ON r-REFLEXIVE BANACH SPACES

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Abstract. A Banach space X is called *r-reflexive* if for any cover $\mathcal{U}$ of X by weakly open sets there is a finite subfamily $\mathcal{V} \subset \mathcal{U}$ covering some ball of radius 1 centered at a point x with $\|x\| \le r$. We prove that an infinite-dimensional separable Banach space X is ∞-reflexive (r-reflexive for some $r \in \mathbb{N}$) if and only if each ε-net for X has an accumulation point (resp., contains a non-trivial convergent sequence) in the weak topology of X. We show that the quasireflexive James space J is r-reflexive for no $r \in \mathbb{N}$. We do not know if each ∞-reflexive Banach space is reflexive, but we prove that each separable ∞-reflexive Banach space X has Asplund dual. As a by-product of the proof we obtain a covering characterization of the Asplund property of Banach spaces.

1. Introduction

In this paper we address the following problem posed by the third author in 2000 on the Winter School in Krištanovice (Čech Republic):

Question 1. *Is a separable Banach space X reflexive if each net in X has an accumulation point in the weak topology of X?*

By a *net* in a Banach space $(X, \|\cdot\|)$ we understand an ε-net $N \subset X$ for some $\varepsilon > 0$. A subset $N \subset X$ is called an *ε-net* for a subset $B \subset X$ if for every point $x \in B$ there is a point $y \in N$ with $\|x - y\| < \varepsilon$.

It turns out that Question 1 is equivalent to an even more intriguing question concerning ∞-reflexive Banach spaces.

Definition 1. A Banach space $(X, \|\cdot\|)$ is called *r-reflexive* where $r \in [0, +\infty]$ if for every cover $\mathcal{U}$ of X by weakly open sets there is a finite subfamily $\mathcal{V} \subset \mathcal{U}$ that covers the open unit ball $x + B_X = \{y \in X : \|x - y\| < 1\}$ centered at some point $x \in X$ with $\|x\| \le r$.

Observe that a Banach space X is reflexive if and only if it is 0-reflexive. We define a Banach space X to be *ω-reflexive* if it is r-reflexive for some $r \in [0, \infty)$.

It turns out that for infinite-dimensional separable Banach spaces the property appearing in Question 1 is equivalent to the ∞-reflexivity.

Theorem 1. *An infinite-dimensional separable Banach space X is ∞-reflexive (resp. ω-reflexive) if and only if every net in X has an accumulation point (resp. contains a non-trivial convergent sequence) in the weak topology of X.*

This theorem is not true for non-separable Banach spaces: for any uncountable set Γ the Banach space $c_0(\Gamma)$ is weakly Lindelöf [Fab, §7.1]. Consequently, each net

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for $c_0(\Gamma)$, being uncountable, has an accumulation point in the weak topology. On the other hand, the space $c_0(\Gamma)$ is not ∞-reflexive by Proposition 2 below. This example also shows that in the realm of non-separable Banach spaces the answer to Question 1 is negative.

Theorem 1 allows us to reformulate and extend Question 1 as follows:

Question 2. *Is a (separable) Banach space X reflexive if it is ∞-reflexive? ω-reflexive? 1-reflexive?*

In light of the last part of this question, it is interesting to mention that a Banach space X is reflexive if and only if X is r-reflexive for some $r < 1$. This equivalence (observed by the referee) follows from the fact that each cover of a 1-ball $x + B_X$ centered at a point $x \in X$ with $\|x\| \le r < 1$ covers also the closed ball of radius $\frac{1}{2}(1-r)$ centered at the origin.

Trying to answer Questions 1 and 2, it is natural to look at the quasireflexive James space J (having codimension 1 in its second dual). We recall that a Banach space X is *quasireflexive* if it has finite codimension in its second dual space X^{**}.

Theorem 2. *The quasireflexive James space J is not ω-reflexive.*

However we do not know if the James space is ∞-reflexive.

Question 3. *Is each quasireflexive Banach space ∞-reflexive? Is the James space ∞-reflexive?*

Our principal result on separable ∞-reflexive Banach spaces asserts that any such a space has Asplund dual. We recall that a Banach space X is *Asplund* if each separable subspace Y of X has separable dual Y^*.

Theorem 3. *Each separable ∞-reflexive Banach space X has Asplund dual X^*.*

Since the Banach space l_1 is not Asplund, Theorem 3 implies the result of [Ba] (asserting that the dual space X^* of a separable ∞-reflexive Banach space X contains no copy of l_1). Theorem 3 has also another corollary related to the Fréchet-Urysohn property of the weak topology on bounded subsets of an ∞-reflexive Banach space.

Following [En, §1.6], we say that a topological space X is *Fréchet-Urysohn* if for each accumulation point $x \in X$ of a subset $A \subset X$ some sequence $\{a_n\}_{n=1}^{\infty} \subset A$ converges to x.

Since Eberlein compact spaces are Fréchet-Urysohn, the weak topology of a reflexive space X is Fréchet-Urysohn on bounded subsets of X. A similar property holds for separable ∞-reflexive Banach spaces.

