PROBLEMS FROM THE LVIV TOPOLOGICAL SEMINAR

TARAS BANAKH, BOHDAN BOKALO, IGOR GURAN, TARAS RADUL, AND MICHAEL ZARICHNYI

Introduction

This collection of problems is formulated by participants and guests of the Lviv topological seminar held at the Ivan Franko Lviv National University (Ukraine).

Asymptotic dimension

We recall that a metric space X is proper if the distance $d(\cdot, x_0)$ to a fixed point is a proper map for any $x_0 \in X$. A map $f : X \to Y$ between metric spaces is called *coarse* if it satisfies the following two conditions [33]:

(*Coarse Uniformity*). There is a monotone increasing function $\lambda : [0, \infty) \to [0, \infty)$ such that $d_Y(f(x), f(x')) \leq \lambda(d_X(x, x'))$;

(*Metric Properness*). The preimage $f^{-1}(B)$ is bounded for every bounded set $B \subset Y$.

Two maps f, g into a metric space (X, d) are *close* if there exists a constant $C > 0$ such that $d(f(x), g(x)) < C$, for every $C > 0$. Two metric spaces X, Y are said to be *coarse equivalent* if there exist coarse maps $f\colon X \to Y$ and $g\colon Y \to X$ such that the maps gf and 1_X are close and also fg and 1_Y are close.

For a proper metric space X the *Higson compactification* $\bar{X}$ is defined by means of the following proximity: $A\delta B$ if and only if $\lim_{r\to\infty} d(A \setminus B_r(x_0), B \setminus B_r(x_0)) < \infty$ if $\operatorname{diam} A = \operatorname{diam} B = \infty$ and $d(A, B) = 0$ otherwise.

Here $x_0 \in X$ is a base point, $B_r(x_0)$ is the r-ball centered at x_0 and $d(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}$.

The remainder $\nu X = \bar{X} \setminus X$ of the Higson compactification is called the *Higson corona* [33].

Asymptotic dimension asdim of a metric space was defined by Gromov for studying asymptotic invariants of discrete groups [25]. This dimension can be considered as asymptotic analogue of the Lebesgue covering dimension dim. Dranishnikov has introduced dimension asInd which is analogous to large inductive dimension Ind (see [19]). It is known that $\operatorname{asdim} X = \operatorname{asInd} X$ for each proper metric space with $\operatorname{asdim} X < \infty$. The problem of coincidence of asdim and asInd is still open in the general case [19].

The addition theorem for asdim is proved in [13]: suppose that a metric space X is presented as a union $A \cup B$ of subspaces. Then $\operatorname{asdim} \leq \max\{\operatorname{asdim} A, \operatorname{asdim} B\}$.

We have also a weaker result for the dimension asInd: let X be a proper metric space and $X = Y \cup Z$ where Y and Z are unbounded sets. Then $\operatorname{asInd} X \leq \operatorname{asInd} Y + \operatorname{asInd} Z$ (see [31]).

We do not know whether this estimation is the best possible.

? 1001 **Question.** *Let X be a proper metric space and $X = Y \cup Z$. Is it true that* $\operatorname{asInd} X \leq \max\{\operatorname{asInd} Y, \operatorname{asInd} Z\}$*?*

Let us note that the negative answer to this question gives us a negative answer to the problem of coincidence of asymptotic dimensions.

Extending codomain of Ind to ordinal numbers we obtain the transfinite extension trInd of the dimension Ind. It is known that there exists a space S_α such that $\operatorname{trInd} S_\alpha = \alpha$ for each countable ordinal number α [22]. This method does not work for asInd: the extension appears to be trivial. However there exists non trivial transfinite extension trasdim of asdim (see [32]): there is a metric space X with $\operatorname{trasdim} X = \omega$.

? 1002 **Question.** *Find for each countable ordinal number ξ a metric space X_ξ with* $\operatorname{trasdim} X_\xi = \xi$.

