounded.*

The following lemma is not trivial and can be found in [3, 3.7].

Lemma 7. *Let X be a dense subspace of a metric space M. The hyperspace $\mathrm{Cld}_H(X)$ (resp. $\mathrm{Bdd}_H(X)$) is an absolute retract if and only if so is the hyperspace $\mathrm{Cld}_H(M)$ (resp. $\mathrm{Bdd}_H(M)$).*

For a metric space X by $\mathrm{Fin}(X)$ we denote the subspace of $\mathrm{Comp}(X)$ consisting of non-empty finite subspaces of X.

Lemma 8. *If Y is a subset of a locally path-connected space X, then the subset $L = \mathrm{Fin}(X) \setminus \mathrm{Fin}(Y)$ is relatively LC^0 in $\mathrm{Comp}(X)$.*

Proof. By the argument of [4] we can show that $\mathrm{Fin}(X)$ is relatively LC^0 in $\mathrm{Comp}(X)$. Consequently, for every compact set $K \in \mathrm{Comp}(X)$ and a neighborhood $U \subset \mathrm{Comp}(X)$ of K there is a neighborhood $V \subset \mathrm{Comp}(X)$ of K such that any two points $A, B \in \mathrm{Fin}(X) \cap V$ can be linked by a path in $\mathrm{Fin}(X) \cap U$. Since $\mathrm{Comp}(X)$ is a Lawson semilattice, we may assume that U and V are subsemilattices of $\mathrm{Comp}(X)$. We claim that any two points $A, B \in L \cap V$ can be connected by a path in $L \cap U$. Since $L \subset \mathrm{Fin}(X)$, there is a path $\gamma : [0,1] \to U \cap \mathrm{Fin}(X)$ such that $\gamma(0) = A$ and $\gamma(1) = B$. Define a new path $\gamma' : [0,1] \to U \cap \mathrm{Fin}(X)$ letting $\gamma'(t) = \gamma(\max\{0, 2t-1\}) \cup \gamma(\min\{2t, 1\})$. Observe that $A \subset \gamma'(t)$ if $t \le 1/2$ and $B \subset \gamma'(t)$ if $t \ge 1/2$. Since $A, B \notin \mathrm{Fin}(Y)$, we conclude that $\gamma'([0,1]) \subset L \cap U$. □

5. Proof of Theorem 2

Let X be a metric space and $\overline{X}$ be its completion. First we prove that $\mathrm{Bdd}_H(X)$ is a separable AR if and only if $\overline{X}$ is proper, connected and locally connected.

To prove the "only if" part, assume that $\mathrm{Bdd}_H(X)$ is a separable absolute retract. By Lemma 7, the hyperspace $\mathrm{Bdd}_H(\overline{X})$ is a separable absolute retract too. By Lemma 6, the separability of $\mathrm{Bdd}_H(X)$ implies that each bounded subset of X is totally bounded, which is equivalent to the properness of the completion $\overline{X}$ of X. In this case $\mathrm{Comp}(\overline{X}) = \mathrm{Bdd}_H(\overline{X})$ is an absolute retract and we can apply the Curtis Theorem [7] to conclude that the locally compact space $\overline{X}$ is connected and locally connected.

Next, we prove the "if" part of Theorem 2. Assume that the completion $\overline{X}$ of X is proper, connected, and locally connected. Then $\mathrm{Bdd}_H(\overline{X}) = \mathrm{Comp}(\overline{X})$ is a separable locally compact absolute retract by [7]. The subsemilattice $\mathrm{Bdd}_H(X)$, being relatively complete in $\mathrm{Bdd}_H(\overline{X})$ is homotopy dense in $\mathrm{Bdd}_H(\overline{X})$ by Proposition 1.

Now we prove that $\mathrm{Cld}_H(X)$ is a separable AR if and only if $\overline{X}$ is compact, connected and locally connected.

If $\overline{X}$ is compact, connected, and locally connected, then $\mathrm{Cld}_H(X) = \mathrm{Bdd}_H(X)$ is a separable AR by the preceding case. Conversely, if $\mathrm{Cld}_H(X)$ is separable AR, then Lemma 6 guarantees that X is totally bounded, and hence $\mathrm{Cld}_H(X) = \mathrm{Bdd}_H(X)$ and we can apply the preceding case to conclude that $\overline{X}$ is connected and locally connected. Also the space $\overline{X}$ is compact, being the completion of a totally bounded metric space X.

6. Proof of Theorems 1

Let X be a metric space. If $\mathrm{Bdd}_H(X)$ (resp. $\mathrm{Cld}_H(X)$) is homeomorphic to ℓ_2, then X is topologically complete and nowhere locally compact by Lemmas 4 and 5. Since ℓ_2 is a separable AR, we may apply Theorem 2 to conclude that the completion $\overline{X}$ of X is connected, locally connected, and proper (resp. compact). This proves the "only if" part of Theorem 1.

