Separation properties between the σ-compactness and Hurewicz property

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Abstract

We introduce and study so-called $\mathcal{C}$-separation properties leading to a fine hierarchy of spaces with the Hurewicz property $\bigcup_{fin}(\mathcal{O},\Gamma)$. By definition, a topological space X has the *$\mathcal{C}$-separation* property for a class $\mathcal{C}$ of spaces if for any embedding $X \subset C$ into a space $C \in \mathcal{C}$ there is a σ-compact subset $A \subset C$ containing X. It turns out that the classical Hurewicz property is equivalent to the $\mathcal{G}_\delta$-separation property for the class $\mathcal{G}_\delta$ of Polish spaces. On the other extreme there are Sierpiński sets having the $\mathcal{UM}$-separation property for the class $\mathcal{UM}$ of universally measurable spaces. We construct several examples distinguishing the $\mathcal{C}$-separation properties for various descriptive classes $\mathcal{C}$ and also study the interplay between the $\mathcal{C}$-separation properties and the selection principles $\bigcup_{fin}(\mathcal{C},\Gamma)$.

Introduction

Trying to define a covering counterpart of the σ-compactness, W. Hurewicz [Hu] introduced spaces currently refered to as Hurewicz spaces or, in the modern terminology, spaces satisfying the selection principle $\bigcup_{fin}(\mathcal{O},\Gamma)$. Namely, a topological space X is called *Hurewicz* (equivalently, satisfies $\bigcup_{fin}(\mathcal{O},\Gamma)$) if for any sequence $(u_n)_{n\in\omega}$ of open covers of X in each cover u_n one can choose a finite subfamily $v_n \subset u_n$ so that $(\cup v_n)_{n\in\omega}$ is a γ-cover of X. The latter means that each point $x \in X$ belongs to all but finitely many sets $\cup v_n$.

It is clear that each σ-compact space is Hurewicz and each Hurewicz space is Lindelöf. Thus working with metrizable Hurewicz spaces, it is natural to restrict ourselves to metrizable separable spaces. That is why in this paper all spaces are assumed metrizable and separable.

Hurewicz spaces need not be σ-compact, see [JMMS]. However, they can be characterized with help of σ-compact sets as follows [JMMS, Theorem 5.7]: a metrizable space X is Hurewicz if and only if for any embedding $X \subset \tilde{X}$ into a complete metric space $\tilde{X}$ there is a σ-compact set A with $X \subset A \subset \tilde{X}$. Having in mind this characterization of the Hurewicz property, let us define a stronger property depending on a class $\mathcal{C}$ of topological spaces and called the $\mathcal{C}$-separation property. Namely, we define a topological space X to have the *$\mathcal{C}$-separation* property if for any embedding of X into a space $C \in \mathcal{C}$ there is a σ-compact subset A with $X \subset A \subset C$. Therefore, in the framework of metrizable separable spaces the Hurewicz property is equivalent to the $\mathcal{G}_\delta$-separation property, where $\mathcal{G}_\delta$ stands for the class of Polish (= absolute G_δ-) spaces.

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In this paper we shall study the $\mathcal{C}$-separation property for some other Borel and projective classes of spaces. The interest to such kind of activity was motivated by Problem 2 [JMMS] concering the Scheepers diagram. This problem was reduced in [Zd] to finding a Hurewicz subspace $H \subset \mathbb{Z}_2^{\omega\times\omega}$ that algebraically generates the subgroup

$$G = \{x \in \mathbb{Z}_2^{\omega\times\omega} : \forall k \in \omega \; \exists n \in \omega \; \forall m \geq n \;\; (x_{k,m} = 0)\}$$

of $\mathbb{Z}_2^{\omega\times\omega}$. The group G is not σ-compact but belongs to the Borel class $\mathcal{F}_{\sigma\delta}$ of absolute $F_{\sigma\delta}$-sets. This implies that the group G cannot be generated by a subset having the $\mathcal{F}_{\sigma\delta}$-separation property. In this situation, taking into account that Hurewicz spaces have the $\mathcal{G}_\delta$-separation property, it is natural to ask if they have the stronger $\mathcal{F}_{\sigma\delta}$-separation property, see [Ts][1]. Surprisingly, the answer to this question is consistently "not", see Theorem 1(1) below. So the $\mathcal{G}_\delta$- and $\mathcal{F}_{\sigma\delta}$-separation properties are distinct and hence $\mathcal{C}$-separation properties for various Borel classes do provide a non-trivial hierarchy of Hurewicz spaces.