In the classical dimension theory of infinite dimensional spaces there is a special class of spaces that have property C. Properties of such spaces are close to those of finite-dimensional spaces. Dranishnikov defined an asymptotic analogue of property C [16].

? 1003 **Question.** *Let X and Y be two metric spaces with property C. Does $X \times Y$ have property C?*

It is known that the dimension trasdim classifies the class of metric spaces with asymptotic property C. Hence a positive answer to the following question gives us the positive answer to the Question 3.

? 1004 **Question.** *Is there a function $\alpha\colon \omega_1 \to \omega_1$ such that* $\operatorname{trasdim} X \times Y \leq \alpha(\xi)$ *for each countable ordinal number ξ and two metric spaces X, Y with* $\operatorname{trasdim} X \leq \xi$ *and* $\operatorname{trasdim} Y \leq \xi$*?*

Arhangelskii introduced the dimension Dind (see [21]). This dimension has an asymptotic counterpart.

For a proper metric space (X, d) we let $\operatorname{asDind} X = -1$ if and only if X is bounded. Suppose that we have already defined the class of proper metric spaces for which $\operatorname{asDind} X \leq n - 1$. We say that $\operatorname{asDind} X \leq n$ if for every finite family $\mathcal{U}$ of open in the Higson compactification $\bar{X}$ sets there exists a finite family $\mathcal{V}$ of open subsets in $\bar{X}$ with the following property: the family $\{V \cap \nu X \mid V \in \mathcal{V}\}$ is a discrete in νX family which refines $\mathcal{U}$ and $\operatorname{asDind} X \setminus \cup\mathcal{V} \leq n - 1$.

? 1005 **Question.** *Find relations between the dimension* Dind *and the other asymptotic dimension functions.*

It is proved in [20] that every proper metric space of asymptotic dimension 0 is coarsely equivalent to an ultrametric space. Recall that a metric d on a set

X is called an *ultrametric* if $d(x,y) \le \max\{d(x,z), d(z,y)\}$ for every $x,y,z \in X$. The mentioned results from [20] is an asymptotic version of the classical de Groot's result characterizing zero-dimensional metric spaces as those admitting a compatible ultrametric.

Nagata [29] introduced a counterpart of the notion of ultrametric: a metric d on a set X is said to satisfy property $(*)_n$ if, for every $x, y_1, \dots, y_{n+2} \in X$, there exist i, j, $i \ne j$, such that $d(y_i, y_j) \le d(x, y_i)$.

Question. *Is every proper metric space (X,d) with* $\operatorname{asdim} X \le n$ *coarsely equivalent to a proper metric space whose metric satisfies $(*)_n$?* **1006 ?**

There are another classes of metrics that characterize covering dimension (see, e.g. [26]).

Question. *Are there metrics that characterize the asymptotic dimension $n \ge 1$?* **1007 ?**

Extension of metrics

The problem of existence of linear regular operators extending (pseudo)metrics was formulated by C. Bessaga [14] and solved by T. Banakh [2].

Question. *Is there a linear operator that extends metrics from a compact metrizable space X to left invariant metrics on a free topological group of X?* **1008 ?**

A similar question can be formulated for extension of metrics from a compact metrizable space X to norms on the free linear space over X.

Let (X,d) be a compact metric space. Given a subset A of X, we say that a pseudometric ϱ on A is *Lipschitz* if there is $C > 0$ such that $d(x,y) \le C\varrho(x,y)$, for any $x,y \in A$. Also, a function $f\colon A \to \mathbb{R}$ is *Lipschitz* if if there is $C > 0$ such that $|f(x) - f(y)| \le Cd(x,y)$, for every $x,y \in A$. Denote by $\mathrm{lpm}(A)$ (resp. $\mathrm{lpf}(A)$) the set of all Lipschitz pseudometrics (resp. functions) on A. The set $\mathrm{lpm}(A)$ (resp. $\mathrm{lpf}(A)$) is a cone (resp. linear space) with respect to the operations of pointwise addition and multiplication by scalar. We endow $\mathrm{lpm}(A)$ with the norm $\|\cdot\|_{\mathrm{lpm}(A)}$,

