INTERPLAY BETWEEN THE ALGEBRAIC STRUCTURE OF A GROUP AND ARITHMETIC PROPERTIES OF ITS SPECTRUM

TARAS BANAKH

Abstract. We investigate the inteplay between the algebraic structure of a group G and arithmetic properties of its spectrum $\sigma(G)$ which consists of the eigenvalues of all the inner automorphisms of G. A complex number λ is called an eigenvalue of a group automorphism $A : G \to G$ if $\varphi \circ A|H = \lambda \cdot \varphi$ for some non-trivial homomorphism $\varphi : H \to \mathbb{C}$ defined on an A-invariant subgroup $H \subset G$.

It is shown that many properties of a group G (such as the presence of a finitely generated subgroup of infinite rank, nilpotence, periodicity, polycyclicity, etc) are coded in its spectrum. In the paper the spectra are applied to investigate the so-called reversive properties of groups. The paper ends with a list of related open problems.

Introduction

In this paper we study the interplay between the algebraic structure of a group G and arithmetric properties of its spectrum $\sigma(G)$. By definition, the spectrum $\sigma(G)$ of a group G is the subset of the complex plane $\mathbb{C}$, consisting of the eigenvalues of the inner automorphisms of G. A complex number $\lambda \in \mathbb{C}$ is defined to be an *eigenvalue* of an automorphism $A : G \to G$ of a group G is there is a non-trivial homomorphism $\varphi : H \to \mathbb{C}$ from an A-invariant subgroup H of G such that $\varphi \circ A|H = \lambda \cdot \varphi$. The homomorphism φ can be thought as an eigenvector corresponding to the eigenvalue λ (for a more deep analysis of the nature of eigenvalues of group automorphisms see Section 6). The set $\sigma(A) \subset \mathbb{C}$ of all eigenvalues of the automorphism $A : G \to G$ is called the *spectrum* of A. As we said, the spectrum of a group G is the union of the spectra of all inner automorphisms $i^g : G \to G$, $g \in G$, of G (here $i^g : x \mapsto gxg^{-1}$).

It should be mentioned that in case of an automorphism $A : G \to G$ of an abelian torsion-free group G of finite rank our notion of an eigenvalue agrees with the classical one: $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if λ is an eigenvalue of the matrix M of A in any basis of G (that is, λ is a root of the characteristic polynomial $\det(zE - M)$ of the matrix M).

This fact allows us to reduce the problem of determining the spectrum of a group automorphism to calculating the roots of the characteristic polynomials of certain induced automorphisms of some torsion-free abelian groups. Such a reduction is described in the second section of the paper. It turns out that the algebraic structure of a group G imposes some restrictions onto the spectra of automorphisms of G. In particular, if G has finite rank r (in the sense that every finitely generated subgroup of G is generated by $\leq r$ elements), then the spectrum $\sigma(A)$ of any automorphism A of G lies in the set $\mathbb{A}^*(r)$ of non-zero algebraic numbers of degree $\leq r$. If G is a solvable group with finite Hirsch rank

1991 *Mathematics Subject Classification.* 20E25, 20E34, 20E36, 20F14, 20F16, 20F18, 20F19, 20F45, 20F50, 20M05.

Key words and phrases. eigenvalue of a group automorphism, spectrum of a group, free semigroup, virtually nilpotent group, collapsing group, solvable group, Hirsch rank.

$h(G)$, then the cardinality of the spectrum $\sigma(A)$ does not exceed $h(G)$. If G is a polycyclic group, then $\sigma(A)$ lies in the set $\mathbb{A}_1^*$ of units of the ring of integer algebraic numbers.