Corollary 1. *If X is a separable ∞-reflexive Banach space, then the unit ball of X endowed with the weak topology is a Fréchet-Urysohn space.*

Proof. By Theorem 3, the space X^* is Asplund and hence X^* contains no isomorphic copy of the space l_1. By Pełczynski Theorem [Pel] (see also [Dis, p.213]), the space X does not contain a copy of l_1. Then the Odell-Rosenthal Theorem [OR] (see also [Dis, p.215]) guarantees that the second dual unit ball $\bar{B}^{**}$ endowed with the weak* topology is Rosenthal compact; more precisely, $\bar{B}^{**}$ is a compact subspace of the space $B_1(B^*) \subset \mathbb{R}^{\bar{B}^*}$ of functions of the first Baire class on the dual unit ball $\bar{B}^*$. Finally, we apply the Bourgain-Fremlin-Talagrand Theorem [BFT] establishing the Fréchet-Urysohn property of separable Rosenthal compacta to conclude that the unit ball $\bar{B}^{**} \supset \bar{B}$ is Fréchet-Urysohn. $\square$

The proof of Theorem 3 relies on a characterization of the Asplund property of a dual Banach space in terms of so-called weak* covering properties.

Definition 2. A Banach space X is defined to satisfy the *τ-covering property*, where τ is a weaker linear topology on X, if for every sequence $(U_i)_{i=1}^{\infty}$ of τ-open sets in X whose intersection $\bigcap_{i=1}^{\infty} U_i$ is a norm-neighborhood of the origin in X there are points $x_1, \dots, x_n \in X$ such that the union $\bigcup_{i=1}^{n}(x_i + U_i)$ contains the open unit ball B_X centered at the origin of X.

If τ is the weak or weak* topology, then we say about *the weak or weak* covering properties*, briefly, WCP and W*CP.

Theorem 3 can be derived from the following theorem that can have an independent value.

Theorem 4. (1) *Each separable ∞-reflexive Banach space has the weak covering property;*
(2) *If a Banach space X has the weak covering property, then the second dual space X^{**} has the weak* covering property;*
(3) *A Banach space X is Asplund if and only if the dual space X^* has the weak* covering property.*

The obtained results fit into the following diagram connecting various reflexivity-like properties and holding for any separable Banach space X:

$$\begin{tikzcd}
X \text{ is 0-reflexive} \arrow[r] \arrow[d, leftrightarrow] & X \text{ is } \omega\text{-reflexive} \arrow[r] & X \text{ is } \infty\text{-reflexive} \arrow[r] & X \text{ has WCP} \arrow[d] \\
X \text{ is reflexive} \arrow[r] & X \text{ is quasireflexive} \arrow[r] \arrow[u, "/" marking] & X^* \text{ is Asplund} \arrow[r, leftrightarrow] & X^{**} \text{ has W}^*\text{CP}
\end{tikzcd}$$

Before passing to proofs of Theorems 1–4 we discuss some stability properties of r-reflexive spaces and ask some related questions.

Proposition 1. *Let Z be a Banach subspace of a Banach space X.*
(1) *If X is r-reflexive for some $r \in [0, +\infty]$, then the quotient space X/Z is r-reflexive too.*
(2) *If X is r-reflexive for some $r \in \{0, \omega, \infty\}$, then each Banach space Y isomorphic to X is r-reflexive.*
(3) *If Z is reflexive and X/Z is r-reflexive for some $r \in [0, +\infty)$, then X is r-reflexive too.*

Question 4. *Is a subspace of a (separable) r-reflexive Banach space r-reflexive (at least for $r \in \{\omega, \infty\}$)?*

Question 5. *Is the second dual X^{**} of an r-reflexive Banach space X r-reflexive? Is a Banach space X r-reflexive if its second dual X^{**} if r-reflexive for some $r \in [0, +\infty]$?*

Since the r-reflexivity is an isomorphic property for $r \in \{0, \omega, \infty\}$, we may also ask:

Question 6. *Is the r-reflexivity an isomorphic property for arbitrary $r \in (0, +\infty)$?*

As we already know, Theorem 1 is not true for non-separable Banach spaces. What about Theorem 3?

Question 7. *Has each ∞-reflexive Banach space Asplund dual?*

We can give a partial answer for Banach spaces with $\aleph_0$-monolithic dual space. We recall that a topological space X is *monolithic* (resp. $\aleph_0$-*monolithic*) if each (separable) subspace Y of X has network weight $\mathrm{nw}(Y)$ equal to the density $\mathrm{dens}(Y)$ of Y. It is easy to see that each Banach space is monolithic in norm and weak topologies.

We shall say that a Banach space X has ($\aleph_0$-)*monolithic dual space*, if the dual space X^* is ($\aleph_0$-)monolithic with respect to the weak* topology. It can be shown that a Banach space X has ($\aleph_0$-)monolithic dual space if and only if for any (separable) subset $Y \subset X^*$ the annulator $Y^\top = \{x \in X : \forall y^* \in Y\ y^*(x) = 0\}$ satisfies $\mathrm{dens}(X/Y^\top) = \mathrm{dens}(Y)$ in X. The latter property was introduced in [BPZ] as the property (1). Since Corson compacta are monolithic, each weakly Lindelöf determined Banach space (=Banach space with Corson dual ball) has monolithic dual. In particular, for each set Γ the Banach space $c_0(\Gamma)$ has monolithic dual.

Proposition 2. *Each ∞-reflexive Banach space with $\aleph_0$-monolithic dual has Asplund dual.*

Proof. Assume that X is an ∞-reflexive Banach space with $\aleph_0$-monolithic dual. To show that X^* is Asplund, take any separable subspace $Y \subset X^*$ and consider its annulator $Y^\top = \{x \in X : \forall y^* \in Y\ y^*(x) = 0\}$ in X. The Hahn-Banach Theorem implies that Y is weak* dense in $(X/Y^\top)^*$ identified with the subspace $(Y^\top)^\perp \subset X^*$ of functionals that annulate $Y^\top$. Since X has $\aleph_0$-monolithic dual, the space $(X/Y^\top)^*$, being separable, has countable network weight in the weak* topology. Consequently, the unit ball of $(X/Y^\top)^*$ in the weak* topology has countable network weight and is metrizable. This is equivalent to the separability of $X/Y^\top$. Being a quotient of the ∞-reflexive space X, the space $X/Y^\top$ is ∞-reflexive. Applying Theorem 3, to the separable ∞-reflexive space $X/Y^\top$, we conclude that the dual space $(X/Y^\top)^*$ is Asplund and consequently, its separable subspace Y has separable dual Y^*. $\square$

Also we do not know if the separability assumption is essential in Corollary 1.

