OPEN PROBLEMS IN INFINITE-DIMENSIONAL TOPOLOGY

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Abstract. We ask some questions lying on the borderline of infinite-dimensional topology and related areas such as Dimension Theory, Descriptive Set Theory, Banach Space Theory, Theory of Retracts, Algebraic Topology.

Contents

Introduction

The development of Infinite-Dimensional Topology was greatly stimulated by three famous Open Problem Lists: that of Geoghegan [57], West [73] and Dobrowolski, Mogilski [55]. We hope that the present list of problems will play a similar role for further development of Infinite-Dimensional Topology.

We expect that the future progress will happen on the intersection of Infinite-Dimensional Topology with neighbor areas of mathematics: Dimension Theory, Descriptive Set Theory, Analysis, Theory of Retracts. According to this philosophy we formed the current list of problems. We tried to select problems whose solution would require creating new methods.

We shall restrict ourselves by separable and metrizable spaces. A *pair* is a pair (X, Y) consisting of a space X and a subspace $Y \subset X$. By ω we denote the set of non-negative integers.

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Higher-Dimensional Descriptive Set Theory

Many results and objects of infinite-dimensional topology have zero-dimensional counterparts usually considered in the Descriptive Set Theory. As a rule, "zero-dimensional" results have simpler proofs comparing to their higher-dimensional counterparts. Some zero-dimensional results are proved by essentially zero-dimensional methods (like those of infinite game theory) and it is an open question to which extent their higher-dimensional analogues are true. We start with two problems of this sort.

For a class $\mathcal{C}$ of spaces and a number $n \in \omega \cup \{\infty\}$ consider the subclasses $\mathcal{C}[n] = \{C \in \mathcal{C} : \dim C \le n\}$ and $\mathcal{C}[\omega] = \bigcup_{n\in\omega} \mathcal{C}[n]$. Following the tradition of Logic and Descriptive Set Theory, by Σ^1_1 we denote the class of *analytic* spaces, that is metrizable spaces which are continuous images of $\mathbb{N}^\omega$. Also Π^0_α and Σ^0_α stand for the multiplicative and additive classes of absolute Borel spaces corresponding to a countable ordinal α. In particular, Π^0_1, Π^0_2, and Σ^0_2 are the classes of compact, Polish, and σ-compact spaces, respectively. In topology those classes usually are denoted by $\mathcal{M}_0, \mathcal{M}_1$, and $\mathcal{A}_1$, respectively.

Following the infinite-dimensional tradition, we define a space X to be $\mathcal{C}$-*universal* for a class $\mathcal{C}$ of spaces, if X contains a closed topological copy of each space $C \in \mathcal{C}$. According to a classical result of the Descriptive Set Theory [61, 26.12], an analytic space X is $\Pi^0_\xi[0]$-universal for a countable ordinal $\xi \ge 3$ if and only if $X \notin \Sigma^0_\xi$. This observation implies that a space X is $\Pi^0_\xi[0]$-universal if and only if the product $X \times Y$ is $\Pi^0_\xi[0]$-universal for some/any space $Y \in \Sigma_\xi$. The philosophy of this result is that a space X is $\mathcal{C}$-universal if $X \times Y$ is $\mathcal{C}$-universal for a relatively simple space Y.

The following theorem proved in [24, 3.2.12] shows that in some cases this philosophy is realized also on the higher-dimensional level.

Theorem. *Let $\mathcal{C} = \Pi^0_\xi[n]$ where $n \in \omega \cup \{\infty\}$ and $\xi \ge 3$ is a countable ordinal. A space X is $\mathcal{C}$-universal if and only if $X \times Y$ is $\mathcal{C}$-universal for some space $Y \in \Sigma^0_2$.*

However we do not know if the condition $Y \in \Sigma^0_2$ can be replaced with a weaker condition $Y \in \Sigma^0_3$ (which means that Y is an absolute $G_{\delta\sigma}$-space).

? 1001 **Question.** *Let $\mathcal{C} = \Pi^0_\xi[n]$ where $n \in \omega \cup \{\infty\}$ and $\xi \ge 3$ be a countable ordinal. Is a space X is $\mathcal{C}$-universal if $X \times Y$ is $\mathcal{C}$-universal for some space $Y \in \Sigma^0_3$? $Y \in \Sigma^0_\xi$?*

As we already know the answer to this problem is affirmative for $n = 0$.

In fact, the affirmative answer to Question 1001 would follow from the validity of the higher-dimensional version of the Separation Theorem of Louveau and Saint-Raymond [61, 28.18]. Its standard formulation says that two disjoint analytic sets A, B in a Polish space X can be separated by a Σ^0_ξ-set with $\xi \ge 3$ iff the pair $(A \cup B, A)$ is $(\Pi^0_1[0], \Pi_\xi)$-universal.

A pair (X, Y) of spaces is defined to be $\vec{\mathcal{C}}$-*universal* for a class of pairs $\vec{\mathcal{C}}$ if for every pair $(A, B) \in \vec{\mathcal{C}}$ there is a closed embedding $f : A \to X$ with $f^{-1}(Y) = B$. For classes $\mathcal{A}$, $\mathcal{B}$ of spaces by $(\mathcal{A}, \mathcal{B})$ we denote the class of pair (A, B) with $A \in \mathcal{A}$ and $B \in \mathcal{B}$. We recall that Π^0_1 stands for the class of compacta.

The Separation Theorem of Louveau and Saint-Raymond implies that for every $\Pi^0_\xi[0]$-universal subspace X of a space $Y \in \Sigma^0_\xi$ the pair (Y,X) is $(\Pi^0_1[0],\Pi^0_\xi)$-universal. The philosophy of this result is clear: if a $\mathcal{C}$-universal space X for a complex class $\mathcal{C}$ embeds into a "relatively simple" space Y, then the pair (Y,X) is $(\mathcal{M}_0[0],\mathcal{C})$-universal. If the "relatively simple" means "σ-compact", then the above zero-dimensional result has a higher-dimensional counterpart proved in [24, 3.1.2] (see also [15] and [49]).

Theorem. *Let $n \in \omega \cup [\infty]$ and $\mathcal{C} \in \{\Pi^0_\xi, \Sigma^0_\xi : \xi \geq 3\}$. For every $\mathcal{C}[n]$-universal subspace X of a space $Y \in \Sigma^0_2$ the pair (Y,X) is $(\Pi^0_1[n],\mathcal{C})$-universal.*

We do not know if $Y \in \Sigma_2$ in this theorem can be replaced with $Y \in \Sigma^0_3$.

Question. *Let $n \in \omega \cup [\infty]$ and $\mathcal{C} = \Pi^0_\xi$ for a countable ordinal $\xi \geq 3$. Is it true that for each $\mathcal{C}[n]$-universal subspace X of a space $Y \in \Sigma^0_3$ the pair (Y,X) is $(\Pi^0_1[n],\mathcal{C})$-universal?* **1002 ?**

As we already know the answer to this question is affirmative for $n = 0$. Using Theorem 3.2.12 of [24] on preservation of the $\mathcal{C}$-universality by perfect maps one can show that the affirmative answer to Question 1002 implies that to Question 1001.

The third our problem that has higher-dimensional descriptive flavor asks if the higher-dimensional Borel complexity can be concentrated on sets of a smaller dimension. First let us make two simple observations: the Hilbert cube $[0,1]^\omega$ is Π^0_1-universal while its pseudointerior $(0,1)^\omega$ is Π^0_2-universal. In light of these observations one could suggest that for each $\xi \geq 1$ there is a one-dimensional space X whose countable power X^ω is Π^0_ξ-universal. But this is not true: no finite-dimensional space X has Σ^0_2-universal countable power X^ω (see [16], [40], [23]). On the other hand, for every meager locally path connected space X the $(2n+1)$-st power X^{2n+1} is $\Sigma^0_2[n]$-universal (which means that X^{2n+1} contains a closed topological copy of each n-dimensional σ-compact space), see [17].

Question. *Let $\mathcal{B} \in \{\Pi^0_\xi, \Sigma^0_\xi : \xi \geq 1\}$ is a Borel class. Is there a one-dimensional space X (in $\mathcal{B}$) with $\mathcal{B}[\omega]$-universal power X^ω?* **1003 ?**

The answer to this problem is affirmative for the initial Borel classes $\mathcal{B} \in \{\Pi^0_1, \Pi^0_2, \Sigma^0_2\}$, see [35], [20], [17]. Moreover, for such a class $\mathcal{B}$ a space X with $\mathcal{B}[n]$-universal power X^{n+1} can be chosen as a suitable subspace of a dendrite with dense set of end-points.