We shall be interested in the following classes of metrizable separable spaces:

- $\mathcal{F}$, the class of all compacta (=absolute closed sets);
- $\mathcal{G}_\delta$, the class of all Polish spaces (= absolute G_δ-sets);
- $\mathcal{G}_\delta\mathcal{F}_\sigma$, the class of spaces that can be written as the union of a Polish and a σ-compact subspaces;
- $\sigma\mathcal{G}_\delta$, the class of σ-complete spaces (= countable unions of closed Polish subspaces);
- $\mathcal{B}$, the class of absolute Borel spaces;
- $\mathcal{A}$, the class of analytic spaces (a space X is *analytic* if it is a continuous image of Polish spaces);
- $\mathcal{UM}$, the class of universally measurable spaces (a space X is *universally measurable* if for any embedding of X to a Polish space Y the set X is measurable with respect to any Borel probability measure μ on Y).

By [Ke, Theorem 21.10] each analytic space is universally measurable, so we have the chain of inclusions

$$\mathcal{F} \subset \mathcal{G}_\delta \subset \mathcal{G}_\delta\mathcal{F}_\sigma \subset \mathcal{B} \subset \mathcal{A} \subset \mathcal{UM}$$

determining the hierarchy of the corresponding separation properties:

$$\begin{array}{ccccccccc}
 & & & & & & & & \sigma\mathcal{G}_\delta\text{-separation} \\
 & & & & & & & & \Updownarrow \\
\mathcal{UM}\text{-separation} & \Rightarrow & \mathcal{A}\text{-separation} & \Rightarrow & \mathcal{B}\text{-separation} & \Rightarrow & \mathcal{G}_\delta\mathcal{F}_\sigma\text{-separation} & \Rightarrow & \mathcal{G}_\delta\text{-separation} \\
\Uparrow \;\; \not\downarrow & & & & & & & & \Updownarrow \\
\sigma\text{-compact} & & & & & & & & \text{Hurewicz}
\end{array}$$

[1] We were not aware of the consistently negative answer to this question (given by Theorem 1(1)) at the moment of publication of [Ts].

The equivalence between the $\mathcal{G}_\delta$- and $\sigma\mathcal{G}_\delta$-separation properties will be proved in Theorem 1(2). An example of a non-σ-compact space with the $\mathcal{UM}$-separation property was actually constructed by W.Sierpiński [Si]. We recall that a subset S of the real line $\mathbb{R}$ is called *Sierpiński* if S is uncountable but has countable intersection with each Lebesque null subset of $\mathbb{R}$. Such sets can be easily constructed under (CH) but do not exist under (MA+$\neg$CH). Having in mind the notion of a Sierpinski set, let us define a subset $S \subset \mathbb{R}$ to be *κ-Sierpinski* for a cardinal κ if $|S \cap N| < \kappa$ for every Lebesgue null set $N \subset \mathbb{R}$. Thus a Sierpiński set is exactly an uncountable $\aleph_1$-Sierpiński set. In Theorem 1(3) we shall show that each $\aleph_1$-Sierpiński set has the $\mathcal{UM}$-separation property. The $\mathcal{A}$-separation property (which is weaker than the $\mathcal{UM}$-separation property) can be established for the wider class of $\mathfrak{b}$-Sierpiński sets with help of the selection principle $\bigcup_{fin}(\mathcal{A},\Gamma)$ defined by analogy with $\bigcup_{fin}(\mathcal{O},\Gamma)$ as follows.

Given a class $\mathcal{C}$ of spaces and a space X, let $\mathcal{C}(X)$ be the family of subsets $C \subset X$ for which there is a metrizable compactification $\tilde{X}$ of X and a subset $\tilde{C} \in \mathcal{C}$ of $\tilde{X}$ with $C = \tilde{C} \cap X$. Observe that $\mathcal{B}(X)$ is the family of all Borel subsets of X while $\mathcal{A}(X)$ is the family of Souslin subsets of X (i.e., subsets obtained by application of the Souslin A-operation to the family of closed subsets fo X).