$$\|d\|_{\mathrm{lpm}(A)} = \sup\left\{\frac{d(x,y)}{\varrho(x,y)} \mid x \ne y\right\}$$

and $\mathrm{lpf}(A)$ with the seminorm $\|\cdot\|_{\mathrm{lpf}(A)}$,

$$\|f\|_{\mathrm{lpf}(A)} = \sup\left\{\frac{|f(x) - f(y)|}{\varrho(x,y)} \mid x \ne y\right\}.$$

We say that a map $u\colon \mathrm{lpm}(A) \to \mathrm{lpm}(X)$ is an *extension operator* for Lipschitz pseudometrics if the following holds:

1. u is linear (i.e. $u(d_1 + d_2) = u(d_1) + u(d_2)$, $u(\lambda d) = \lambda u(d)$ for every $d_1, d_2 \in \mathrm{lpm}(A)$, $\lambda \in \mathbb{R}_+$);
2. $u(d)|(A \times A) = d$, for every $d \in \mathrm{lpm}(A)$;
3. u is continuous in the sense that $\|u\| = \sup\{\|u(d)\|_{\mathrm{lpm}(X)}X \mid \|d\|_{\mathrm{lpm}(A)} \le 1\}$ is finite.

This definition is a natural counterpart of those introduced in [15] for the extensions of Lipschitz functions. The following notation is introduced in [15]:

$$\lambda(A, X) = \inf\{\|u\| \mid u \text{ is a linear extension operator from } \mathrm{lpf}(A) \text{ to } \mathrm{lpf}(X)\}.$$

Similarly, we put

$$\Lambda(A, X) = \inf\{\|u\| \mid u \text{ is a linear extension operator from } \mathrm{lpm(A)} \text{ to } \mathrm{lpm}(X)\}.$$

? 1009 **Question.** *Let A be a closed subspace of a compact metric space X. Is there an extension operator for Lipschitz pseudometrics $u\colon \mathrm{lpm}(A) \to \mathrm{lpm}(X)$?*

? 1010 **Question.** *Compare $\Lambda(S, X)$ and $\lambda(S, X)$.*

Questions in General Topology

All topological spaces in this section are assumed to be Hausdorff, see [5] and [6] for undefined notions used below.

? 1011 **Question.** *Is there an interplay between topological properties of a compact topological inverse semigroup S and those of the maximal Clifford semigroup $C \subset S$ and the maximal sublattice E? In particular:*

(a) *Is S countably cellular (or separable) if so is the space C?*
(b) *Is S countably cellular if the maximal semilattice E is second countable?*
(c) *Is S (hereditary) separable if all maximal groups of S are (hereditary) separable and the maximal semilattice is Lawson and (hereditary) separable?*
(d) *Is S fragmentable (resp. Corson, Eberlein, Gul'ko, Radon-Nikodym, or Rosenthal) compact if so is the Clifford semigroup C?*

By a *mean* on a space X we understand any commutative idempotent operation $m : X \times X \to X$. Associative means are also called *semilattice operations.* Each scattered metrizable compact space, being homeomorphic to an ordinal interval $[0, \alpha]$, admits a continuous assiciative mean (just take the operations min or max on $[0, \alpha]$).

Question. *Does any scattered compact Hausdorff space X admits a (separately) continuous mean?*

It should be noted that there exist scattered compacta admitting no separately continuous associative means, see [7].

The other our question is due to V. Maslyuchenko, V. Mykhaylyuk and O. Sobchuk and relates to the classical theorem of Baire on functions of the first Baire class. We recall that a function $f : X \to Y$ between topological spaces is called

- *of the first Baire class* if f is the pointwise limit of a sequence of continuous functions;
- *F_σ-measurable* if the preimage $f^{-1}(U)$ of any open set $U \subset Y$ is of type F_σ in X.