A related question concerns the universality in classes of compact spaces. It is well-known that the n-dimensional cube $[0,1]^n$ is not $\Pi^0_1[n]$-universal. On the other hand, for any dendrite D with dense set of end-points the product the power D^{n+1} is $\Pi^0_1[n]$-universal [35],and the product $D^{n+1} \times [0,1]^{2n}$ is $\Pi^0_1[2n]$-universal, see [27].

Question. *Is $X \times [0,1]^{2n}$ $\Pi^0_1[2n]$-universal for any $\Pi^0_1[n]$-universal space X? Equivalently, is $\mu^n \times [0,1]^{2n}$ $\Pi^0_1[2n]$-universal (where μ^n denotes the n-dimensional Menger cube)?* **1004 ?**

For $n \le 1$ the answer to this problem is negative. We expect that this is so for all n.

Z_n-sets and related questions

In this section we consider some problems related to Z_n-sets, where $n \in \omega \cup \{\infty\}$. By definition, a subset A of a space X is a *Z_n-set* in X if A is closed and the complement $X \setminus A$ is *n-dense* in X in the sense that each map $f : [0,1]^n \to X$ can be uniformly approximated by maps into $X \setminus A$. In particular, the 0-density is equivalent to the usual density and a subset $A \subset X$ is a Z_0-set in X if and only if it is closed and nowhere dense in X.

A set $A \subset X$ is a *σZ_n-set* if A is the countable union of Z_n-sets in X. A subset $A \subset X$ is called *n-meager* if $A \subset B$ for some σZ_n-set B in X. A space X is a *σZ_n-space* (or else *n-meager*) if X is a σZ_n-set (equivalently n-meager) in itself. In particular, a space is 0-meager if and only if it is of the first Baire category.

According to a classical result of S.Banach [5], an analytic topological group either is complete or else is 0-meager. It is natural to ask about the infinite version of this result. Namely, Question 4.4 in [55] asks if any incomplete Borel pre-Hilbert space is ∞-meager. The answer to this question turned out to be negative: the linear span(E) of the Erdös set $E \subset \{(x_i) \in l_2 : \forall i\ x_i \in \mathbb{Q}\}$ is meager but not ∞-meager, see [24, 5.5.19]. Moreover, span(E) cannot be written as the countable union $\bigcup_{n\in\omega} Z_n$ where each set Z_n is a Z_n-set in span(E). On the other hand, for every $n \in \omega$, span(E) can be written as the countable union of Z_n-sets, see [9].

? 1005 **Question.** *Is an (analytic) linear metric space X a σZ_∞-space if X can be written as the countable union $X = \bigcup_{n\in\omega} X_n$ where each set X_n is a Z_n-set in X.*

By its spirit this problem is related to Selection Principles, a branch of Combinatorial Set Theory considered in the papers [71], [68].

Another feature of span(E) leads to the following problem, first posed in [6].

? 1006 **Question.** *Is every analytic non-complete linear metric space X a σZ_n-space for every $n \in \omega$? Is this true if X is a linear subspace of l^2 or $\mathbb{R}^\omega$?*

With help of the Multiplication Formula for σZ_n-spaces [26] or [19], the (second part of the) above problem can be reduced to the following one.

? 1007 **Question.** *Let X be a non-closed analytic linear subspace of the space $L = l^2$ or $L = \mathbb{R}^\omega$. Can L be written as the direct sum $L = L_1 \oplus L_2$ of two closed subspaces $L_1, L_2 \subset L$ so that for every $i \in \{1,2\}$ the projection X_i of X onto L_i is a proper subspace of L_i?*

Let us note that the zero-dimensional counterpart of this question has an affirmative solution: for each meager subset $H \subset \{0,1\}^\omega$ there is a partition $\omega = A \cup B$ of ω into two disjoint sets A, B such that the projections of H onto $\{0,1\}^A$ and $\{0,1\}^B$ are not surjective. This partition can be easily constructed by induction.

The following weaker problem related to Question 1007 also is open.

? 1008 **Question.** *Let X be a non-closed analytic linear subspace in l^2. Is there a closed infinite-dimensional linear subspace $L \subset l^2$ such that $X + L \neq l^2$?*

We recall that a space X is (*strongly*) *countable-dimensional* if X can be written as the countable union $X = \bigcup_{n=1}^{\infty} X_n$ of (closed) finite-dimensional subspaces of X. The space $\mathrm{span}(E)$ is countable-dimensional but not strongly countable-dimensional, see [9].

Question. *Is each (analytic) strongly countable-dimensional linear subspace of l^2 ∞-meager? equivalently, 2-meager?* 1009 ?

In light of this question it should be mentioned that each closed finite-dimensional subspace of the Hilbert space l^2 is a Z_1-set. On the other hand, l^2 contains a closed zero-dimensional subsets failing to be a Z_2-set in l^2. Yet, each finite-dimensional Z_2-set in l^2 is a Z_∞-set in l^2, see [63].

Let $\mathcal{M}_n$ be the σ-ideal consisting of n-meager subsets of the Hilbert cube Q. In particular, $\mathcal{M}_0$ coincides with the ideal $\mathcal{M}$ of meager subsets of Q well studied in Set Theory. For each non-trivial ideal $\mathcal{I}$ of subsets of a set X we can study four cardinal characteristics:

- $\mathrm{add}(\mathcal{I}) = \min\{|\mathcal{J}| : J \subset \mathcal{I} \ \ \bigcup \mathcal{J} \notin \mathcal{I}\}$;
- $\mathrm{cov}(\mathcal{I}) = \min\{|\mathcal{J}| : J \subset \mathcal{I} \ \ \bigcup \mathcal{J} = X\}$;
- $\mathrm{non}(\mathcal{I}) = \min\{|A| : A \subset X \ A \notin \mathcal{I}\}$;
- $\mathrm{cof}(\mathcal{I}) = \min\{|\mathcal{C}| : \mathcal{C} \subset \mathcal{I} \ \forall A \in \mathcal{I} \ \exists C \in \mathcal{C} \text{ with } A \subset C\}$.

The cardinal characteristics of the ideal $\mathcal{M}_0 = \mathcal{M}$ are calculated in various models of ZFC and can vary between $\aleph_1$ and the continuum $\mathfrak{c}$, see [72]. In [29] it is shown that $\mathrm{cov}(\mathcal{M}_n) = \mathrm{cov}(\mathcal{M})$ and $\mathrm{non}(\mathcal{M}_n) = \mathrm{non}(\mathcal{M})$ for every $n \in \omega \cup \{\infty\}$.

Question. *Is $\mathrm{add}(\mathcal{M}_n) = \mathrm{add}(\mathcal{M})$ and $\mathrm{cof}(\mathcal{M}_n) = \mathrm{cof}(\mathcal{M})$ for every $n \in \omega \cup \{\infty\}$?* 1010 ?

It is well-known that Z_∞-sets in ANR-spaces can be characterized as closed sets with homotopy dense complement. A subset D of a space X is called *homotopy dense* if there is a homotopy $h : X \times [0,1] \to X$ such that $h(x,0) = x$ and $h(x,t) \in D$ for all $(x,t) \in X \times (0,1]$.

One of the problems from [73] and [55] asked about finding an inner characterization of homotopy dense subspaces of s-manifold. In [24, 1.3.2] (see also [7] and [52]) it was shown that such subspaces can be characterized with help of SDAP, the Toruńczyk's Strong Discrete Approximation Property. This characterization allowed to apply powerful tools of the theory of Hilbert manifolds to studying spaces with SDAP.

Question. *Find an inner characterization of homotopy dense subspaces of Q-manifolds.* 1011 ?

The problem of characterization of locally compact spaces homeomorphic to homotopy dense subsets of compact ANR's (or compact Q-manifolds) was considered in [45] and [50].

It is known that each homotopy dense subspace X of a locally compact ANR-space has LCAP, Locally Compact Approximation Property. The latter means that for every open cover $\mathcal{U}$ of X the identity map of X can be uniformly approximated by maps $f : X \to X$ whose range $f(X)$ has locally compact closure in X.

? 1012 **Question.** *Is each space X with LCAP homeomorphic to a homotopy dense subspace of a locally compact ANR.*

Let us note that LCAP appears as an important ingredient in many results of infinite-dimensional topology, see [24], [8].

? 1013 **Question.** *Let X be a convex set in a linear metric space. Has X LCAP? Has X LCAP if X is an absolute retract?*

The answer to the latter question is affirmative if the completion of X is an absolute retract, see [24, 5.2.5].

The topological structure of convex sets and topological groups

One of classical applications of infinite-dimensional topology is detecting the topological structure of convex sets in linear metric spaces. As a rule, convex sets are absolute retracts and have many other nice features facilitating applications of powerful methods of infinite-dimensional topology. Among such methods let us recall the theory of Q- and l^2-manifolds and the theory of absorbing and coabsorbing spaces. The principal notion unifying these theories is the notion of a strongly universal space.