We shall say that a topological space X satisfies the selection principle $\bigcup_{fin}(\mathcal{C},\Gamma)$ if for any sequence $(u_n)_{n\in\omega}$ of countable covers $u_n \subset \mathcal{C}(X)$ of X in each cover u_n we can choose a finite subcollection $v_n \subset u_n$ such that the sequence $(\cup v_n)_{n\in\omega}$ is a γ-cover of X. Let us mention that the selection principle $\bigcup_{fin}(\mathcal{B},\Gamma)$ (denoted by $\bigcup_{fin}(\mathcal{B},\mathcal{B}_\Gamma)$) was introduced and studied in [ST]. It turns out that in a suitable model of ZFC each space $X \in \bigcup_{fin}(\mathcal{B},\Gamma)$ is countable!

In the following diagram we collect all known relations between the $\mathcal{C}$-separation properties and the corresponding selection principles $\bigcup_{fin}(\mathcal{C},\Gamma)$ for various classes $\mathcal{C}$.

$$
\begin{tikzcd}
 & & & & & \sigma\mathcal{G}_\delta\text{-separation} \arrow[d, Leftrightarrow] \\
\sigma\text{-compact} \arrow[r, Rightarrow] & \mathcal{UM}\text{-separation} \arrow[r, Rightarrow] & \mathcal{A}\text{-separation} \arrow[r, Rightarrow] \arrow[l, bend right, "|" marking] & \mathcal{B}\text{-separation} \arrow[r, Rightarrow] & \mathcal{G}_\delta\mathcal{F}_\sigma\text{-separation} \arrow[r, Rightarrow] & \mathcal{G}_\delta\text{-separation} \arrow[l, bend right, "|" marking] \\
\aleph_1\text{-Sierpiński} \arrow[r, Rightarrow] \arrow[u, "|" marking] \arrow[ur] & \mathfrak{b}\text{-Sierpiński} \arrow[r, Rightarrow] \arrow[u, "|" marking] & \bigcup_{fin}(\mathcal{UM},\Gamma) \arrow[r, Rightarrow] \arrow[u, Rightarrow] & \bigcup_{fin}(\mathcal{B},\Gamma) \arrow[r, Leftrightarrow] \arrow[u, Rightarrow] & \bigcup_{fin}(\mathcal{F},\Gamma) \arrow[r, Rightarrow] \arrow[u, Rightarrow] & \bigcup_{fin}(\mathcal{O},\Gamma) \arrow[u, Leftrightarrow]
\end{tikzcd}
$$

The equivalence $\bigcup_{fin}(\mathcal{B},\Gamma) \Leftrightarrow \bigcup_{fin}(\mathcal{F},\Gamma)$ was proved in [BRR]. All other non-trivial implications from the above diagram are proved in the following

Theorem 1. *1. Under $\mathfrak{b} = \mathfrak{d}$ there is Borel space X and a Hurewicz subspace H of X such that X is the union of a Polish space and a σ-compact space, but no σ-compact subspace of X contains H. Consequently, there exists a Hurewicz space H failing to have the $\mathcal{G}_\delta\mathcal{F}_\sigma$-separation property.*

2. The selection principle $\bigcup_{fin}(\mathcal{O},\Gamma)$ is equivalent to the $\mathcal{G}_\delta$-separation property and is equivalent to the $\sigma\mathcal{G}_\delta$-separation property.

3. Every $\aleph_1$-Sierpiński set has the $\mathcal{UM}$-separation property.

4. $\mathfrak{b}$-Sierpiński sets satisfy the selection principle $\bigcup_{fin}(\mathcal{UM},\Gamma)$.

5. *Every topological space $X \in \bigcup_{fin}(\mathcal{B},\Gamma)$ has the $\mathcal{B}$-separation property.*

6. *Every topological space $X \in \bigcup_{fin}(\mathcal{UM},\Gamma)$ has the $\mathcal{A}$-separation property.*

7. *Each uncountable space X of size $|X| < \min\{\mathfrak{b}, \mathrm{non}(\mathcal{L})\}$ is $\mathfrak{b}$-Sierpiński and hence has the $\mathcal{A}$-separation property but fails to have the $\mathcal{UM}$-separation property.*

Here by $\mathrm{non}(\mathcal{L})$ we denote the smallest cardinality of a subset $S \subset \mathbb{R}$ that does not belong to the σ-ideal $\mathcal{L}$ of Lebesgue null subsets of $\mathbb{R}$.