To prove the "if" part, assume that X is topologically complete and nowhere locally compact and the completion $\overline{X}$ of X is proper, connected and locally connected. First we consider the case of compact $\overline{X}$. By Curtis-Shori Theorem [5], the hyperspace $\mathrm{Cld}_H(\overline{X}) = \mathrm{Comp}(\overline{X})$ is homeomorphic to Q. Now consider the map $e : \mathrm{Cld}_H(X) \to \mathrm{Cld}_H(\overline{X})$ assigning to each closed subset $F \subset X$ its closure $\overline{F}$ in $\overline{X}$ and note that this map is an isometric embedding, which allows us to identify the hyperspace $\mathrm{Cld}_H(X)$ with the subspace $\{F \in \mathrm{Cld}_H(\overline{X}) : F = \mathrm{cl}(F \cap X)\}$ of $\mathrm{Cld}_H(\overline{X})$. It is easy to check that this subspace is dense and relatively complete in the Lawson semilattice $\mathrm{Cld}_H(\overline{X})$. Then it is homotopically dense in $\mathrm{Cld}_H(\overline{X})$ by Proposition 1 and Lemma 3. By Lemma 4, the subset $\mathrm{Cld}_H(X)$, being topologically complete, is a G_δ-set in $\mathrm{Cld}_H(\overline{X})$. Since X is nowhere locally compact, the complement $\overline{X} \setminus X$ is dense in $\overline{X}$. By Lemmas 4 and 8, the dense subsemilattice $L = \mathrm{Fin}(\overline{X}) \setminus \mathrm{Fin}(X)$ is homotopy dense in $\mathrm{Cld}_H(\overline{X})$. Since $L \cap \mathrm{Cld}_H(X) = \emptyset$, we get that $\mathrm{Cld}_H(X)$ is a homotopy dense G_δ-subset in $\mathrm{Cld}_H(\overline{X})$ with homotopy dense complement. Applying Lemma 1 we conclude that the space $\mathrm{Cld}_H(X)$ is homeomorphic to ℓ_2.

Next, we consider the case of non-compact $\overline{X}$. It follows from the properness of $\overline{X}$ that $\mathrm{Bdd}_H(\overline{X}) = \mathrm{Comp}_H(\overline{X})$ and hence $\mathrm{Bdd}_H(\overline{X})$ is homeomorphic to the Hilbert cube with deleted point $Q \setminus \{pt\}$ by the Curtis Theorem [7]. Repeating the preceding argument, we can prove that $\mathrm{Bdd}_H(X)$ can be identified with homotopy dense G_δ-set with homotopy negligible complement in $\mathrm{Bdd}_H(\overline{X})$. Since the one-point compactification of $\mathrm{Bdd}_H(\overline{X})$ is homeomorphic to the Hilbert cube, we can apply Lemma 1 to conclude that $\mathrm{Bdd}_H(X)$ is homeomorphic to ℓ_2.

References

[1] H.A. Antosiewicz, A. Cellina, Continuous extensions of multifunctions, *Ann. Polon. Math.* **34**:1 (1977), 107–111.

[2] G. Beer, *Topologies on closed and closed convex sets*, MIA 268, Kluwer Acad. Publ., Dordrecht, 1993.

[3] T. Banakh, R. Voytsitskyy, Characterizing metric spaces whose hyperspaces are absolute neighborhood retracts, Topology Appl. (to appear).

[4] D. Curtis, N.T.-Nhu, Hyperspaces of finite subsets which are homeomorphic to $\aleph_0$-dimensional linear metric spaces, *Topology Appl.* **19** (1985), 251–260.

[5] D.W. Curtis, R.M. Schori, Hyperspaces of Peano continua are Hilbert cubes, *Fund. Math.* **101** (1978), 19–38.

[6] D.W. Curtis, Hyperspaces homeomorphic to Hilbert space, *Proc. Amer. Math. Soc.* **75** (1979), 139–152.

[7] D.W. Curtis, Hyperspaces of noncompact metric spaces, *Compositio Math.* **40** (1980), 126–130.

[8] D.W. Curtis, Boundary sets in the Hilbert cube, *Topology Appl.* **20** (1985), 201–221.

[9] R. Engelking, *General Topology.* PWN, Warszawa, 1977.

[10] W. Kubiś, K. Sakai, Hausdorff hyperpsaces of $\mathbb{R}^m$ and their dense subspaces, Preprint.

[11] K.Kuratowski, *Topology, II*, Academic Press and PWN, 1968.

[12] J.D. Lawson, Topological semilattices with small semilattices, *J. London Math. Soc.* (2) **1** (1969) 719–724.

[13] W. Kubiś, K. Sakai, M. Yaguchi, Hyperspaces of separable Banach space with the Wijsman topology, *Topology Appl.* **148** (2005), 7–32.

Instytut Matematyki, Akademia Świętokrzyska, Kielce, Poland and
Department of Mathematics, Ivan Franko Lviv National University, Lviv, Ukraine
E-mail address: tbanakh@yahoo.com and voytsitski@mail.lviv.ua