It is well-known that each function $f : X \to Y$ of the first Baire class with values in a perfectly normal space is F_σ-measurable. The converse is true if X is Polish and Y is a linear metric space. For non-metric Y the situtation is not so definite.

Question. *Is each F_σ-measurable function $f : [0,1] \to C_p[0,1]$ a function of the first Baire class?*

This question is equivalent to the original question of V. Maslyuchenko, V. Mykhaylyuk, and O. Sobchuk [28]:

Question. *Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be a function continuous with respect to the first variable and of the first Baire class with respect to the second variable. Is f the pointwise limit of separately continuous functions?*

Let $\mathcal{P}$ be a property of a subset in a topological space. A topological space is called an *$A\mathcal{P}$-space* (resp. *$WA\mathcal{P}$-space*) if for every $A \subset X$ and every (resp. some) point $x \in \bar{A} \setminus A$ there exists a subset $B \subset A$ with the property $\mathcal{P}$ in X such that $x \in \bar{B}$.

For example, a space X has countable tightness iff it is an $A\mathcal{P}$-space for the property $\mathcal{P}$ of being a countable subset. A space X is Frechet-Urysohn (resp. sequential) if and only if X is an $A\mathcal{P}$-space (resp. $WA\mathcal{P}$-space) where $\mathcal{P}$ is the property of a subset $A \subset X$ to have compact metrizable closure. A space X is a *k'-space* in the sense of Arkhangelski [1] if and only if X is an $A\overline{\mathcal{C}}$-space where $\overline{\mathcal{C}}$ is the property of a subset $A \subset X$ to have compact closure in X.

Question. *Find an example of a countably compact $WA\overline{\mathcal{C}}$-space which is not an $A\overline{\mathcal{C}}$-space.* **1012 ?**

Let $\mathcal{D}$ (resp. $\mathcal{M}$) denote the properties of a subspace to be discrete (resp. metrizable).

Question. *Is every topological group of countable tightness an $A\mathcal{M}$-space? $A\mathcal{D}$-space?* **1013 ?**

Question. *Let X be an $A\mathcal{M}$-space. Is free topological group of X an $A\mathcal{M}$-space?* **1014 ?**

Question. *Characterize the class of monothetic $A\mathcal{M}$-groups ($A\mathcal{M}$-paratopological groups)?* **1015 ?**

Question. *Is every countable regular space an $A\mathcal{M}$-space?* **1016 ?**

Questions on functors in the category of compact Hausdorff spaces

We denote by **Comp** the category of compact Hausdorff spaces and continuous maps. In the sequel, all the functors are assumed to be covariant endofunctors in **Comp**.

First, we mention few examples of functors. The *hyperspace functor* exp assigns to every compact Hausdorff space X the set $\exp X$ of nonempty closed subsets in X endowed with the Vietoris topology. A base of this topology consists of the sets of the form $\langle U_1, \dots, U_n \rangle = \{A \in \exp X \mid A \subset U_1 \cup \dots \cup U_n,\ A \cap U_i \neq \emptyset$ for all $i\}$. Given a map $f\colon X \to Y$ in **Comp**, the map $\exp f\colon \exp X \to \exp Y$ is definerd by $\exp F(A) = f(A)$.

The *probability measure functor* P assigns to every compact Hausdorff space X the set $P(X)$ of probability measures endowed with the weak* topology.

Let G be a subgroup of the permutation group S_n. The *G-symmetric power of* X, $SP^n_G(X)$, is the quotient space of X^n with respect to the natural action of G on X^n by permutation of coordinates. One can easily see that this construction determines a functor.

Some properties of the mentioned functors and other known functors were used by E.V. Shchepin [34] in order to introduce the notion of a normal functor. It turned out that normal functors, and some close to normal functors, found important applications in the topology of nonmetrizable compact Hausdorff spaces and other areas of topology (see, e.g., [35, 23, 34]).