A topological space X is defined to be *strongly $\mathcal{C}$-universal* for a class $\mathcal{C}$ of spaces if for every cover $\mathcal{U}$, every space $C \in C$ and a map $f : C \to X$ whose restriction $f|B : B \to X$ onto a closed subset $B \subset C$ is a Z-embedding there is a Z-embedding $\tilde{f} : C \to X$ which is $\mathcal{U}$-near to f and coincides with f on B. A map $f : C \to X$ is called a *Z-embedding* if it is a topological embedding and $f(C)$ is a Z_∞-set in X. A topological space X is *strongly universal* if it is strongly $\mathcal{Z}(X)$-universal for the class $\mathcal{Z}(X)$ of spaces homeomorphic to Z_∞-sets of X. Many natural spaces are strongly universal.

In [24], [7], [12], [14], [36], [37], [41], [44], [47], [54] many results on the strong universality of convex sets in linear metric space was obtained. Nonetheless the following problem still is open.

? 1014 **Question.** *Let X be an infinite-dimensional closed convex set in a locally convex linear metric space L. Is X strongly universal?*

The answer is not known even for the case when X is a pre-Hilbert space. A bit weaker question also is open.

? 1015 **Question.** *Let X be an infinite-dimensional closed convex set in a locally convex linear metric space L. Is X strongly $\mathcal{Z}_{tb}(X)$-universal for the class of spaces homeomorphic to totally bounded Z_∞-subsets of X?*

The answer to this problem is affirmative if X has an almost internal point $x_0 \in X$ (the latter means that the set $\{x \in X : \exists z \in X\ \exists t \in (0,1) \text{ with } x_0 = tx + (1-t)z\}$ is dense in X), see [12].

The strong universality enters as one of important ingredients into the definition of a (co)absorbing space. A topological space X is called *absorbing* (resp. *coabsorbing*) if X is an ∞-meager (resp. ∞-comeager) strongly universal ANR with SDAP. We recall that a space is n-meager where $n \in \omega \cup \{\infty\}$ if it is a σZ_n-set in itself. A space X is defined to be *n-comeager* if X contains an absolute G_δ-subset G with n-meager complement $X \setminus G$ in X. In particular an analytic space is 0-comeager if and only if it is Baire.

Question. *Let X be a closed convex subset of a locally convex linear metric space. Assume that X is 0-comeager. Is it ∞-comeager? Is X n-comeager for all $n \in \omega$?* **1016 ?**

Question. *Assume that $X \in AR$ is an ∞-(co)meager closed convex set in a linear metric space. Is X a (co)absorbing space?* **1017 ?**

The principal result of the theory of (co)absorbing spaces is the Uniqueness Theorem [24, §1.6] asserting that two (co)absorbing spaces X, Y are homeomorphic if and only if X, Y are homotopically equivalent and $\mathcal{Z}(X) = \mathcal{Z}(Y)$. This fact helps to establish the topological structure of many infinite-dimensional (co)absorbing spaces appearing in “nature”, see [24], [41], [66].

In particular, in [24] it was shown that a closed convex subset X of a locally convex linear metric space is Π^0_ξ-(co)absorbing for $\xi \geq 2$ if and only if X is a Π^0_ξ-universal ∞-(co)meager space and $X \in \Pi^0_\xi$. The same result is true for additive Borel classes Σ^0_ξ with $\xi \geq 3$. Surprisingly, but for the class Π^0_1 of compacta we still have an open question.

Question. *Let X be a Π^0_1-universal convex (closed σ-compact) subset of l^2. Is X strongly Π^0_1-universal?* **1018 ?**

This question has an affirmative answer if X contains an almost internal point.

A similar situation holds for metrizable topological groups. It is shown in [24, 4.2.3] that a topological group G is a Π^0_ξ-absorbing space for $\xi \geq 2$ if and only if $G \in \Pi^0_\xi$ is a Π^0_ξ-universal ANR.

Question. *Let $G \in \mathsf{ANR}$ be a Π^0_1-universal σ-compact metrizable group. Is G an Π^0_1-absorbing space?*

A similar question for the class $\Pi^0_1[\omega]$ of finite-dimensional compacta is also open, see [55, 5.7].

Question. *Let G be an infinite-dimensional σ-compact strongly countable-dimensional locally contractible group (containing a topological copy of each finite-dimensional compactum). Is G a $\Pi^0_1[\omega]$-absorbing space? Equivalently, is G an l^2_f-manifold?*

In fact, the method of absorbing sets works not only for spaces and pairs but also for order-preserving systems $(X_\gamma)_{\gamma \in \Gamma}$ of topological spaces, indexed by a partially ordered set Γ with largest element $\max \Gamma$. The order-preserving property of (X_γ)

means that $X_\gamma \subset X_{\gamma'}$ for any elements $\gamma \le \gamma'$ in Γ. So each X_γ is a subspace of $X_{\max\Gamma}$.

Such systems (X_γ) are called Γ-systems. For a Γ-system $\mathcal{X} = (X_\gamma)_{\gamma\in\Gamma}$, a subset $F \subset X_{\max\Gamma}$, and a map $f : Y \to X$ we let $F \cap \mathcal{X} = (F \cap X_\gamma)_{\gamma\in\Gamma}$ and $f^{-1}(\mathcal{X}) = (f^{-1}(X_\gamma))_{\gamma\in\Gamma}$.

The notion of the strong universality extends to Γ-systems as follows: A Γ-system $\mathcal{X} = (X_\gamma)_{\gamma\in\Gamma}$ is called *strongly $\vec{\mathcal{C}}$-universal* for a class $\vec{\mathcal{C}}$ of Γ-systems if given: an open cover $\mathcal{U}$ of $X_{\max\Gamma}$, a Γ-system $\mathcal{A} = (A_\gamma)_{\gamma\in\Gamma} \in \vec{\mathcal{C}}$, a closed subset $F \subset A_{\max\Gamma}$ and a map $f : A_{\max\Gamma} \to X_{\max\Gamma}$ such that $f|F$ is a Z-embedding with $F \cap f^{-1}(\mathcal{X}) = F \cap \mathcal{A}$, there is a Z-embedding $\tilde{f} : A_{\max\Gamma} \to X_{\max\Gamma}$ such that $\tilde{f}$ is $\mathcal{U}$-near to f, $\tilde{f}|F = f$ and $\tilde{f}^{-1}(\mathcal{X}) = \mathcal{A}$.

A system $\mathcal{X}$ is called *$\vec{\mathcal{C}}$-absorbing* in E if $\mathcal{X}$ is strongly $\vec{\mathcal{C}}$-universal and $X_{\max\Gamma} = \bigcup_{n\in\omega} Z_n$ where each E_i is a Z_∞-set in $X_{\max\Gamma}$ and $Z_n \cap \mathcal{X} \in \vec{\mathcal{C}}$. For more information on absorbing systems, see [4].

Given a class $\vec{\mathcal{C}}$ of Γ-systems and a non-negative integer number n consider the subclass

$$\vec{\mathcal{C}}[n] = \{\mathcal{X} \in \vec{\mathcal{C}} : \dim(X_{\max\Gamma}) \le n\}.$$

The following question is related to the results on existence of absorbing sets for n-dimensional Borel classes [76].

? 1019 **Question.** *For which classes $\vec{\mathcal{C}}$ of Γ-systems the existence of a $\vec{\mathcal{C}}$-absorbing Γ-system implies the existence of a $\vec{\mathcal{C}}[n]$-absorbing Γ-system for every $n \in \omega$?*

One can formulate this question also for another types of dimensions, in particular, for extension dimension introduced by Dranishnikov [56].

Topological characterization of particular infinite-dimensional spaces

The theory of (co)absorbing spaces is applicable for spaces which are either ∞-meager or ∞-comeager. However some natural strongly universal spaces do not fall into either of these two categories. One of such spaces is $\mathrm{span}(E)$, the linear hull of the Erdös set in l^2 which is a meager strongly universal AR with SDAP that fails to be ∞-meager, see [9], [53].

? 1020 **Question.** *Give a topological characterization of* $\mathrm{span}(E)$. *Is* $\mathrm{span}(E)$ *homeomorphic to the linear hull* $\mathrm{span}(\mathbb{Q}^\omega)$ *of* $\mathbb{Q}^\omega$ *in* $\mathbb{R}^\omega$*? to the linear hull* $\mathrm{span}(E_p)$ *of the Erdös set* $E_p = \{(x_i) \in l^p : (x_i) \in \mathbb{Q}^\omega$ *and* $\lim_{i\to\infty} x_i = 0\}$ *in the Banach space* l^p, $1 \le p \le \infty$*?*

Another problem of this sort concerns the countable products X^ω of finite-dimensional meager absolute retracts X. Using [24, 4.1.2] one can show that for the countable product of such a space X is a strongly universal AR with SDAP. The space X^ω is a n-meager for all $n \in \omega$ but unfortunately is not ∞-meager.