Question 8. *Let X be an ∞-reflexive Banach space (with $\aleph_0$-monolithic dual). Is the unit ball of X, endowed with the weak topology a Fréchet-Urysohn space?*

Finally, we ask:

Question 9. *Let X be a separable ∞-reflexive Banach space. Is the dual space X^* separable? Equivalently, is the second dual X^{**} separable?*

Now we present the proofs of the results announced in the introduction.

2. Proof of Theorem 4

The first item of Theorem 4 is established in

Lemma 1. *A separable ∞-reflexive Banach space X has the weak covering property.*

Proof. To show that X has the weak covering property, take any sequence $(U_n)_{n\in\omega}$ of weakly open sets in X such that $\bigcap_{n\in\omega} U_n$ has non-empty interior in X. Let $\{x_n : n \in \omega\}$ be a countable dense set in X. It follows that $\{x_n + U_n : n \in \omega\}$ is a cover of X by weakly open sets. The ∞-reflexivity of X yields a point $x \in X$ such that the open unit ball $x + B_X$ centered at x lies in the finite union $\bigcup_{n=0}^m x_n + U_n$ for some $m \in \omega$. Then $B_X \subset \bigcup_{n=0}^m (x_n - x + U_n)$ witnessing the weak covering property of X. □

The second item of Theorem 4 is established in

Lemma 2. *If a Banach space has the weak covering property, then the second dual space X^{**} has the weak* covering property.*

Proof. Suppose that $(V_i)_{i=1}^\infty$ is a sequence of weak* open sets in X^{**} whose intersection $\bigcap_{i=1}^\infty V_i$ contains a closed ε-ball $\varepsilon\bar{B}^{**}$. To show that X^{**} has the weak* covering property, it suffices to find points $x_1, \dots, x_n \in X^{**}$ such that $\bigcup_{i=1}^n (x_i + V_i) \supset \bar{B}^{**}$.

By the compactness of $\varepsilon\bar{B}^{**}$ and the regularity of the weak* topology, for every $i \in \mathbb{N}$, there is a weak* open subset $W_i \subset X^{**}$ such that $\varepsilon\bar{B}^{**} \subset W_i \subset \overline{W_i} \subset V_i$ where the closure is taken in the weak* topology of X^{**}.

Consider the sequence $(U_i)_{i=1}^\infty$, $U_i = W_i \bigcap X$, of weakly open sets in X. Note that $\bigcap_{i=1}^\infty U_i = (\bigcap_{i=1}^\infty W_i) \bigcap X \supset \varepsilon\bar{B}^{**} \bigcap X = \varepsilon\bar{B}$. By definition of the weak covering property of X, there exist points $x_1, \dots, x_n \in X$ such that the union $\bigcup_{i=1}^n (x_i + U_i)$ contains the open unit ball B centered at the origin. According to Goldstine Theorem [HHZ, p.46], $\overline{B} = \bar{B}^{**}$. Thus we obtain $\bar{B}^{**} = \overline{B} \subset \bigcup_{i=1}^n (x_i + \overline{U_i}) \subset \bigcup_{i=1}^n (x_i + \overline{W_i}) \subset \bigcup_{i=1}^n (x_i + V_i)$, and hence X^{**} has the weak* covering property. □

For the proof of the third item of Theorem 4 we need an auxiliary

Lemma 3. *Let K be a weak* compact subset of a weak* open set U of a dual Banach space X^*. Then there is a weak* open set V in X^* such that $K \subset V \subset U$ and $V = V + L$ for some weak* closed linear subspace L of finite codimension in X^*.*

Proof. By definition, the weak* topology on X^* has a base consisting of sets W such that $W = W + F^\perp$ for some finite subset $F \subset X$. Here, as expected, $F^\perp = \{x^* \in X^* : \forall x \in F\ x^*(x) = 0\}$. Consequently, for every $x \in K$ we may find a weak* open subset $O(x) \subset X^*$ such that $x \in O(x) \subset U$ and $O(x) = O(x) + F_x^\perp$ for some finite subset $F_x \subset X$. Using the weak* compactness of K, choose a finite subcover $\{O(x_1), \dots, O(x_n)\}$ of the cover $\{O(x) : x \in K\}$ of K. Then the weak* open set $V = \bigcup_{i=1}^n O(x_i)$ has the properties $K \subset V \subset U$ and $V = V + F^\perp$, where $F = \bigcup_{i=1}^n F_{x_i}$. □

The following characterization establishes the third item of Theorem 4.

Proposition 3. *For a Banach space X the following conditions are equivalent:*

1. *X is Asplund;*
2. *X^* has W*CP;*
3. *for each sequence $(U_i)_{i=1}^\infty$ of weak* open subsets of X^* whose intersection $\bigcap_{i=1}^\infty U_i$ is a norm-neighborhood of the origin there is a sequence of points $\{a_i^*\}_{i=1}^\infty \subset X^*$ such that $X^* = \bigcup_{i=1}^\infty (a_i^* + U_i)$.*

Proof. (1) ⇒ (3) Fix a sequence $(U_i)_{i=1}^\infty$ of weak* open sets in X^* whose intersection $\bigcap_{i=1}^\infty U_i$ contains the closed ε-ball $\varepsilon\bar{B}^*$ centered at the origin.