Taking into account that the class $\bigcup_{fin}(\mathcal{B},\Gamma)$ can consistently be equal to the class of countable spaces, we can ask

Problem 2. *Is it consistent that each space with the $\mathcal{B}$-separation (resp. $\mathcal{A}$-, $\mathcal{UM}$-separation) property is σ-compact?*

At the moment we have only a consistent example distinguishing the $\mathcal{G}_\delta$- and $\mathcal{F}_{\sigma\delta}$-separation properties and a consistent example distinguishing the $\mathcal{A}$- and $\mathcal{UM}$-separation properties. What about the other separation properties?

Problem 3. *Is there a space with the $\mathcal{F}_{\sigma\delta}$-separating property but without the $\mathcal{B}$-separation property? Is there a space with the $\mathcal{B}$-separating property but without the $\mathcal{A}$-separation property?*

1 Proof of Theorem 1

First we introduce some notations. As expected, ω stands for the discrete space of all finite ordinals; $\bar{\omega} = \omega \cup \{\infty\}$ is a convergent sequence with the limit point ∞, which is assumed to be larger than all elements of ω. We denote by

- ω^ω the space of all functions from ω to ω, endowed with the Tychonov topology;
- ω^ω_0 the dense subspace of ω^ω, consisting of all eventually zero sequences;
- $\omega^{\uparrow\omega}$ the closed subspace of ω^ω, consisting of non-decreasing functions;
- $\omega^{\uparrow\omega}_b$ the dense subspace of $\omega^{\uparrow\omega}$, consisting of all bounded non-decreasing functions;
- $\bar{\omega}^{\uparrow\omega}$ the space of all non decreasing elements of $\bar{\omega}^\omega$;
- $\bar{\omega}^{\uparrow\omega}_\infty = \{x \in \bar{\omega}^{\uparrow\omega} : \exists n \in \omega\ \forall m \geq n\ \ x_m = \infty\}$ the countable set of all "eventually infinite" elements of $\bar{\omega}^{\uparrow\omega}$.

For $x, y \in \omega^\omega$, the notation $x \leq y$ (resp. $x \leq^* y$) means that $x_n \leq y_n$ for all (but finitely many) $n \in \omega$. The smallest cardinalities of an unbounded and dominating subsets of ω^ω with respect to $\leq^*$ are standardly denoted by $\mathfrak{b}$ and $\mathfrak{d}$, respectively (see [Va] for more details).

By a $\mathfrak{b}$-*scale* we understand a transfinite function sequence $B = \{b^\alpha : \alpha < \mathfrak{b}\} \subset \omega^{\uparrow\omega}$ such that $b^\alpha \leq^* b^\beta$ for all $\alpha < \beta$, and B is unbounded with respect to $\leq^*$.

Theorem 4. (Bartoszynski-Tsaban [BT]). *For any $\mathfrak{b}$-scale $B \subset \omega^{\uparrow\omega}$ the union $B \bigcup \bar{\omega}^{\uparrow\omega}_\infty$ has the Hurewicz property $\bigcup_{fin}(\mathcal{O},\Gamma)$.*

We are in a position now to present a proof of Theorem 1.

1. The desired space X will be constructed as a subspace of $\omega^{\uparrow\omega}$. The latter space is homeomorphic to the countable product ω^ω with help of the homeomorphism $h : \omega^{\uparrow\omega} \to \omega^\omega$ defined by $h : (x_i)_{i\in\omega} \mapsto (x_i - x_{i-1})_{i\in\omega}$ (we put $x_{-1} = 0$). Decomposing ω into the countable union $\omega = \bigcup_{i\in\omega} \Omega_i$ of pairwise disjoint infinite subsets, we obtain a coordinate-permutating homeomorphism $\Psi : \omega^\omega \to \prod_{i\in\omega} \omega^{\Omega_i}$. It follows that

$$P = (\Psi \circ h)^{-1}(\prod_{i\in\omega} \omega^{\Omega_i} \setminus \omega_0^{\Omega_i}) = \{x \in \omega^{\uparrow\omega} : \forall i \ \exists^\infty n \in \Omega_i \ (x_n < x_{n+1})\}$$

is a dense G_δ-subset of $\omega^{\uparrow\omega}$, where $\omega_0^{\Omega_i}$ is the set of all elements of ω^{Ω_i} whose all but finitely many coordinates are equal to 0. Let $X = P \cup \omega_b^{\uparrow\omega}$.