? 1021 **Question.** *Let X, Y be finite-dimensional σ-compact absolute retracts of the first*

Baire category. Are X^ω and Y^ω homeomorphic? (Applying [17] one can show that each of the spaces X^ω, Y^ω admits a closed embedding into the other space).

Our third pathologic (though natural) example is the hyperspace $\exp_H(\mathbb{Q}_I)$ of closed subsets of the space of rationals $\mathbb{Q}_I = [0,1] \cap \mathbb{Q}$ on the interval, endowed with the Hausdorff metric. This space has many interesting features similar to those of span$E$: $\exp_H(\mathbb{Q}_I)$ is n-meager for all $n \in \omega$ but fails to be ∞-meager; $\exp_H(\mathbb{Q}_I)$ is homeomorphic to its square and belongs to the Borel class Π^0_3 of absolute $F_{\sigma\delta}$-subsets; $\exp_H(\mathbb{Q}_I)$ is $\Pi^0_3[\omega]$-universal but fails to be Π^0_3-universal, see [22].

Question. *Give a topological characterization of the space* $\exp_H(\mathbb{Q}_I)$*. In particular, are the spaces* $\exp_H(\mathbb{Q}_I)$ *and* $\exp_H(\mathbb{Q}_I \times \mathbb{Q}_I)$ *homeomorphic?* **1022 ?**

The three preceding examples were meager but not ∞-meager. Because of that they cannot be treated by the theory of absorbing spaces. The other two our problems concern spaces that are 0-comeager but not ∞-comeager and hence cannot be treated by the theory of coabsorbing spaces. These spaces are defined with help of the operation of weak product

$$W(X,Y) = \{(x_i) \in X^\omega : \exists n \in \omega\ \forall i \geq n\ x_i \in Y\}$$

where Y is a subspace of X. The classical space of the form $W(X,Y)$ is the Nagata space $N = W(\mathbb{R},\mathbb{P})$ well-known in Dimension Theory as a universal space in the class of countable-dimensional (absolute $G_{\delta\sigma}$-)spaces. Here $\mathbb{P} = \mathbb{R} \setminus \mathbb{Q}$ stands for the space of irrationals. The countable product $\mathbb{P}^\omega$ is a dense absolute G_δ-set in $W(\mathbb{R},\mathbb{P})$. Nonetheless, $W(\mathbb{R},\mathbb{P})$ contains no ∞-dense absolute G_δ-set (because $W(\mathbb{R},\mathbb{P})$ is countable-dimensional) and thus $W(\mathbb{R},\mathbb{P})$ is not a coabsorbing space (but is strongly universal and has SDAP).

Question. *Give a topological characterization of the Nagata space* $N = W(\mathbb{R},\mathbb{P})$. **1023 ?**

To pose a (possibly) more tractable question, let us note that N is homeomorphic to $W(N,\mathbb{P}^\omega)$ (by a coordinate-permutating homeomorphism).

Question. *Is* $N = W(\mathbb{R},\mathbb{P})$ *homeomorphic to* $W(N,(\mathbb{P} \setminus \{\sqrt{2})^\omega)$*?* **1024 ?**

Next, we shall ask about the characterization of the pair $(I^\omega, \mathbb{P}_I^\omega)$ where $\mathbb{P}_I = I \cap \mathbb{P}$ is the set of irrational numbers on the interval $I = [0,1]$. Topological characterizations of the Hilbert cube I^ω and irrational numbers $\mathbb{P}_I^\omega$ are well-known.

Question. *Give a topological characterization of the pair* $(I^\omega, \mathbb{P}_I^\omega)$*. In particular, is* $(I^\omega, \mathbb{P}_I^\omega)$ *homeomorphic to* (I^ω, G) *for every dense zero-dimensional* G_δ*-subset* $G \subset I^\omega$ *with homotopy dense complement in* I^ω*?* **1025 ?**

A similar question concerns also the pair $(I^\omega, \mathbb{Q}_I^\omega)$ where $\mathbb{Q}_I = I \cap \mathbb{Q}$. Since $\mathbb{Q}_I^\omega$ is not ∞-meager in I^ω, this pair can not be treated by the theory of absorbing pairs, see [24].

Question. *Give a topological characterization of the pair* $(I^\omega, \mathbb{Q}_I^\omega)$. **1026 ?**

Problems on ANR's

On of the principal problems on ANR's from the preceding two lists [57], [73], the classical Borsuk's Problem on the AR-property of linear metric spaces, has been resolved in negative by R.Cauty in [38] who constructed a σ-compact linear metric space that fails to be an absolute retract. However, the "compact' version of Borsuk's problem still is open.

? 1027 **Question.** *Let C be a compact convex set in a linear metric space. Is C an absolute retract?*

There are also many other natural spaces whose ANR-property is not established. Some of them are known to be divisible by the Hilbert space l^2 in the sense that they are homeomorphic to the product with l^2 (and hence are l^2-manifolds if and only if they are ANR's). A classical example of this sort is the homeomorphism group of an n-manifolds for $n \geq 2$, see [73, HS4].

Another example is the space H_B of Brouwer homeomorphisms of the plane, endowed with the compact-open topology. A homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ is a *Brouwer homeomorphism* if h preserves the orientation and has no fixed point.

? 1028 **Question.** *Is the space H_B an ANR?*

It is known that H_B is locally contractible [33], is homotopically equivalent to the circle S^1 and is divisible by l^2 [67]. So, H_B is homeomorphic to $S^1 \times l^2$ if and only if H_B is an ANR.

In spite of the existence of a linear metric space failing to be an AR, R. Cauty proved that each convex subset of a linear metric space is an algebraic ANR (algebraic ANR's are defined with help of a homological counterpart of the Lefschetz condition, see [43]). This follows from even more general fact asserting that each metrizable locally equiconnected space is an algebraic ANR, see [43].

We recall that a topological space X is *locally equiconnected* if there are an open neighborhood $U \subset X \times X$ of the diagonal and a continuous function $\lambda : U \times [0,1] \to X$ such that $\lambda(x,y,0) = x$, $\lambda(x,y,1) = y$ and $\lambda(x,x,t) = x$ for every $(x,y,t) \in U \times [0,1]$. If $U = X \times X$, then X is called *equiconnected*. It is easy to see that each (locally) contractible topological group G is (locally) equiconnected and so is any retract of G. We do not know if the converse is true.

? 1029 **Question.** *Let X be a (locally) equiconnected metrizable space. Is X a (neighborhood) retract of a contractible metrizable topological group?*

It should be mentioned that this question has an affirmative answer for compact X, see [39]. The proof of this particular case exploits the fact that each metrizable equiconnected space X admits a Mal'tsev operation (which is a continuous map $\mu : X^3 \to X$ such that $\mu(x,x,y) = \mu(y,x,x) = y$ for all $x, y \in X$). Due to Sipacheva [70] we know that each compact space X admitting a Mal'tsev operation is a retract of the free topological group $F(X)$ over X. Therefore, each equiconnected compact metrizable space X has a Mal'tsev operation and hence is a retract of the free topological group $F(X)$. Moreover, it can be shown that the connected component of $F(X)$ containing X is contractible. Now it is easy to select a metrizable group topology τ on $F(X)$ inducing the original topology on X and such that X still is a retract of $(F(X),\tau)$ and the component of $(F(X),\tau)$ containing X is contractible. This resolves the "compact" version of Question 1029. The non-compact version of this problem is related to the following question (discussed also in [62]):

Question. *Is a metrizable space admitting a Mal'tsev operation a retract of a metrizable topological group?* **1030 ?**

By definition, an *n-mean* on a topological space X is a continuous map $m : X^n \to X$ such that $m(x,\dots,x) = x$ for all $x \in X$ and $m(x_{\sigma(1)},\dots,x_{\sigma(n)}) = m(x_1,\dots,x_n)$ for any vector $(x_1,\dots,x_n) \in X^n$ and any permutation σ of $\{1,\dots,n\}$. Let us note that each convex subset C of a linear topological space admits an n-means and so does any retract of C.

Question. *Let $n \geq 2$. Is there a metrizable equiconnected compact space X admitting no n-mean?* **1031 ?**

If such a compact space X exists then it is a retract of a contractible group but fails to be a retract of a convex subset of a linear topological space.