By Lemma 3, for every $i \ge 1$ there exists a weak* open set $V_i \subset X^*$ such that $\varepsilon\bar{B}^* \subset V_i \subset U_i$ and $V_i = V_i + F_i^\perp$ for some finite subset $F_i \subset X$. Let Y be the closed linear hull of the set $F = \bigcup_{i=1}^\infty F_i$ in X. As X is Asplund, Y^* is separable. Since Y^* is isomorphic to $X^*/Y^\perp = X^*/F^\perp$, the latter quotient space is separable. Since the quotient map $\pi : X^* \to X^*/F^\perp$ is open, the set $\pi(\varepsilon B^*)$ has non-empty interior in $X^*/F^\perp$. The separability of $X^*/F^\perp$ yields a countable subset $C = \{c_i : i \ge 1\}$ of $X^*/F^\perp$ such that $C + \pi(\varepsilon B^*) = X^*/F^\perp$. For every $i \ge 1$ find a point $a_i^* \in X^*$ with $\pi(a_i^*) = c_i$. Then

$$X^* = \bigcup_{i=1}^\infty \big(a_i^* + \pi^{-1}(\pi(\varepsilon B^*))\big) = \bigcup_{i=1}^\infty (a_i^* + \varepsilon B^* + F^\perp) \subset \bigcup_{i=1}^\infty (a_i^* + (V_i + F_i^\perp)) \subset \bigcup_{i=1}^\infty (a_i^* + U_i).$$

The implication (3) ⇒ (2) trivially follows from the weak* compactness of the unit ball $\bar{B}^* \subset X^*$.

(2) ⇒ (1) Assume that X is not Asplund. Then by Theorem 5.2.3 of [Fab], the dual Banach space X^* contains a bounded subset D such that every non-empty relatively weak* open subset U of D has norm diameter $> 8\varepsilon$ for some $\varepsilon > 0$. Without loss of generality, $0 \in D$ and $\|x^*\| < 1$ for every $x^* \in D$.

Let $2 = \{0,1\}$ and $2^{<\omega} = \bigcup_{n\in\omega} 2^n$ be the set of all finite binary sequences. For each sequence $s = (s_0, \dots, s_{n-1}) \in 2^{<\omega}$ by $|s| = n$ we denote its length and by $s|k = (s_0, \dots, s_{k-1})$ the initial segment of s of length $k \le |s|$. For $i \in \{0,1\}$ let $s\hat{\ }i = (s_0, \dots, s_{n-1}, i)$ be the concatenation of s and i.

The set $2^{<\omega}$ is a (binary) tree with respect to the partial order: $s \le t$ if $s = t|n$ for some $n \le |t|$. The empty sequence is the smallest element of $2^{<\omega}$.

Let $x^*_\emptyset = 0$ and $x_\emptyset = 0$. By induction on the tree $2^{<\omega}$, we shall construct sequences $(x^*_t)_{t\in 2^{<\omega}} \subset D$ and $(x_t)_{t\in 2^{<\omega}} \subset X$ such that for every $t \in 2^{<\omega}$ the following conditions are satisfied:

(1) $x^*_{t\hat{\ }0} = x^*_t$ and $x_{t\hat{\ }0} = x_t$;
(2) $|x^*_{t\hat{\ }1}(x_s) - x^*_t(x_s)| < 2^{-|t|}\varepsilon$ for all $s \in 2^{<\omega}$ with $|s| \le |t|$;
(3) $\|x_{t\hat{\ }1}\| = 1$;
(4) $x^*_{t\hat{\ }1}(x_{t\hat{\ }1}) - x^*_t(x_{t\hat{\ }1}) \ge 4\varepsilon$.

Suppose for some $t \in 2^{<\omega}$ the functionals x^*_s and points x_s have been constructed for all $s \in 2^{<\omega}$ with $|s| < |t|$. If $t = \tau\hat{\ }0$ for some $\tau \in 2^{<\omega}$, then we put $x^*_t = x^*_\tau$ and $x_t = x_\tau$.

Now consider the other case: $t = \tau\hat{\ }1$ for some $\tau \in 2^{<\omega}$. Consider the weak* open set

$$W = \{x^* \in D : \forall s \in 2^{<\omega}\ |s| < |t| \ \Rightarrow\ |x^*(x_s) - x^*_\tau(x_s)| < \varepsilon\}$$

in D. Since $W \ne \emptyset$, we have $\mathrm{diam} W > 8\varepsilon$. Consequently there exists a functional $x^*_t \in W$ such that $\|x^*_t - x^*_\tau\| > 4\varepsilon$. Choose a point $x_t \in X$ with $\|x_t\| = 1$ and $(x^*_t - x^*_\tau)(x_t) \ge 4\varepsilon$. This completes the inductive construction.

For every $i \in \mathbb{N}$ let

$$U_i = \{x^* \in X^* : |x^*(x_s)| < \varepsilon \text{ for every } s \in 2^{<\omega} \text{ with } |s| \le i\}.$$

Evidently that U_i are weak* open sets in X^* and their intersection $\bigcap_{i=1}^\infty U_i$ contains the open ε-ball εB^*. To see that W*CP fails for the space X^* it suffices to check that $B^* \not\subset \bigcup_{i=1}^n a_i^* + U_i$ for every $n \in \mathbb{N}$ and points $a_1^*, \dots, a_n^* \in X^*$. This will follow as soon as we find $t \in 2^n$ with $x_t^* \notin \bigcup_{j=1}^n (a_j^* + U_j)$.