Claim 1. *For every unbounded function $f \in \omega^{\uparrow\omega}$ the intersection $X \cap \downarrow f$ is not σ-compact, where $\downarrow f = \{g \in \omega^{\uparrow\omega} : g \le f\}$. Therefore, $X \cap \downarrow f$ is not contained in any σ-compact subspace of X.*

Proof. By the definition of P, $P \cap \downarrow f$ is a dense G_δ-subset of $\downarrow f$, and hence $\downarrow f \setminus X \subset \downarrow f \setminus P$ is a meager subset of $\downarrow f$. The set $\downarrow f \setminus X$ contains all unbounded elements $y \in \downarrow f$, such that there exists $i \in \omega$ with the property $y_n = y_{n+1}$ for all $n \in \Omega_i$. Consequently, $\downarrow f \setminus X$ is dense in $\downarrow f$.

Assume, contrary to our claim, that $X \cap \downarrow f = \bigcup_{n\in\omega} C_n$, where C_n is compact for all $n \in \omega$. Since $C_n \cap (\downarrow f \setminus X) = \emptyset$, C_n is nowhere dense in $\downarrow f$ for all $n \in \omega$, and hence $X \cap \downarrow f$ is a meager subset of $\downarrow f$. Thus the Polish space $\downarrow f$ is meager being a union of two of its meager subsets $X \cap \downarrow f$ and $\downarrow f \setminus X$, a contradiction. $\square$

Now we return to the proof of the first item of Theorem 1 and assuming $\mathfrak{b} = \mathfrak{d}$ we shall construct a Hurewicz subspace $H \subset X$. The equality $\mathfrak{b} = \mathfrak{d}$ yields a transfinite sequence $(f^\alpha)_{\alpha<\mathfrak{b}}$ of non-decreasing unbounded functions such that $f^\alpha \ge^* f^\beta$ for any $\alpha < \beta < \mathfrak{b}$, and for any non-decreasing unbounded function $f \in \omega^{\uparrow\omega}$ there is an ordinal $\alpha < \mathfrak{b}$ with $f^\alpha \le^* f$. By [vD, Theorem 8.10(d)] (see also [vE]), the family of σ-compact subsets of X has cofinality $\mathfrak{d}$[2], which means that there is a family $\{K_\alpha\}_{\alpha<\mathfrak{d}}$ of σ-compact subsets of X such that each σ-compact subset of X lies in some K_α. By transfinite induction we can construct a sequence of functions $(x^\alpha)_{\alpha<\mathfrak{d}} \subset X \subset \omega^{\uparrow\omega}$ such that

(i) $x^\alpha \le f^\alpha$;
(ii) $x^\alpha \le^* x^\beta$ for all $\beta < \alpha$; and
(iii) $x^\alpha \notin K_\alpha$

for every $\alpha < \mathfrak{b}$. The choice of x^α is always possible by Claim 1 and $\mathfrak{b} = \mathfrak{d}$. Moreover, the way K_α and x^α were chosen guarantees that H is contained in no σ-compact subset of X. Therefore it suffices to prove the subsequent

Claim 2. *$H = \omega_b^{\uparrow\omega} \cup \{x^\alpha : \alpha < \mathfrak{b}\}$ has the Hurewicz property.*

[2] As it was shown in [vD], the cofinality of the family of compact subsets of any absolute $F_{\sigma\delta}$ space is at most $\mathfrak{d}$. This result was extended in [vE] to the family of all coanalytic spaces. Now, if $\{L_\alpha : \alpha < \mathfrak{d}\}$ is a cofinal family of *compact* subsets of X^ω, then $\{\bigcup_{n\in\omega} \mathrm{pr}_n(L_\alpha) : \alpha < \mathfrak{d}\}$ is a cofinal family of *σ-compact* subspaces of X.