For a compact space X let $L(X)$ be the free topological linear space over X and $P(X)$ be the convex hull of X in $L(X)$ (it can be shown that $P(X)$ is a free convex set over X). Let $(U(X),\lambda_X)$ be the free equiconnected space over X (where $\lambda : U(X)\times U(X)\times[0,1] \to U(X)$ is the equiconnected map of $U(X)$), see [42]. Let $\mathcal{T}_v(X)$ be the family of metrizable linear topologies on $L(X)$ inducing the original topology on X. (The family $\mathcal{T}_v(X)$ was essentially used in [38] for constructing the example of a linear metric space failing to be an AR). Let $\mathcal{T}_c(X) = \{\tau|P(X) : \tau \in \mathcal{T}_v(X)\}$ be the family consisting of the restrictions of the topologies $\tau \in \mathcal{T}_v(X)$ onto $P(X)$, and $\mathcal{T}_u(X)$ be the family of metrizable topologies on $U(X)$ which induce the initial topology on X and preserve the continuity of the equiconnected map $\lambda_X : U(X)\times U(X)\times[0,1] \to U(X)$.

It is interesting to study the classes $\mathcal{A}_v$ ($\mathcal{A}_c$, $\mathcal{A}_u$) of metric compacta X such that the spaces $(L(X),\tau)$ $(P(X),\tau)$, $(U(X),\tau)$) are absolute retracts for all topologies τ in $\mathcal{T}_v(X)$ ($\mathcal{T}_c(X)$, $\mathcal{T}_u(X)$). It is known that $\mathcal{A}_u \subset \mathcal{A}_c \subset \mathcal{A}_v$, see [42].

Question. *Is it true that $\mathcal{A}_u = \mathcal{A}_c = \mathcal{A}_v$?* **1032 ?**

The class $\mathcal{A}_u$ contains all metrizable compact ANR's and all metrizable compact C-spaces, see [42].

Question. *Is it true that each weakly infinite-dimensional compact metrizable space belongs to $\mathcal{A}_v$? to $\mathcal{A}_u$?* **1033 ?**

In light of this question let us mention that there is a strongly infinite-dimensional compact space D of finite cohomological dimension with $D \notin \mathcal{A}_v$, see [38]. In fact, the free linear space $L(D)$ over D, endowed with a suitable metrizable topology, gives the mentioned example of a linear metric space which is not an AR.

According to [42] the class $\mathcal{A}_c$ is closed with respect to countable products. We do not know if the same is true for the class $\mathcal{A}_u$.

1034 ? **Question.** *Is the class $\mathcal{A}_u$ closed with respect to countable products?*

Infinite-Dimensional Problems from Banach Space Theory

In this section we survey some open problems lying in the intersection of infinite-dimensional topology and the theory of Banach spaces. Our principal object is the unit ball $B_X = \{x \in X : \|x\| \le 1\}$ of a Banach space X, endowed with the weak topology. It is well-known that the weak ball B_X is metrizable (and separable) if and only if the Banach space X has separable dual. So, till the end of this section by a "Banach space" we understand an infinite-dimensional Banach space with separable dual. In [10] the following general problem was addressed:

? 1035 **Question.** *Investigate the interplay between geometric properties of a Banach space X and topological properties of its weak unit ball B_X. Find conditions under which two Banach spaces have homeomorphic weak unit balls.*

It turns out that answers to these questions depend on (1) the class $\mathcal{W}(X)$ of topological spaces homeomorphic to closed bounded subsets of a Banach space X endowed with the weak topology, and (2) (in case of complex $\mathcal{W}(X)$) on properties of the norm of X. Let us remark that $\mathcal{W}(X)$ coincides with the class $\mathcal{F}_0(B_X)$ of topological spaces homeomorphic to closed subsets of the weak unit ball B_X of X.

In [10] it was observed that the class $\mathcal{W}(X)$ is not too large: it lies in the class Π^0_3 of absolute $F_{\sigma\delta}$-sets. For reflexive infinite-dimensional Banach spaces X the class $\mathcal{W}(X)$ coincides with the class Π^0_1 of compacta. On the other hand, for the Banach space $X = c_0$ the class $\mathcal{W}(X)$ is the largest possible and coincides with the Borel class Π^0_3. An intermediate case $\mathcal{W}(X) = \Pi^0_2$ happens if and only if X is a non-reflexive Banach space with PCP, the Point Continuity Property (which means that for each bounded weakly closed subset $B \subset X$ the identity map $(B, weak) \to B$ has a continuity point).

For Banach spaces with PCP the weak unit ball B_X is homeomorphic either to the Hilbert cube Q (if X is reflexive) or to the pseudointerior $s = (0,1)^\omega$ of Q (if X is not reflexive). In two latter cases, the topology of B_X does not depend on the particular choice of an equivalent norm on X. In this case we say that the Banach space X has BIP, the Ball Invariance Property. More precisely, X has BIP if the weak unit ball B_X of X is homeomorphic to the weak unit ball B_Y of any Banach space Y, isomorphic to X. It is known [10] that PCP implies BIP and BIP implies CPCP, the Convex Point Continuity Property, which means that each closed convex bounded subset of the Banach space has a point at which the norm topology coincides with the weak topology. It is known that the properties PCP and CPCP are different: the Banach space B_∞ constructed in [59] has CPCP but not PCP.

? 1036 **Question.** *Is BIP equivalent to PCP? To CPCP? Has the Banach space B_∞ BIP?*

In fact, the geometric properties PCP, BIP, and CPCP of a Banach space X can be characterized via topological properties of the weak unit ball B_X: X has PCP (resp. CPCP, BIP) if and only if B_X is Polish (0-comeager, ∞-comeager).

The norm of a Banach space X will be called *n-(co)meager* if the respective weak unit ball B_X is n-(co)meager. Let us remark that each Kadec norm is ∞-comeager (since the unit sphere is an ∞-dense absolute G_δ-subset in the weak unit ball). It is well-known that each separable Banach space admits an equivalent Kadec (and hence ∞-comeager) norm.

Question. *Give a geometric characterization of Banach spaces admitting an equivalent ∞-meager norm.* **1037 ?**

For n-meager norms with $n \in \omega$ the answer is known: a Banach space X admits an equivalent n-meager norm if and only if X fails to have the CPCP. On the other hand, a Banach space X has an equivalent ∞-meager norm if X fails to be strongly regular, see [10]. We recall that a Banach space X is called *strongly regular* if for every $\varepsilon > 0$ and every non-empty convex bounded subset $C \subset X$ there exist non-empty relatively weak-open subsets $U_1, \dots, U_n \subset C$ such that the norm diameter of $\frac{1}{n}\sum_{i=1}^{n} U_i$ is less than ε. An example of a strongly regular Banach space S_*T_∞ failing to have CPCP was constructed in [58]. This space has an equivalent norm which is n-meager for every $n \in \omega$, see [10].

Question. *Is there a strongly regular Banach space admitting an equivalent ∞-meager norm? Has the space S_*T_∞ an equivalent ∞-meager norm?* **1038 ?**

If a Banach space admits a 0-meager norm (equivalently, X fails CPCP), then the class $\mathcal{W}(X)$ contains all finite-dimensional absolute $F_{\sigma\delta}$-spaces. If, moreover, the norm of a Banach space X is ∞-meager, then $\mathcal{W}(X) = \Pi^0_3$ and the weak unit ball B_X is homeomorphic to the weak unit ball of the Banach space c_0 endowed with the standard sup-norm. We do not know if the Banach space S_*T_∞ has an equivalent ∞-meager norm, but we know that $\mathcal{W}(S_*T_\infty) = \Pi^0_3$ and the weak unit ball of S_*T_∞ endowed with a Kadec norm is homeomorphic to the weak unit ball of c_0 endowed with a Kadec norm. The space S_*T_∞ is an example of a strongly regular space with $\mathcal{W}(S_*T_\infty) = \Pi^0_3$. However, S_*T_∞ fails to have CPCP.

Question. *Is there a Banach space X with $\mathcal{W}(X) = \Pi^0_3$ admitting no ∞-meager norm? having CPCP?* **1039 ?**

In light of this question it should be mentioned that each Banach space with PCP has $\mathcal{W}(X) = \Pi^0_i$ for $i \in \{1, 2\}$. Also a Banach space with CPCP admits no 0-meager norm. It is known that the Banach space c_0 contains no conjugate subspaces and has $\mathcal{W}(c_0) = \Pi^0_3$.

Question. *Suppose X is a Banach space with separable dual, containing no subspace isomorphic to a dual space. Is $\mathcal{W}(X) = \Pi^0_3$?* **1040 ?**

For a Banach space X with an ∞-meager norm the weak unit ball is an absorbing space (in fact, a Π^0_3-absorbing space).