If a functor F preserves embeddings, then, for a compact Hausdorff space X and a closed subspace A of X, we always identify the space $F(A)$ with a subspace in $F(X)$ along the embedding $F(i)$, where $i\colon A \to X$ is the inclusion map.

Let a functor F preserve embeddings. We say that F *preserves preimages* if $F\left(f^{-1}(A)\right) = F(f)^{-1}\left(F(A)\right)$ for every map $f\colon X \to Y$ and every closed subset A of Y. We say that F *preserves intersections* whenever $F\left(\bigcap\{A_\alpha \mid \alpha \in \Gamma\}\right) = \bigcap\{F(A_\alpha) \mid \alpha \in \Gamma\}$ for every family of closed subsets $\{A_\alpha \mid \alpha \in \Gamma\}$ in X.

An endofunctor F in **Comp** is called *normal* (Shchepin[34]) if F preserves embeddings, surjections, weight of infinite compacta, intersections, preimages, singletons, the empty set, and the limites of inverse systems $\mathcal{S} = \{X_\alpha, p_{\alpha\beta}; \mathcal{A}\}$ over directed sets $\mathcal{A}$. More precisely, the latter condition means that the map $h = (F(p_\alpha)_{\alpha\in\mathcal{A}}$ is a homeomorphism of $F(\varprojlim \mathcal{S})$ onto $\varprojlim F(\mathcal{S})$, where $p_\alpha\colon \varprojlim \mathcal{S} \to X_\alpha$ is the limit projection.

A functor F is said to be *weakly normal (almost normal)* if it satisfies all the properties from the previous definition except perhaps the property of being epimorphic (respectively, the preimage preserving property).

The hyperspace functor exp, the probability measure functor P, and the G-symmetric power functor SP^n_G are examples of normal functors.

For a functor F and a compact Hausdorff space X denote by $F_n(X)$ the subspace $\bigcup\{F(f)\left(F(n)\right) \mid f \in C(n, X)\}$ of $F(X)$ (here $C(n, X)$ denotes the set of all maps from the discrete space n to X). Clearly, such a construction determines a subfunctor F_n of F. A functor F is of *finite degree* if there exists $n \in \mathbb{N}$ such that $F = F_n$.

If $\varphi = (\varphi_X)\colon F \to F'$ is a natural transformation of functors then we say that F is a *subfunctor* of F' if all the components of φ are inclusion maps and we say that F' is a *quotient functor* of F if all the components of φ are onto maps.

The characteristic map of a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g\big\downarrow & & \big\downarrow h \\ Z & \xrightarrow{u} & T \end{array}$$

in the category **Comp** is the map $\chi\colon X \to Y \times_T Z = \{(y,z) \in Y \times Z \mid h(y) = u(z)\}$ defined by the formula $\chi(x) = (f(x), g(x))$. A diagram is bicommutative if its characteristic map is onto. A diagram is open-bicommutative if its characteristic map is open and onto.

A functor $F\colon$ **Comp** $\to$ **Comp** is said to be *bicommutative* (resp. open-bicommutative) if F preserves the class of bicommutative (resp. open-bicommutative) diagrams.

A functor is *open* if it preserves the class of open surjective maps. E.V. Shchepin proved that every open functor is bicommutative.

Question (Shchepin)**.** *Is every normal bicommutative functor open?* **1017 ?**

This problem wass formulated more then 25 years ago. The notions of open and bicommutative functors were introduced by E.V. Shchepin [34]. The problem was solved in [38] for normal functors of finite degree.

Question. *Is every normal bicommutative (open) functor open-bicommutative?* **1018 ?**

It is proved in [35] that natural transformations of (weakly, almost) normal functors form a set and therefore one can introduce the category of normal functors and their natural transformations.

A (weakly, almost) normal functor F is called *universal* if every (weakly, almost) normal functor is isomorphic to a subfunctor of F.

Question. *Is there a universal (weakly, almost) normal functor?* **1019 ?**

A normal functor F is called *couniversal* if every normal functor F' is a quotient functor of F.