Proof. Set $\min(\emptyset) = \infty$. Given any $x \in \bar\omega^{\uparrow\omega}$ and $n \in \omega$, set $\phi(x)_n = \min\{m \in \omega : x_m > n\}$. Then ϕ is easily seen to be a autohomeomorphism of $\bar\omega^{\uparrow\omega}$. In addition, $\phi(\omega_b^{\uparrow\omega}) = \bar\omega_\infty^{\uparrow\omega}$ and $\{\phi(x^\alpha) : \alpha < \mathfrak{b}\}$ is a $\mathfrak{b}$-scale. It remains to apply Theorem 4. □

2. At first we show that the Hurewicz property $\bigcup_{fin}(\mathcal{O},\Gamma)$ is equivalent to the $\mathcal{G}_\delta$-separation one.

Fix $X \subset Y$, where Y is a Polish space and X has the property $\bigcup_{fin}(\mathcal{O},\Gamma)$. Let Z be a metrizable compactification of Y. Since Y is a G_δ-subset of Z, there exists a decreasing sequence $(O_n)_{n\in\omega}$ of open subsets of Z such that $Y = \bigcap_{n\in\omega} O_n$. For every $x \in X$ and $n \in \omega$ find an open subset $U(n,x)$ of Z such that $x \in U(n,x) \subset \mathrm{cl}_Z(U(n,x)) \subset O_n$ and consider the sequence $(u_n = \{U_{n,x} : x \in X\})$ of open covers of X by open subsets of Z. (Here $\mathrm{cl}_Z(A)$ denotes the closure in Z of a subset $A \subset Z$.) Since X has the property $\bigcup_{fin}(\mathcal{O},\Gamma)$, there exists a sequence $(K_n)_{n\in\omega}$ of finite subsets of X such that $X \subset \bigcup_{n\in\omega}\bigcap_{m\ge n}\cup_{x\in K_m} U(m,x)$. Now, it sufficies to observe that

$$F = \bigcup_{n\in\omega}\bigcap_{m\ge n}\bigcup_{x\in K_m} \mathrm{cl}_Z U(m,x)$$

is a σ-compact subset of Y containing X.

Now, assume that X has the $\mathcal{G}_\delta$-separation property. Let (w_n) be a sequence of open covers of X and Y be a Polish space containing X. Let u_n be a family of open subsets of Y with the property $w_n = \{U \cap X : U \in u_n\}$. For every $n \in \omega$ denote by O_n the union $\cup u_n$ and set $G = \bigcap_{n\in\omega} O_n$. Since $G \supset X$ is a G_δ-subset of Y, there exists a σ-compact subset F of Y such that $X \subset F \subset G$. Let us write F as a countable union $\bigcup_{n\in\omega} F_n$ of its compact subspaces with the property $F_n \subset F_{n+1}$ for all $n \in \omega$. Then for every $n \in \omega$ we can find a finite subset v_n of u_n such that $F_n \subset \cup v_n$. From the above it follows that $\{\cup v_n : n \in \omega\}$ is a γ-cover of $F \supset X$, and hence $\{\bigcup\{U \cap X : U \in v_n\} \ : \ n \in \omega\}$ is a γ-cover of X, which finishes our proof.

After we know that the properties $\bigcup_{fin}(\mathcal{O},\Gamma)$ and the $\mathcal{G}_\delta$-separation coincide, the equivalence of the $\mathcal{G}_\delta$-separation and $\sigma\mathcal{G}_\delta$-separation properties follows from the fact that the Hurewicz property is inherited by closed subsets.

3. Let X be a $\aleph_1$-Sierpiński subset of an universally measurable subset Z of $\mathbb{R}$. Since the Lebesque measure λ is regular [Ke, Theorem 17.10], there exists a σ-compact $F \subset Z$ such that $\lambda(Z \setminus F) = 0$. Since $|X \cap (Z \setminus F)| \le \aleph_0$, the union $F \bigcup X \cap (Z \setminus F)$ is a σ-compact subset of Z containing X.