Similarly, for a Banach spaces with ∞-comeager norm the weak unit ball B_X is a coabsorbing space and its topology is completely determined by the class $\mathcal{W}(X)$. The same concerns the topology of the pair (B^{**}_X, B_X), where B^{**}_X is the unit ball in the second dual Banach space X^{**}, endowed with the $*$-weak topology. The topology of this pair is completely determined by the class $\mathcal{W}(X^{**},X)$ of pairs (K,C) homeomorphic to pairs of the form $(B,B\cap X)$ where $B\subset X^{**}$ is w^*-closed bounded subset of the second dual space $(X^{**}, weak^*)$.

More precisely, we have the following classification theorem of Cantor-Bernstein type proved in [10].

Theorem (Classification Theorem)**.** *Let X,Y be Banach spaces with separable dual and ∞-comeager norms.*

1. *The weak unit balls B_X and B_Y are homeomorphic if and only if $\mathcal{W}(X)=\mathcal{W}(Y)$.*
2. *The pairs (B^{**}_X,B_X) and (B^{**}_Y,B_Y) are homeomorphic if and only if $\mathcal{W}(X^{**},X)=\mathcal{W}(Y^{**},Y)$.*

It is clear that the topological equivalence of the pairs (B^{**}_X,B_X) and (B^{**}_Y,B_Y) implies the topological equivalence of the weak unit balls B_X and B_Y. We do not know if the converse is also true.

? 1041 **Question.** *Assume that X,Y are Banach spaces with homeomorphic weak unit balls B_X and B_Y. Are the pairs (B^{**}_X,B_X) and (B^{**}_Y,B_Y) homeomorphic?*

The answer to this question is affirmative provided $\mathcal{W}(X)=\mathcal{W}(Y)=\Pi^0_\xi$ for some $\xi\in\{1,2,3\}$.

The classification Theorem suggests introducing the partially ordered set

$$\mathfrak{W}^s_\infty=\{\mathcal{W}(X) : X \text{ is an infinite-dimensional Banach space with separable dual}\}$$

inducing the following preorder of the family of Banach spaces: $X\le_{\mathcal{W}} Y$ if $\mathcal{W}(X)\subset\mathcal{W}(Y)$ (equivalently, if the weak unit ball of X admits a closed embedding into the weak unit ball of Y).

Since each separable Banach space is isomorphic to a subspace of $C[0,1]$, the set $\mathfrak{W}^s_\infty$ contains at most continuum elements. Note that the set $\mathfrak{W}^s_\infty$ is partially ordered by the natural inclusion relation.

It is easy to see that the poset $\mathfrak{W}^s_\infty$ has the smallest and largest elements: $\Pi^0_1=\mathcal{W}(l_2)$ and $\Pi^0_3=\mathcal{W}(c_0)$ corresponding to classes $\mathcal{W}(X)$ of the Hilbert space l^2 and the Banach space c_0. Also it is known that the class $\Pi^0_2=\mathcal{W}(J)$ where J is the James quasireflexive space is a unique immediate successor of Π^0_1. For some time there was a conjecture that $\mathfrak{W}^s_\infty$ consists just of these three elements: Π^0_1, Π^0_2, and Π^0_3. However it was discovered in [10] that for the Banach space B_∞ (distinguishing the properties PCP and CPCP) the class $\mathcal{W}(B_\infty)$ is intermediate between $\mathcal{W}(J)$ and $\mathcal{W}(c_0)$. So the poset $\mathfrak{W}^s_\infty$ appeared to be reacher than expected.

? 1042 **Question.** *Investigate the ordered set $\mathfrak{W}^s_\infty$. In particular, is it infinite? Is it linearly ordered?*

The pathological class $\mathcal{W}(B_\infty)$ contains the class $\Pi^0_3[0]$ of all zero-dimensional absolute $F_{\sigma\delta}$-spaces but not the class $\Pi^0_3[1]$, see [10]. This suggests the following (probably difficult)

? 1043 **Question.** *Let $n \in \omega$. Is there a Banach space X such that $\Pi^0_3[n] \subset \mathcal{W}(X)$ but $\Pi^0_3[n+1] \not\subset \mathcal{W}(X)$?* (Such a space X if exists has CPCP but not PCP).

Question. *Is there a Banach space X such that $\Pi^0_3[\omega] \subset \mathcal{W}(X)$ but $\Pi^0_3 \not\subset \mathcal{W}(X)$?* 1044 ? (Such a space X if exists is strongly regular but fails to have PCP).

In fact, the pathological space B_∞ is one of the spaces $J_*T_{\infty,n}$, $n \geq 0$, constructed in [58].

Question. *Is $\mathcal{W}(J_*T_{\infty,n}) \neq \mathcal{W}(J_*T_{\infty,m})$ for $n \neq m$?* 1045 ?

Another two questions concern the influence of operations over Banach spaces on the classes $\mathcal{W}(X)$.

Question. *Is $\mathcal{W}(X \oplus Y) = \max\{\mathcal{W}(X), \mathcal{W}(Y)\}$ for infinite-dimensional Banach* 1046 ? *spaces X and Y with separable duals?*

Question. *Let X be an infinite-dimensional Banach space. Is $\mathcal{W}(X \oplus X) =$* 1047 ? *$\mathcal{W}(X)$? Is $\mathcal{W}(X \oplus \mathbb{R}) = \mathcal{W}(X)$?*

Note that an infinite-dimensional Banach space X with separable dual need not be isomorphic to $X \oplus X$ or $X \oplus \mathbb{R}$, see [60].

In the following diagram we collect all known information on the relationship between geometric properties of a Banach space X with separable dual, topological properties of the weak unit ball B_X and properties of the class $\mathcal{W}(X)$. In the first line of the diagram FD means "finite-dimensional", R "reflexive", and SR "strongly regular". The second line of the diagram means that *every* equivalent weak unit ball B of X has the corresponding property; the third line means that the class $\mathcal{W}(X)$ *does not* contain the corresponding class of absolute $F_{\sigma\delta}$-spaces. The slashed and curved arrows indicate that the corresponding implication is false (with a counterexample written near the slashed arrow).

$$
\begin{tikzcd}
X \text{ has:} & (\mathrm{FD}) \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & (\mathrm{R}) \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & (\mathrm{PCP}) \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & (\mathrm{BIP}) \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & (\mathrm{CPCP}) \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & (\mathrm{SR}) \arrow[r, Rightarrow] & (c_0 \not\subset X) \arrow[ddl, bend left=60, "S_*T_\infty", "|" marking] \\
B_X \text{ is:} & (\text{ f.d.}) \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & (\text{compact}) \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & (\text{Polish}) \arrow[r, Rightarrow] \arrow[d, Leftrightarrow] & (\infty\text{-comeager}) \arrow[r, Rightarrow] \arrow[d, phantom, "?"] & (\text{Baire}) \arrow[r, Rightarrow] & (\text{not } \infty\text{-meager}) \arrow[u, Rightarrow] & \\
\mathcal{W}(X) \not\supset\text{:} & \Pi^0_1 \arrow[r, Rightarrow] & \Pi^0_2 \arrow[r, Rightarrow] & \Pi^0_3[0] \arrow[r, Rightarrow] & \Pi^0_3[1] \arrow[r, Rightarrow] \arrow[l, bend left, "B_\infty", "|" marking] & \Pi^0_3[\omega] \arrow[r, Rightarrow] \arrow[u, Rightarrow] & \Pi^0_3 \arrow[u, Rightarrow] &
\end{tikzcd}
$$

Finally we ask some questions on the topological structure of operator images. By an *operator image* we understand an infinite-dimensional normed space of the form TX for a suitable linear continuous operator $T : X \to Y$ between separable Banach spaces. In [21] it was shown that each operator image belongs to a Borel class $\Pi^0_{\alpha+2} \setminus \Sigma^0_{\alpha+2}$, $\Sigma^0_{\alpha+1} \setminus \Pi^0_{\alpha+1}$ or $D_2(\Pi^0_{\alpha+1}) \setminus (\Pi^0_\alpha \cup \Sigma^0_\alpha)$ for a suitable ordinal α (here $D_2(\Pi^0_{\alpha+1})$ is the class consisting of differences $X \setminus Y$ with $X, Y \in \Pi^0_{\alpha+1})$

and each such a Borel class contains an operator image. Moreover, up to a homeomorphism each class Π^0_2, Σ^0_2, $D_2(\Pi^0_2)$ contains exactly one operator image. On the other hand, the class $\Pi^0_3 \setminus \Sigma^0_3$ contains at least two topologically distinct operator images, see [21] and [11].

? 1048 **Question.** *Does the class $\Sigma^0_3 \setminus \Pi^0_3$ contain two topologically distinct operator images? The same question for other Borel classes.*

The image $T : X \to Y$ of a Banach space under a compact operator T always is an absorbing space, see [21]. Moreover, for every countable ordinal $\alpha \geq 1$ the multiplicative Borel class $\Pi^0_{\alpha+2}$ contains an operator image which is a $\Pi^0_{\alpha+2}$-absorbing space. We do not know if the same is true for the additive Borel classes.