Since $|(x_1^* - x_0^*)(x_1)| \ge 4\varepsilon$, there is $t_0 \in \{0,1\}$ with $|x_{t_0}^*(x_1) - a_1^*(x_1)| \ge 2\varepsilon$. By the same reason, the inequality $(x_{(t_0,1)} - x_{(t_0,0)})(x_{(t_0,1)}) \ge 4\varepsilon$ yields a number $t_1 \in \{0,1\}$ such that $|(x^*_{(t_0,t_1)} - a_2^*)(x_{(t_0,1)})| \ge 2\varepsilon$. Proceeding by finite induction and using (4), we may construct a sequence $t = (t_0, t_1, \dots, t_{n-1}) \in 2^n$ such that for every $k \le n$

(5) $|(x^*_{t|k} - a_k^*)(x_{s_k})| \ge 2\varepsilon$ for some sequence $s_k \in 2^k$.

Let us show that $x_t^* \notin \bigcup_{j=1}^n (a_j^* + U_j)$. Assuming the converse, we would find a number $p \le n$ with $x_t^* - a_p^* \in U_p$ which implies

(6) $|(x_t^* - a_p^*)(x_{s_p})| < \varepsilon$.

It follows from (2) that

$$|(x_t^* - x_{t|p}^*)(x_{s_p})| \le \sum_{k=p}^{n-1} |(x^*_{t|k+1} - x^*_{t|k})(x_{s_p})| \le \sum_{k=p}^{n-1} 2^{-k}\varepsilon < 2^{-p+1}\varepsilon \le \varepsilon$$

which together with (6) yields the inequality $|(x^*_{t|p} - a_p^*)(x_{s_p})| < 2\varepsilon$ that contradicts (5). □

3. Proof of Theorem 1

The following two lemmas yield the "∞-reflexive" part of Theorem 1.

Lemma 4. *Each net in an infinite-dimensional ∞-reflexive Banach space has an accumulation point in the weak topology.*

Proof. Assume that some ε-net N in X has no accumulating points in the weak topology. Replacing N by a suitable homothetic copy, we can assume that $\varepsilon = \frac{1}{8}$. Since N has no accumulation points in the weak topology, there is a cover $\mathcal{U}$ of X by weakly open subsets such that each set $U \in \mathcal{U}$ has at most one common point with the net N. Since X is ∞-reflexive, there is a finite subfamily $\mathcal{V} \subset \mathcal{U}$ whose union $\cup\mathcal{V}$ contains some ball B of radius 1. Then $B \cap N \subset \bigcup_{V \in \mathcal{V}} V \cap N$ is finite. One can easily check that $B \cap N$ is a $\frac{1}{4}$-net for B, which implies that X is finite-dimensional according to the classical Riesz Lemma on an almost orthogonal element, see [HHZ, Lemma 15]. □

Lemma 5. *A separable Banach space X is ∞-reflexive if each net in X has an accumulation point in the weak topology.*

Proof. Assuming that X is not ∞-reflexive, find a cover $\mathcal{U}$ of X by weakly open sets such that for every finite subfamily $\mathcal{V} \subset \mathcal{U}$ the union $\cup\mathcal{V}$ contains no ball of radius 1. Using the separability of X, we can assume that the cover $\mathcal{U}$ is countable and hence can be enumerated as $\mathcal{U} = \{U_n : n \in \omega\}$. Let $\{a_n : n \in \omega\}$ be a countable dense set in X. For every $n \in \omega$ we can find a point $x_n \in X \setminus \bigcup_{i<n} U_i$ with $\|x_n - a_n\| \le 1$. Such a point x_n exists by the choice of the cover $\mathcal{U}$. Then $\{x_n : n \in \omega\}$ is a 2-net in X having no accumulation point in the weak topology. □

The "ω-reflexive" part of Theorem 1 is established in the following more general characterization of the ω-reflexivity. However we shall need a more general meaning for an ε-net: a subset N of a Banach space $(X, \|\cdot\|)$ is called an *ε-net* for a subset $B \subset X$ if for every $x \in B$ there is $y \in N$ with $\|x - y\| < \varepsilon$.

Lemma 6. *For a separable infinite-dimensional Banach space X the following conditions are equivalent:*

1. *X is ω-reflexive;*
2. *each net for X contains a non-trivial sequence convergent in the weak topology of X;*
3. *there are a bounded set $D \subset X$ and $\varepsilon > 0$ such that each ε-net $N \subset X$ for D has an accumulation point in the weak topology of X.*

Proof. We shall prove the equivalences (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (2).

(1) $\Rightarrow$ (3). Assume that X is ω-reflexive and find $r \in \mathbb{N}$ such that X is r-reflexive. We claim that each $\frac{1}{4}$-net for the ball $(r+1)B = \{x \in X : \|x\| < r+1\}$ has an accumulating point in the weak topology of X. Assuming that it is not so, find a $\frac{1}{4}$-net $N \subset X$ for $(r+1)B$ having no accumulation point in the weak topology. This allows us to construct a cover $\mathcal{U}$ of X by weakly open sets having at most one-point intersection with the net N. The r-reflexivity of X yields a finite subfamily $\mathcal{V} \subset \mathcal{U}$ covering the 1-ball $x + B = \{y \in X : \|x - y\| < 1\}$ centered at a point $x \in X$ with $\|x\| \le r$. Then the intersection $(x + B) \cap N \subset \bigcup_{V \in \mathcal{V}} V \cap N$ is finite and thus lies in a finite-dimensional subspace $F \subset X$. The Riesz almost orthogonality Lemma 15 in [HHZ] allows us to find a point $y \in B_x$ such that $\|y - x\| = \frac{1}{2}$ but $\mathrm{dist}(y, F) > \frac{1}{4}$. Using the fact that N is a $\frac{1}{4}$-net for $(r+1)B \supset x + B$, find a point $z \in N$ with $\|z - y\| < \frac{1}{4}$. Then $z \in (x + B) \cap N \setminus F$ which is not possible because $(x + B) \cap N \subset F$.