4. Let S be a $\mathfrak{b}$-Sierpiński subset of $\mathbb{R}$ and $(u_n)_{n\in\omega}$ be a sequence of countable covers of S by universally measurable sets. Let us write u_n in the form $\{U_{n,k} : k \in \omega\}$. The regularity of λ implies that for every $n,k \in \omega$ there exists a Borel (even F_σ) subset $W_{n,k}$ of $U_{n,k}$ such that $\mu(U_{n,k}\setminus W_{n,k}) = 0$. Therefore $|S\cap(U_{n,k}\setminus W_{n,k})| < \mathfrak{b}$ for all $n,k \in \omega$. Set $C = \bigcup_{n,k\in\omega}(U_{n,k}\setminus W_{n,k})$. Since $\mathfrak{b}$ is regular, we conclude that $|C| < \mathfrak{b}$. In addition, for every $n \in \omega$ the family $w_n = \{W_{n,k} : k \in \omega\}$ is a Borel cover of the $\mathfrak{b}$-Sierpiński subset $S' = S \setminus C$ of $\mathbb{R}$. As it was noted in [ST, p. 376], every $\mathfrak{b}$-Sierpiński set has the property $\bigcup_{fin}(\mathcal{B},\Gamma)$, and consequently $\{\bigcup_{k\le k_n} W_{n,k} : n \in \omega\}$ is a γ-cover of S' for some number sequence $(k_n)_{n\in\omega}$.

Since u_n is a cover of S for all $n \in \omega$, for every $c \in C$ we can find a number sequence $(m_n(c))_{n\in\omega}$ such that $c \in \bigcup_{k\le m_n(c)} U_{n,k}$ for all $n \in \omega$. By the definition of the cardinal $\mathfrak{b}$ we can find a sequence $(m_n)_{n\in\omega}$ with the property $(m_n(c))_{n\in\omega} \le^* (m_n)_{n\in\omega}$ for all $c \in C$. A direct verification shows that

$$\{\bigcup_{k\le\max\{k_n,m_n\}} U_{n,k} : n\in\omega\}$$

is a γ-cover of S, which finishes our proof.

5. Let X be topological space with the property $\bigcup_{fin}(\mathcal{B},\Gamma)$, (Y,τ) be a Polish space, and B be a Borel subset of (Y,τ) containing X. According to [Ke, Theorem 13.1], there exists a topology τ_1 on Y such that $\tau_1\supset\tau$, (Y,τ_1) is a Polish space, $B\in\tau_1$, and the σ-algebras of Borel subsets of Y generated by τ and τ_1 coincide. From the above it follows that the space $(X,\tau_1|X)$ has the property $\bigcup_{fin}(\mathcal{O},\Gamma)$, and hence it has the $\mathcal{G}_\delta$-separation property by the already proven second item. Consequently there exists a σ-compact subspace F of (Y,τ_1) such that $X\subset F\subset B$ (B is open in (Y,τ_1) by our choice of τ_1). Then F is σ-compact subspace of (Y,τ) witnessing for the $\mathcal{B}$-separation property of X.

6. Let X be topological space with the property $\bigcup_{fin}(\mathcal{UM},\Gamma)$, Y be a Polish space, and A be an analytic subset of Y containing X. By [Ke, Exercise 14.3], there exists a Polish space Z and a closed subset C of $Y\times Z$ such that $A=\mathrm{pr}_Y(C)$, where $\mathrm{pr}_Y:(y,z)\mapsto y$ is a projection onto Y. Then there exists a map $f:A\to C$ such that $(\mathrm{pr}_Y\circ f)(y)=y$ for all $y\in A$, and $f^{-1}(B)\in\sigma(\mathcal{A}(Y))$ for all $B\in\mathcal{B}(Y\times Z)$, see [Ke, Theorem 18.1]. Recall that $\mathcal{A}(Y)\subset\mathcal{UM}(Y)$. Applying the standard arguments, we conclude that $f(X)$ has the property $\bigcup_{fin}(\mathcal{B},\Gamma)$, and hence it has the $\mathcal{B}$-separation property by the fifth item. Thus there exists a σ-compact subspace F of C with the property $f(X)\subset F$. Then $\mathrm{pr}_Y(F)$ is a witness for the $\mathcal{A}$-separation property of X.

7. This item is a direct consequence of the following

Claim 3. *A space X is countable if and only if X has the $\mathcal{UM}$-separation property and $|X|<\mathrm{non}(\mathcal{L})$.*

Proof. Only the "if" part requires proof. Assume that $X\subset\mathbb{R}$ has size $|X|<\mathrm{non}(\mathcal{L})$ and it has the $\mathcal{UM}$-separation property. From the above it follows that X is universally measurable (moreover, $\mu(X)=0$ for every atomless Borel measure on $\mathbb{R}$). Therefore there exists a σ-compact space F such that $X\subset F\subset X$, which means that X is σ-compact. Since $|X|<\mathrm{non}(\mathcal{L})\le|\mathbb{R}|$, we conclude that X is countable. □

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