Question. *Is there a couniversal (weakly, almost) normal functor?* **1020 ?**

A normal functor F is called *zero-dimensional* if $\dim F(X) = 0$ for every compact Hausdorff space X with $\dim X = 0$.

Question. *Is every normal functor a quotient functor of a zero-dimensional normal functor?* **1021 ?**

Let $\tau > \omega$ be a cardinal number. A functor F is called *τ-normal* if F satisfies all the properties from the definition of normality except the preserving of weight and, in addition, the weight of $F(X)$ is $\leq \tau$, for every compact metrizable X (the minimal τ for which a functor F is τ-normal is called the *weight* of F.

Actually, one can find the prototype of the notion of τ-normal functor in Shchepin's paper [34] as he considered the so-called normal functor-powers, i.e., the spaces of the form $F(X^\tau)$.

We say that a map $f\colon X\to Y$ satisfies the *homeomorphism-lifting property* if, for every homeomorphism $h\colon Y\to Y$ there exists a homeomorphism $h'\colon X\to X$ such that $fh'=hf$.

? 1022 **Question** (Shchepin)**.** *Let X be a metric compact space and F a normal functor. Does the map $F\left((pr)^\tau\right):F\left((X\times X)^\tau\right)\to F(X^\tau)$ satisfy the homeomorphism-lifting property?*

? 1023 **Question.** *Is every multiplicative τ-normal functor isomorphic to the power functor $(\cdot)^\tau$?*

For normal functors, this problem was posed by Shchepin and solved in [36].

Shchepin proved the so-called spectral theorem, which states that, under some reasonable conditions, if a nonmetrizable compact Hausdorff space is represented as the inverse limit of two systems consisting of spaces of smaller weight then these systems contain isomorphic cofinal subsystems (see [34] for details). One can consider representations of τ-normal functors as the limits of inverse systems consisting of functors of smaller weight and their natural transformations.

? 1024 **Question.** *Is there a counterpart of Shchepin's spectral theorem in the category of τ-normal functors?*

Of special interest are functors of finite degree that preserve the class of compact metric ANR spaces (i.e., absolute neighborhood retracts). Basmanov [12] established such a property for a wide enough class of functors. Such functors are known to preserve other classes of spaces too: Q-manifolds (i.e., manifolds modeled on the Hilbert cube $Q=[0,1]^\omega$) [23], n-movable spaces [37], compact metric absolute neighborhood extensors in dimension n [17].

We are going to formulate a few questions on the preservation of some geometric properties by functors of finite degree.

Let P be a CW-complex. For any compact metric space X the *Kuratowski notation* $X\,\tau\,P$ means the following: for every continuous map $f\colon A\to P$ defined on a closed subset A of X there is a continuous extension of f onto X.

Denote by $\mathcal{L}$ the class of all countable CW-complexes. Following [18], we define a preorder relation $\le$ on $\mathcal{L}$. For $L_1,L_2\in\mathcal{L}$, we have $L_1\le L_2$ if and only if $X\,\tau\,L_1$ implies $X\,\tau\,L_2$ for all compact metric spaces X. This preorder relation determines the following equivalence relation $\sim$ on $\mathcal{L}$: $L_1\sim L_2$ if and only if $L_1\le L_2$ and $L_2\le L_1$. We denote by $[L]$ the equivalence class containing $L\in\mathcal{L}$.

For a compact metric space X, we say that its *extension dimension does not exceed* $[L]$ (briefly $\text{ext-dim}\,X\le[L]$) whenever $X\,\tau\,L$.

A compact metric space X is said to be an absolute (neighborhood) extensor in extension dimension $[L]$ if for any compact metric pair (A,B) with $\text{ext-dim}\,A\le[L]$ and any continuous map $f\colon B\to X$ there exists a continuous extension $\bar f\colon A\to X$ (respectively $\bar f\colon U\to X$, where U is a neighborhood of B in A) of f.