(3) $\Rightarrow$ (1) Assume that for some bounded set $D \subset X$ and some $\varepsilon > 0$ each ε-net $N \subset X$ for D has an accumulation point in the weak topology. Replacing D by its homothetic copy, we can assume that $\varepsilon = 1$. Let $r = \sup\{\|x\| : x \in D\}$. We claim that the space X is r-reflexive. Otherwise we can find an open cover $\mathcal{U}$ of X by weakly open subsets such that no finite subfamily of $\mathcal{U}$ covers the open unit ball centered at a point $x \in X$ with $\|x\| \le r$. Use the separability of X to find a countable subcover $\{U_n : n \in \omega\} \subset \mathcal{U}$ of X and let $\{x_n : n \in \omega\}$ be a countable dense set in the r-ball $rB = \{x \in X : \|x\| < r\}$. For every $n \in \omega$ select a point $y_n \in X \setminus \bigcup_{k<n} U_k$ with $\|x_n - y_n\| < 1$ (such a point y_n exists by the choice of the cover $\mathcal{U}$). Then $N = \{y_n : n \in \omega\}$ is a 2-net for D without accumulation points in the weak topology of X. This a contradiction.

(3) $\Rightarrow$ (2) Assume that for some bounded set $D \subset X$ and some $\varepsilon > 0$ each ε-net $N \subset X$ for D has an accumulation point in the weak topology. Then given any ε-net N in X we can find a bounded subset $A \subset N$ having an accumulation point $a \in X$ in the weak topology of X. The implication (3) $\Rightarrow$ (1) ensures that X is ω-reflexive and hence ∞-reflexive. By Corollary 1, the bounded subset $A \cup \{a\}$, being Fréchet-Urysohn, contains a sequence $\{a_n\}_{n=1}^{\infty} \subset A \setminus \{a\}$ that converges to a.

(2) $\Rightarrow$ (3). Assume conversely that for each bounded set D and every $\varepsilon > 0$ there is an ε-net $N \subset X$ for D having no accumulation point in the weak topology of X. In particular, for every $r \in \omega$, there is an 1-net N_r for the r-ball $B_r = \{x \in X : \|x\| \le r\}$ having no accumulating point in the weak topology of X. Now consider the union $N = \bigcup_{r \in \omega} N_r \setminus B_{r-2}$ and note that it is an 1-net for X. Indeed, given any $x \in X$ find $r \in \omega$ with $r - 1 < \|x\| \le r$ and $y \in N_r$ with $\|x - y\| < 1$. Then $\|y\| > \|x\| - 1 > r - 2$ and hence $y \in N_r \setminus B_{r-2} \subset N$. Assuming that N contains a non-trivial weakly convergent sequence $S \subset N$, find $R \in \omega$ with $S \subset B_{R-2}$ and

observe that $S \subset N \cap B_{R-2} \subset \bigcup_{r \le R} N_r$. Then for some $r \le R$ the intersection $S \cap N_r$ is infinite and hence $N_r \supset \bar{S} \cap N_r$ has an accumulation point in the weak topology, which contradicts the choice of N_r. $\square$

4. Proof of Proposition 1

Let Z be a subspace of a Banach space X and let $\pi : X \to X/Z$ denote the quotient operator.

1. Assuming that X is r-reflexive for some $r \in [0, +\infty]$, we shall prove that the quotient space X/Z is r-reflexive too. Given a cover $\mathcal{U}$ of X/Z by weakly open sets, consider the cover $\pi^{-1}(\mathcal{U}) = \{\pi^{-1}(U) : U \in \mathcal{U}\}$ of X. By the r-reflexivity of X there is a finite subfamily $\mathcal{V} \subset \mathcal{U}$ whose preimage $\pi^{-1}(\mathcal{V})$ covers some ball $x + B_X = \{y \in X : \|x - y\| < 1\}$ centered at a point $x \in X$ with $\|x\| \le r$. Then the family $\mathcal{V}$ covers the image $\pi(x + B_X)$ which coincides with the ball $\pi(x) + B_{X/Z} = \{z \in X/Z : \|z - \pi(x)\| < 1\}$ according to the definition of the quotient norm on X/Z. Taking into account that $\|\pi(x)\| \le \|x\| \le r$, we conclude that the space X/Z is r-reflexive.

2. Assume that X is ω-reflexive and a Banach space Y is isomorphic to X. Let $T : X \to Y$ be an isomorphism between X and Y and $M = \max\{\|T\|, \|T^{-1}\|\}$. Let B_X, B_Y denote the open unit balls centered at the origins of the spaces X, Y, respectively. It follows that $\frac{1}{M}B_Y \subset T(B_X) \subset M \cdot B_Y$.