? 1049 **Question.** *Is there an operator image which is a Σ^0_3-absorbing space? a $\Sigma^0_{\xi+1}$-absorbing space with $\xi \geq 1$?*

For $\xi = 1$ the answer is affirmative: the image $T : X \to Y$ of any reflexive Banach space under a compact bijective operator $T : T \to Y$ is Σ^0_2-absorbing.

Some Problems in Dimension Theory

In this section we address some problems related to distinguishing between certain classes of infinite-dimensional compacta intermediate between the class cd of countable-dimensional compacta and the class wid of weakly-infinite-dimensional compacta:

$$\mathsf{fd} \subset \mathsf{cd} \subset \sigma\mathsf{hd} \subset \mathsf{trt} \subset \mathsf{C} \subset \mathsf{wid}$$

In this diagram we by fd and C we denote the classes of finite-dimensional compacta and compact with the property C. The classes $\sigma\mathsf{hd}$ and trt are less known and consist of σ-hereditarily disconnected and trt-dimensional compacta, respectively. A topological space X is called *σ-hereditarily disconnected* if X can be written as the countable union of hereditarily disconnected subspaces.

The definition of trt-dimensional compacta is a bit longer and relies on the transfinite dimension trt introduced by Arenas, Chatyrko, and Puertas in [3]. For a space X they put

(1) $\mathsf{trt}(X) = -1$ iff $X = \emptyset$;
(2) $\mathsf{trt}(X) \leq \alpha$ for an ordinal α iff each closed subset $A \subset X$ with $|A| \geq 2$ can be separated by a closed subset $B \subset A$ with $\mathsf{trt}(B) < \alpha$.
(3) $\mathsf{trt}(X) = \alpha$ if $\mathsf{trt}(A) \leq \alpha$ and $\mathsf{trt}(A) \not\leq \beta$ for any $\beta < \alpha$.

A space X is called trt-*dimensional* if $\mathsf{trt}(X) = \alpha$ for some ordinal α.

In [3] it was proved that each trt-dimensional compactum is a *C-space*, which gives the inclusion $\mathsf{trt} \subset \mathsf{C}$. The inclusion $\sigma hd \subset \mathsf{trt}$ was proved in [30] with help of a game characterization of trt-dimensional spaces.

The classes cd and $\sigma\mathsf{hd}$ of countable-dimensional and σ-hereditarily disconnected compacta are distinguished by the famous Pol's compactum. We do not know if the other considered classes also are different.

? 1050 **Question.** *Is each trt-dimensional compactum σ-hereditarily disconnected? Is each C-compactum trt-dimensional?*

Recently, P. Borst [34] announced an example of a weakly infinite-dimensional compact metric space which fails to be a C-space, thus distinguishing the classes wid and C.

Some immediate questions still are open for the transfinite dimension trt.

Question. *Is the ordinal* $\mathsf{trt}(X)$ *countable for each* trt*-dimensional metrizable compactum* X*?* **1051 ?**

Homological methods in Dimension Theory

In this section we discuss some problems lying in the intersection of Infinite-Dimensional Topology, Dimension Theory, and Algebraic Topology. With help of (co)homologies we shall define two new dimension classes AZ_∞ and hsp of compacta including all trt-dimensional compacta.

The starting point is the homological characterization of Z_n-sets in ANR's due to Daverman and Walsh [48]: a closed subset A of an ANR-space X is a Z_n-set in X for $n \geq 2$ if and only if A is a Z_2-set in X and $H_k(U, U \setminus A) = 0$ for all $k \leq n$ and all open subsets $U \subset X$.

Having this characterization in mind we define a closed subset $A \subset X$ to be a *G-homological Z_n-set* in X for a coefficient group G if the singular relative homology groups $H_k(U, U \setminus A; G)$ are trivial for all $k \leq n$ and all open subsets $U \subset X$. If $G = \mathbb{Z}$, we shall omit the notation of the coefficient group and will speak about homological Z_n-sets. Thus a subset A of an ANR-space X is a Z_n-set for $n \geq 2$ if and only if it is a Z_2-set and a homological Z_n-set in X. Another characterization of Z_n-sets from [19] asserts that a closed subset A of an ANR-space is a homological Z_n-set in X if and only if $A \times \{0\}$ is a Z_{n+1}-set in $X \times [-1, 1]$.

It is more convenient to work with homological Z_n-sets than with usual Z_n-sets because of the absence of many wild counterexample like wild Cantor sets in Q (these are topological copies of the Cantor set in Q that fail to be Z_2-sets, see [74]). According to an old result of Kroonenberg [63] any finite-dimensional closed subset $A \subset Q$ is a homological Z_∞-set in Q. A more general result was proved in [19]: *each closed* trt*-dimensional subset* $A \subset Q$ *is a homological* Z_∞*-set*. We do not know if the same is true for other classes of infinite-dimensional spaces like C or wid.

Question. *Is a closed subset* $A \subset Q$ *a homological* Z_∞*-set in* Q *if* A *is weakly infinite-dimensional?* A *is a* C*-space?* **1052 ?**

This question is equivalent to the following one.

Question. *Let* $W \subset Q$ *be a closed weakly-infinite dimensional subset (with the property* C*). Is the complement* $Q \setminus W$ *homologically trivial?* **1053 ?**

The preceding discussion suggests introducing new dimension classes AZ_n consisting of so-called absolute Z_n-compacta. Namely, we define a compact space K to be *an absolute Z_n-compactum* if for every embedding $e : K \to Q$ of K into the Hilbert cube Q the image $e(K)$ is a homological Z_n-set in Q. Among the classes

AZ_n the most interesting are the extremal classes AZ_0 and AZ_∞. Both of them are hereditary with respect to taking closed subspaces.

In fact, the class AZ_0 coincides with the class of all compact spaces containing no copy of the Hilbert cube and thus AZ_0 is the largest possible non-trivial hereditary class of compact spaces. The class AZ_0 is strictly larger than the class AZ_1: the difference $\mathsf{AZ}_0 \setminus \mathsf{AZ}_1$ contains all hederitarily indecomposable continua $K \subset Q$ separating the Hilbert cube Q (such continua exist according to [32]). Observe also that $\mathsf{AZ}_\infty = \bigcap_{n\in\omega} \mathsf{AZ}_n$.

? 1054 **Question.** *What can be said about the classes AZ_n for $n \in \mathbb{N}$. Are they hereditary with respect to taking closed subspaces? Are they pairwise distinct?*

The class AZ_∞ is quite rich and contains all trt-dimensional compacta. Besides being absolute Z_∞-compacta, trt-dimensional compacta have another interesting property: they contain many (co)homologically stable points. A point x of a space X will be called *homologically* (resp. *cohomologically*) *stable* if for some $k \ge 0$ the singular homology group $H_k(X, X\setminus\{x\})$ (resp. Čech cohomology group $\check{H}^k(X, X \setminus \{x\})$) is not trivial. For locally contractible spaces both notions are equivalent due to the duality between singular homologies and Čech cohomologies in such spaces). But it seems that Čech cohomologies work better beyond the class of locally contractible spaces.

According to [30], each trt-dimensional space contains a (co)homologically stable point and by [18] or [19] the same is true for every locally contractible C-compactum. The local contractibility is essential for the proof of the latter result and we do not known if it can be removed.

? 1055 **Question.** *Has each weakly infinite-dimensional (C-)compactum a cohomologically stable point?*

According to a classical result of Aleksandrov, each compact space X of finite cohomological dimension $\dim_{\mathbb{Z}}(X)$ contains a cohomologically stable point. This implies that the class $\mathsf{fd}_{\mathbb{Z}}$ of compacta with finite cohomological dimension lies in the class hsp of compacta whose any closed subspace has a cohomologically stable point. The class $\mathsf{fd}_{\mathbb{Z}}$ is also contained in the class afd of all almost finite-dimensional compacta, where a space X is called *almost finite-dimensional* if there is $n \in \omega$ such that each closed finite-dimensional subspace $F \subset X$ has dimension $\dim(F) \le n$. By [13], each almost finite-dimensional compactum is an absolute Z_∞-space. Therefore we obtain the following diagram describing the (inclusion) relations between the considered classes of compacta (the arrow $\mathsf{x} \to \mathsf{y}$ means that $\mathsf{x} \subset \mathsf{y}$:

$$\begin{tikzcd}
 & \mathsf{fd}_{\mathbb{Z}} \arrow[r] \arrow[dr] & \mathsf{afd} \arrow[r] & \mathsf{AZ}_\infty \arrow[r] & \mathsf{AZ}_n \arrow[r] & \mathsf{AZ}_1 \arrow[r] & \mathsf{AZ}_0 \\
 & & \mathsf{hsp} & & & & \\
\mathsf{fd} \arrow[r] \arrow[uur] & \mathsf{cd} \arrow[r] & \sigma\mathsf{hd} \arrow[r] & \mathsf{trt} \arrow[r] \arrow[ul] \arrow[uu] & \mathsf{C} \arrow[r] & \mathsf{wid} \arrow[uu] &
\end{tikzcd}$$

Question. *What is the relation between the class* hsp *and other dimension classes from the diagram? In particular, has a (locally contractible) compact space X a cohomologically stable point if X is almost finite-dimensional? weakly infinite-dimensional? an absolute Z_∞-space?* **1056 ?**

Question. *Is a compact space X an absolute Z_∞-space if* **1057 ?**

- *all closed subspaces of X have a cohomologically stable point?*
- *all almost finite-dimensional closed subspaces of X are finite-dimensional?*

We have defined absolute Z_∞-compacta with help of their embedding into the Hilbert cube. What about embeddings into other spaces resembling the Hilbert cube?