In the sequel, we suppose that F is a normal functor of finite degree that preserves the class of compact metrizable ANR-spaces.

? 1025 **Question.** *Does F preserve the class of absolute (neighborhood) extensors in ex-*

tension dimension $[L]$*?*

Two maps $f_0, f_1 \colon X \to Y$ are said to be $[L]$-*homotopic* if there exists a space Z with ext-dim $Z \le [L]$, a map $\alpha \colon Z \to X \times [0,1]$ which is $[L]$-invertible (i.e., satisfies the property of lifting of maps from spaces of extension dimension $\le [L]$), and a map $H \colon Z \to Y$ such that $f_i\alpha(z) = H(z)$, for every $z \in \alpha^{-1}(X \times \{i\})$, $i = \{0,1\}$.

Question. *Does F preserve the relation of $[L]$-homotopy of maps?* **1026 ?**

We finish with the following question.

Question. *Does F preserve the class of essential Q-M-factors, i.e., the class of spaces X such that $X \times A$ is a Q-manifold for some A with* $\dim A < \infty$*?* **1027 ?**

1. Some problems in Ramsey Theory

In this section we ask some problems on symmetric subsets in colorings of groups. By an r-coloring of a set X we understand any map $\chi : X \to \{1, \dots, r\}$, which can be identified with a partition $X = X_1 \cup \dots \cup X_r$ of X into r disjoint pieces $X_i = \chi^{-1}(i)$. As a motivation for subsequent questions let us mention the following result of T. Banakh [3].

Theorem BP. *For any n-coloring of the group $\mathbb{Z}^n$ there is an infinite monochromatic subset $S \subset \mathbb{Z}^n$ symmetric with respect to some point $c \in \{0,1\}^n$.*

A subset S of a group G is called *symmetric* with respect to a point $c \in G$ if $S = cS^{-1}c$.

This theorem suggests to introduce the cardinal function $\nu(G)$ assigning to each group G the smallest cardinal number r of colors for which there is an r-coloring of G without infinite monochromatic subsets.

In [8] the value $\nu(G)$ was calculated for any abelian group G:

$$\nu(G) = \begin{cases} r_0(G) + 1 & \text{if } G \text{ is finitely generated} \\ r_0(G) + 2 & \text{if } G \text{ is infinitely generated and } |G_2| < \aleph_0 \\ \max\{|G_2|, \log|G|\} & \text{if } |G_2| \ge \aleph_0 \end{cases}$$

where $r_0(G)$ is the free rank of G and $G_2 = \{x \in G : 2x = 0\}$ is the Boolean subgroup of G.

Much less is known for non-commutative groups.

Question. *Investigate the cardinal $\nu(G)$ for non-commutative groups G. In particular, is $\nu(F_2)$ finite for the free group F_2 with two generators?*

The only information on $\nu(F_2)$ is that $\nu(F_2) > 2$, see [24].

Question. *Does for every finite coloring of an infinite group G there is a monochromatic symmetric subset $S \subset G$ of arbitrarily large finite size?* (The answer is affirmative if G is Abelian).

For every uncountable abelian group G with $|G_2| < |G|$ there is a 2-coloring of G without symmetric monochromatic subsets of size $|G|$, see [30].

Question. *Is it true that for every 2-coloring of an uncountable abelian group G with $|G_2| < |G|$ and for every cardinal $\kappa < |G|$ there is a monochromatic symmetric subset $S \subset G$ of size $|S| > \kappa$?* (The answer is affirmative under GCH, see [24]).

There is another interesting concept suggested by Theorem BP on colorings of the group $\mathbb{Z}^n$. Let us define a subset $C \subset \mathbb{Z}^n$ to be *central* if for any n-coloring of $\mathbb{Z}^n$ there is an infinite monochromatic subset $S \subset \mathbb{Z}^n$ symmetric with respect to a point $c \in C$. A central set $C \subset \mathbb{Z}^n$ is called *minimal* if it does not lie in any smaller central set.