The space X, being ω-reflexive, is r-reflexive for some r. We claim that Y is M^2r-reflexive. Indeed, given a cover $\mathcal{U}$ of Y by weakly open sets, consider the covers $\mathcal{W} = T^{-1}(\mathcal{U}) = \{T^{-1}(U) : U \in \mathcal{U}\}$ and $\frac{1}{M}\mathcal{W} = \{\frac{1}{M} \cdot W : W \in \mathcal{W}\}$ of X. The r-reflexivity of X implies the existence of a finite subfamily $\mathcal{V} \subset \mathcal{U}$ such that $\bigcup_{V \in \mathcal{V}} \frac{1}{M}T^{-1}(V)$ covers the unit ball $x + B_X$ centered at some point $x \in X$ with $\|x\| \le r$. Letting $y = M \cdot T(x)$, observe that $\|y\| = M \cdot T(x) \le M^2 r$ and $y + B_Y \subset T(Mx + MB_X) = M \cdot T(x + B_X) \subset M \cdot T\big(\bigcup_{V \in \mathcal{V}} \frac{1}{M}T^{-1}(V)\big) = \cup\mathcal{V}$, witnessing the M^2r-reflexivity of the space Y.

By analogy we can prove that the ∞-reflexivity of X implies the ∞-reflexivity of Y. Finally the 0-reflexivity coincides with the usual reflexivity and also is preserved by isomorphisms.

3. Assume that the space Z is reflexive and X/Z is r-reflexive for some $r \in [0, \infty)$.

First we show that for each bounded weakly closed subset $F \subset X$ the image $\pi(F)$ is weakly closed in X/Z. This follows from the fact that the $\bar{F}$ closure of F in the weak*-topology of X^{**} is compact and so is its image $\pi^{**}(\bar{F}) \subset (X/Z)^{**}$. Observing that $\pi(F) = \pi^{**}(\bar{F}) \cap (X/Z)$, we see that the set $\pi(F)$ is weakly closed in X/Z. This fact implies that the image $\pi(\bar{B}_X)$ of the closed unit ball centered at the origin of X coincides with the closed unit ball $\bar{B}_{X/Z}$ centered at the origin of X/Z.

Now we are ready to show that the space X is r-reflexive. Take any weakly open cover $\mathcal{U}$ of X. For every point $y \in (r+1)\bar{B}_{X/Z}$ the set $(r+1)\bar{B}_X \cap \pi^{-1}(y)$ is weakly compact and hence can be covered by a finite subfamily $\mathcal{U}_y \subset \mathcal{U}$. The set $F_y = (r+1)\bar{B}_X \setminus \cup\mathcal{U}_y$ is bounded and weakly closed in X. Consequently, its projection $\pi(F_y)$ is weakly closed in X/Z while the complement $V_y = (r+1)\bar{B}_{X/Z} \setminus \pi(F_y)$ is a weakly open neighborhood of y in $(r+1)\bar{B}_{X/Z}$. Since the space X/Z is r-reflexive the cover $\{V_y : y \in (r+1)B_{X/Z}\}$ of the closed ball $(r+1)B_{X/Z}$ contains a finite subcollection $\{V_{y_1}, \dots, V_{y_n}\}$ whose union contains the open 1-ball $y+B_{X/Z}$ centered at some point $y \in X/Z$ with $\|y\| \le r$. Take any point $x \in X$ with $\|x\| = \|y\|$ and $\pi(x) = y$ and observe that $\mathcal{W} = \bigcup_{i=1}^n \mathcal{U}_{y_i}$ is a finite cover of the open 1-ball $x + B_X$ centered at x. This witnesses that the space X is r-reflexive.

5. Proof of Theorem 2

In this section we prove that the James space J fails to be ω-reflexive. We recall that the James space J is the Banach space consisting of all real sequences $(x_n)_{n\in\omega}$ that tend to zero and have norm

$$\|(x_i)\| = \sup_{n_0<\dots<n_k} \sqrt{\sum_{i=1}^k (x_{n_i} - x_{n_{i-1}})^2} < \infty.$$

Let J_0 denote the set of all eventually zero sequences.

First we prove

Lemma 7. *For every $M > 0$ there is $\varepsilon > 0$ such that for every $x \in J_0$ with $\|x\| \le M$ there is $y = (y_n) \in J$ such that $\|x - y\| < 1$ and $|y_n - 1| \ge \varepsilon$ for all $n \in \omega$.*

Proof. Given $M > 0$ find an integer $m \ge 2$ with $\frac{20M}{\sqrt{2m+1}} < \frac{1}{2}$ and $4M^2(2m+1) > 1$, and let $\varepsilon = \frac{1}{4m+2}$.

Take any point $x = (x_n) \in J_0$ with $\|x\| \le M$. By induction, construct an increasing finite number sequence $(k_i)_{i=0}^r$ such that for $k_{r+1} = \infty$ we get

- $k_0 = 0$;
- $|x_p - x_q| \le \varepsilon$ for all numbers $p, q \in [k_i, k_{i+1})$ and all $0 \le i \le r$;
- for every $0 < i \le r$ there is a number $p_i \in [k_{i-1}, k_i)$ with $|x_{k_i} - x_{p_i}| > \varepsilon$.