Question. *Let A be a compact subset of an absolute retract X whose all points are homological Z_∞-points. Is A a homological Z_∞-set in X if A is an absolute Z_∞-space?* **1058 ?**

Compact absolute retracts whose all points are homological Z_∞-points seem to be very close to being Hilbert cubes. By [19] all such spaces fail to be C-spaces and have infinite cohomological dimension with respect to any coefficient group.

Question. *Let X be a compact absolute retract whose all points are homological Z_∞-points. Is X strongly infinite-dimensional? Is $X \times [0,1]^2$ homeomorphic to the Hilbert cube? Is X homeomorphic to Q if X has DDP, the Disjoint Disks Property?* **1059 ?**

In light of this question we should mention an example of a fake Hilbert cube constructed by Singh [69]. He constructed a compact absolute retract X such that (i) all points of X are homological Z_∞-points, (ii) $X \times [0,1]^2$ and $X \times X$ are homeomorphic to Q but (iii) X contains no proper closed ANR-subspace of dimension greater than one.

A bit weaker question of the same spirit asks if the Square Root Theorem holds for the Hilbert cube.

Question. *Is a space X homeomorphic to the Hilbert cube if X has DDP and X^2 is homeomorphic to Q.* **1060 ?**

Let us note that for the Cantor and Tychonov cubes the square Root Theorem is true, see [25].

The Singh's example shows that the class AZ_0 of absolute Z_0-compacta is not multiplicative. An analogous question for the class AZ_∞ is open.

1061 ? **Question.** *Is the class AZ_∞ closed with respect to taking finite products?*

It should be noted that the product $X \times Y$ of a compact absolute Z_∞-space X and a trt-dimensional compact space Y is an absolute Z_∞-space, see [13].

Infinite-Dimensional Spaces in Nature

The *Gromov-Hausdorff distance* between compact metric spaces (X_1, d_1) and (X_2, d_2) is the infimum of the Hausdorff distance between the images of isometric embeddings of these spaces into a metric space. Let C denote the set of all compact metric spaces (up to isometry) endowed with the Gromov-Hausdorff metric. We call C the Gromov-Hausdorff hyperspace. It is well-known that C is a complete separable space.

? 1062 **Question.** *Is the Gromov-Hausdorff hyperspace homeomorphic to ℓ^2?*

? 1063 **Question.** *Is the subspace of the Gromov-Hausdorff hyperspace consisting of all finite metric spaces homeomorphic to σ?*

? 1064 **Question.** *What is the Borel type of the subspace of the Gromov-Hausdorff hyperspace consisting of all compact metric spaces of dimension $\leq n$?*

A *convex metric compactum* is a convex compact subspace of a normed space.

? 1065 **Question.** *Is the subspace of the Gromov-Hausdorff hyperspace consisting of all convex metric compacta homeomorphic to ℓ^2?*

? 1066 **Question.** *Is the subspace of the Gromov-Hausdorff hyperspace consisting of all convex finite polyhedra homeomorphic to σ?*

A *tree* is a connected acyclic graph endowed with the path metric.

? 1067 **Question.** *Is the subspace of the Gromov-Hausdorff hyperspace consisting of all finite trees homeomorphic to σ?*

For any metric space X, one can consider the Gromov-Hausdorff space $GH(X)$ — the subspace of C consisting of the (isometric copies of the) nonempty compact subsets of X. Note that the properties of $GH(X)$ can considerably differ from those of the Hausdorff hyperspace $\exp X$: as L. Bazylevych remarked, the space $GH(X)$ need not be zero-dimensional for zero-dimensional X.

? 1068 **Question.** *Is the Gromov-Hausdorff hyperspace $GH([0,1]$ homeomorphic to the Hilbert cube?*

Recall that the Banach-Mazur compactum $Q(n)$ is the space of isometry classes of n-dimensional Banach-spaces. The space $Q(n)$ is endowed with the distance $d(E,F) = \mathrm{Log}\inf\{\|T\| \cdot \|T^{-1}\| : T : E \to F \text{ is an isomorphism}\}$. Let $\{\mathrm{Eucl}\} \in Q(n)$ denote the Euclidean point to which corresponds the isometry class of standard n-dimensional Euclidean space. It is proved in [1] (see also [2]) that the space $Q_E(n) = Q(n) \setminus \{\mathrm{Eucl}\}$ is a Q-manifold.

? 1069 **Question.** *Are the Q-manifolds $Q_E(n)$ and $Q_E(m)$ homeomorphic for $n \neq m$?*

Question. *Is the subspace of the Banach-Mazur compactum $Q_{\mathrm{pol}}(n)$ consisting of classes of equivalence of polyhedral norms a σ-manifold? If so, is the pair $(Q_E(n), Q_{\mathrm{pol}}(n))$ a (Q, σ)-manifold?* ? 1070

Question. *Is the subspace of $Q_E(n) = Q(n) \setminus \{Eucl\}$ consisting of classes of equivalence of (smooth) strictly convex norms an ℓ^2-manifold?* 1071 ?

Question. *Is there a topological field homeomorphic to the Hilbert space l^2?* 1072 ?

Let (X, d) be a complete metric space. By $\mathrm{CL}_W(X)$ we denote the set of all nonempty closed subsets in X endowed with the Wijsman topology τ_W generated by the weak topology $\{d(x, \cdot) \mid x \in X\}$.

Question. *Let X be a Polish space. What is the Borel type of the subspace $\{A \in \mathrm{CL}_W(X) \mid \dim A \geq n\}$?* 1073 ?

Question. *Characterize metric spaces X whose hyperspace $\mathrm{CL}_W(X)$ is an ANR.* 1074 ?

Some partial results concerning the latter question can be found in [64].

For a metric space X by $\mathrm{Bdd}_H(X)$ we denote the hyperspace of closed bounded subsets of X endowed with the Hausdorff distance.

A metric space (X, d) is called *almost convex* if for any points $x, y \in X$ with $d(x, y) < s + t$ for some positive reals s, t there is a point $z \in X$ with $d(x, z) < s$ and $d(z, y) < t$. In particular, each subspace of the real line is almost convex.

By [46] or [65] for each almost convex metric space X the hyperspace $\mathrm{Bdd}_H(X)$ is an ANR. We do not know if the converse is true.

Question. *Let X be a metric space whose hyperspace $\mathrm{Bdd}_H(X)$ is an ANR. Is the topology (the uniformity) of X generated by an almost convex metric?* 1075 ?

Metric spaces X whose hyperspaces $\mathrm{Bdd}_H(X)$ are ANR's were characterized in [28]. This characterization implies that the hyperspace $\mathrm{Bdd}_H(X_\#)$ of the 1-dimensional subspace

$$X_\# = \{(x_n) \in c_0 : \exists n \in \omega \text{ such that } x_i \in \tfrac{1}{i}\mathbb{Z} \text{ for all } i \neq n\}$$

of the Banach space c_0 is an ANR.

Question. *Is the topology (the uniformity) of the space $X_\#$ generated by an almost convex metric?* 1076 ?

A metric space X is defined to be an *absolute neighborhood uniform retract* (briefly ANUR) if for any metric space $Y \supset X$ there is a uniformly continuous retraction $r : O_\varepsilon(X) \to X$ defined on an ε-neighborhood of X in Y. It is known that each uniformly convex Banach space X is ANUR. In particular, the Hilbert space l^2 is ANUR.

Question. *Is $\mathrm{Bdd}_H(l^2)$ an absolute neighborhood uniform retract?* 1077 ?

It is known that $\mathrm{Bdd}_H(l^2)$ is an ANR [46] and the closed subspace of $\mathrm{Bdd}_H(l^2)$ consisting of closed bounded convex subsets of l^2 is an ANUR, see [31].

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