? 1028 **Question.** *Describe the geometric structure of (minimal) cenral subsets of $\mathbb{Z}^n$. Is each minimal central subset of $\mathbb{Z}^n$ finite? What is the smallest size $c(\mathbb{Z}^n)$ of a central set in $\mathbb{Z}^n$?*

It was proved in [4] that $\frac{n(n+1)}{2} \le c(\mathbb{Z}^n) < 2^n$ and $c(\mathbb{Z}^n) = \frac{n(n+1)}{2}$ for $n \le 3$.

Question. *Calculate the number $c(\mathbb{Z}^4)$.* (It is known that $12 \le c(\mathbb{Z}^4) \le 14$, see [4]).

Concerning the first (geometric) part of Question 1028 the following information is available for shall n, see [4]:

(1) a subset $C \subset \mathbb{Z}$ is central if and only if C contains a point;
(2) a subset $C \subset \mathbb{Z}^2$ is central if and only if it contains a triangle $\{a, b, c\} \subset C$ (by which we understand a three-element affinely independent subset of $\mathbb{Z}^2$);
(3) each central subset $C \subset \mathbb{Z}^3$ of size $|C| = c(\mathbb{Z}^3) = 6$ is an octahedron $\{c \pm e_i : i \in \{1, 2, 3\}\}$ where $c \in \mathbb{Z}^n$ and $e_1, e_2, e_3 \in \mathbb{Z}^n$ are linearly independent vectors;
(4) there is a minimal central subset $C \subset \mathbb{Z}^3$ of size $|C| > 6$ containing no octahedron.

There is another number invariant $\mathrm{ms}(X, \mathcal{S}, r)$ related to colorings and defined for any space X endowed with a probability measure μ and a family $\mathcal{S}$ of measurable sets called symmetric subsets of X. By definition, $\mathrm{ms}(X, \mathcal{S}, r) = \inf\{\varepsilon > 0 :$ for every measurable r-coloring of X there is a monochromatic subset $S \in \mathcal{S}$ of measure $\mu(S) \ge \varepsilon\}$. The notation "ms" reads as the *m*aximal *m*easure of a *m*onochromatic *s*ymmetric *s*ubset and was suggested by Ya. Vorobets. If the family $\mathcal{S}$ is clear from the context (as it is in case of groups), we will write $\mathrm{ms}(X, r)$ instead of $\mathrm{ms}(X, \mathcal{S}, r)$.

The number invariant $\mathrm{ms}(X, r)$ is defined for many natural algebraic and geometric objects: compact topological groups, spheres, balls etc. For such objects, typically, $\mathrm{ms}(X, r)$ is equal to $\frac{1}{r^2}$, see [9], [10]. However, the most intriguing is the case of the unit interval $[0, 1]$ carrying the standard Lebesgue measure. A subset $S \subset [0, 1]$ is symmetric if $S = 2c - S$ for some point $c \in \mathbb{R}$. Surprisingly, but we do not know the value of $ms([0, 1], r)$ even for $r = 2$. Only some boundes are known: $\frac{1}{r^2 + r\sqrt{r^2 - r}} \le \mathrm{ms}([0, 1], r) < \frac{1}{r^2}$ for $r > 1$ and $\frac{1}{4 + \sqrt{6}} \le \mathrm{ms}([0, 1], 2) < \frac{5}{24}$, see also [27].

Question. *Calculate the value* $ms([0,1],r)$ *(at least for* $r=2$*). Can* $\mathrm{ms}([0,1],2)$ *be expressed via some known mathematic constants?*

It was proved in [10] that the limit $\lim_{r\to\infty} r^2\cdot \mathrm{ms}([0,1],r)$ exists and lies in the interval $[\frac{1}{2},\frac{5}{6}]$.

Question. *Calculate the constant* $c=\lim_{r\to\infty} r^2\cdot \mathrm{ms}([0,1],r)$.

More detail information on these problems can be found in the surveys [9][1] and [11].

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