It follows that

$$M \ge \|x\| \ge \sqrt{\sum_{0<i\le r} |x_{k_i} - x_{p_i}|^2} > \sqrt{r\varepsilon^2}$$

and hence $r < \frac{M^2}{\varepsilon^2}$. Let $A = 2\varepsilon \cdot \mathbb{Z}$ be the arithmetic progression with step 2ε and let $f : \mathbb{R} \to A$ be a function assigning to each real number $t \in \mathbb{R}$ a number $f(t) \in A$ with $|t - f(t)| \le \varepsilon$. Given a number $a \in A$, let

$$r_a = \big|\big\{i \le r : f(x_{k_i}) \in \{a - 2\varepsilon, a, a + 2\varepsilon\}\big\}\big|.$$

Since $|A \cap [\frac{1}{2}, \frac{3}{2}]| = \frac{1}{2\varepsilon} = 2m + 1$, there is a point $a \in A \cap [\frac{1}{2}, \frac{3}{2}]$ with $r_a \le \frac{3r}{2m+1} \le \frac{3M^2}{(2m+1)\varepsilon^2} = 12M^2(2m + 1)$. Taking into account that $1 < 4M^2(2m + 1)$, we get $r_a + 1 \le 16M^2(2m + 1)$.

Consider the sequence $z = (z_n)_{n\in\omega}$ such that $z_n = 0$ for $n \in [k_r, \infty)$ and for every $i < r$ and $n \in [k_i, k_{i+1})$ we have

$$z_n = \begin{cases} 0 & \text{if } f(x_{k_i}) \notin \{a - 2\varepsilon, a, a + 2\varepsilon\}; \\ -5\varepsilon & \text{if } f(x_{k_i}) = a - 2\varepsilon; \\ 5\varepsilon & \text{if } f(x_{k_i}) \in \{a, a + 2\varepsilon\}. \end{cases}$$

The definition of the norm on the James space J implies that

$$\|z\| \le \sqrt{(r_a + 1)(10\varepsilon)^2} \le \sqrt{16M^2(2m + 1)100\varepsilon^2} = \sqrt{\frac{1600M^2}{4(2m + 1)}} = \frac{20M}{\sqrt{2m + 1}} < \frac{1}{2}.$$

Let $e = (e_n)_{n\in\omega}$ be the element of J such that $e_n = 1$ for all $i < k_r$ and $e_n = 0$ for all $n \geq k_r$. It is clear that $\|e\| = 1$.

Finally, consider the point $y = x + z + (1 - a) \cdot e$. Observe that

$$\|y - x\| = \|z + (1 - a) \cdot e\| \leq \|z\| + |1 - a| \cdot \|e\| < \frac{1}{2} + \frac{1}{2} = 1.$$

Now, we show that $|y_n - 1| \geq \varepsilon$ for all $n \in \omega$. Indeed, if $n \geq k_r$, then $y_n = x_n$ and $|y_n - 1| \geq 1 - |x_n| \geq 1 - \varepsilon \geq \varepsilon$.

Next, assume that $n \in [k_i, k_{i+1})$ for some $i < r$. If $f(x_{k_i}) \notin \{a - 2\varepsilon, a, a + 2\varepsilon\}$, then

$$\begin{aligned}|y_n - 1| =& |x_n + z_n + (1 - a) - 1| = |x_n - a| = |x_n - f(x_{k_i}) + f(x_{k_i}) - a| \geq \\ \geq& |f(x_{k_i}) - a| - |x_{k_i} - f(x_{k_i})| - |x_n - x_{k_i}| \geq 4\varepsilon - \varepsilon - \varepsilon \geq \varepsilon.\end{aligned}$$

If $f(x_{k_i}) = a - 2\varepsilon$, then

$$\begin{aligned}|y_n - 1| =& |x_n + z_n + (1 - a) - 1| = |x_n - x_{k_i} + x_{k_i} + f(x_{k_i}) - f(x_{k_i}) + z_n - a| \geq \\ \geq& |z_n + f(x_{k_i}) - a| - |x_n - x_{k_i}| - |f(x_{k_i}) - x_{k_i}| \geq 3\varepsilon - \varepsilon - \varepsilon = \varepsilon.\end{aligned}$$

The case $f(x_{k_i}) \in \{a, a + 2\varepsilon\}$ can be considered by analogy. □

The following lemma combined with Lemma 6 implies that the James space is not ω-reflexive.

Lemma 8. *For every $R \in \mathbb{N}$ the ball $B_R = \{x \in J : \|x\| \leq R\}$ possesses a 2-net in J which is closed and discrete in the weak topology of J.*

Proof. Using Lemma 7, find $\varepsilon > 0$ such that the set

$$A_\varepsilon = \{(y_n) \in J : |y_n - 1| \geq \varepsilon \text{ for all } n \in \omega\}$$

intersects each open ball of unit radius centered at a point $x \in J_0$ with $\|x\| \leq R$. Fix a countable dense set $D = \{x_n : n \in \omega\}$ in $A_\varepsilon \cap B_{R+1}$. It follows that D is a 1-net for the ball B_R. For every $n \in \omega$ consider the sequence $\vec{e}_n = (1, \dots, 1, 0, \dots)$ with first n units. Since $\|e_n\| = 1$ for all $n \in \omega$ the set $D' = \{x_n - e_n : n \in \omega\}$ is a 2-net for the ball B_R in J. We claim that D' is closed and discrete in the weak topology of J. Assuming the converse and using the metrizability of the weak topology of J on bounded subsets, find an increasing number sequence (n_k) such that the sequence $(x_{n_k} - e_{n_k})_{k\in\omega}$ weakly converges to some point $z \in J$. The weak convergence implies the coordinate convergence. Now it is convenient to think of the elements of J as functions defined on ω. It follows that for every $i \in \omega$ $|z(i)| = \lim_{k\to\infty} |x_{n_k}(i) - e_{n_k}(i)| = \lim_{k\to\infty} |x_{n_k}(i) - 1| \geq \varepsilon$, which is not possible because $\lim_{i\to\infty} z(i) = 0$. □

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