In the third section these results on spectra of group automorphisms are applied to studying the interplay between the structure of a group G and the properties of its spectrum $\sigma(G)$. It is shown that the spectrum $\sigma(G)$ of a group G of finite rank r lies in the set $\mathbb{A}^*(r)$ while the spectrum of a polycyclic group lies in the set $\mathbb{A}_1^*$. Moreover, the inclusion $\sigma(G) \subset \mathbb{A}_1^*$ characterizes polycyclic groups in the class of solvable groups of finite rank. The condition $\sigma(G) \subset \sqrt[\infty]{1}$ where $\sqrt[\infty]{1} = \{z \in \mathbb{C} : z^n = 1 \text{ for some } n \in \mathbb{N}\}$ characterizes virtually nilpotent groups in the class of polycyclic groups while the inclusion $\sigma(G) \subset \{1\}$ characterizes so-called periodically Engelian groups in the class of locally solvable groups. Finally, the condition $\sigma(G) = \emptyset$ characterizes periodic groups G. On the other hand, a group G has the maximal possible spectrum $\sigma(G) = \mathbb{C}^*$ if and only if $\sigma(G)$ contains a transcendental number (in this case the group G must have infinite rank). In the third section we also study the algebraic structure of the spectrum of a solvable group G of finite rank. Using the Kolchinov-Malcev Theorem on triangulable matrix groups, we show that each solvable group G of finite rank contains a subgroup H of finite index whose spectrum $\sigma(H)$ is finitely splittable in the sense that there is a family Φ of homomorphisms $\varphi : H \to \mathbb{C}$, whose size $|\Phi|$ does not exceed the Hirsch rank of H, such that $\sigma(H) = \bigcup_{\varphi\in\Phi} \varphi(H)$. In this case, $\sigma(H) \subset \sigma(G) \subset \sqrt[n]{\sigma(H)}$ for some $n \in \mathbb{N}$.

In light of the last result it should be mentioned that the spectrum behaves nicely with respect to many operations over groups. In particular, $\sigma(G \times H) = \sigma(G) \cup \sigma(H)$ for any groups G, H. If H is a (normal) subgroup of G, then $\sigma(H) \subset \sigma(G)$ (and $\sigma(G/H) \subset \sigma(G)$). If H has finite index $[G : H]$ in G, then $\sigma(H) \subset \sigma(G) \subset \sqrt[n]{\sigma(H)}$ for some $n \in \mathbb{N}$ depending only on $[G : H]$. By its nature, the spectrum belongs to local properties of a groups. Namely, $\sigma(G) = \cup\{\sigma(H) : H \text{ is a 2-generated subgroup of } G\}$.

In the forth section we show that the spectrum $\sigma(G)$ of a group G is completely determined by the relation between G and the so-called test groups $\mathrm{Aff}_\lambda(\mathbb{C})$, $\lambda \in \mathbb{C}^*$. By definition, $\mathrm{Aff}_\lambda(\mathbb{C}) = \langle z+1, \lambda z\rangle$ is the abelian-by-cyclic subgroup of the "$az + b$" group $\mathrm{Aff}(\mathbb{C})$ of affine transformations of the complex plane $\mathbb{C}$, generated by two tarnsformations: $w = z + 1$ and $w = \lambda z$. It should be mentioned that the groups $\mathrm{Aff}_\lambda(\mathbb{C})$, $\lambda \in \mathbb{C}^*$, have been appeared in Combinatorial Group Theory [MKS, §3.4] (see also [FM] for their asymptotic properties). The algebraic structure of a group $\mathrm{Aff}_\lambda(\mathbb{C})$ depends essentially on the arithmetic properties of the complex number λ. In particular, for a transcendental λ the group $\mathrm{Aff}_\lambda(\mathbb{C})$ has infinite rank and satisfies $\sigma(\mathrm{Aff}_\lambda(\mathbb{C})) = \mathbb{C}^*$. On the other hand, for an algebraic number λ the spectrum of $\mathrm{Aff}_\lambda(\mathbb{C})$ consists of the numbers $\{\lambda_1^n, \dots, \lambda_d^n : n \in \mathbb{Z}\}$ where $\lambda_1, \dots, \lambda_d$ are the roots of the minimal polynomial of λ. In fact, the spectrum completely determines the algebraic structure of the groups $\mathrm{Aff}_\lambda(\mathbb{C})$: two groups $\mathrm{Aff}_\lambda(\mathbb{C})$ and $\mathrm{Aff}_\mu(\mathbb{C})$ are isomorphic if and only if $\sigma(\mathrm{Aff}_\lambda(\mathbb{C})) = \sigma(\mathrm{Aff}_\mu(\mathbb{C}))$. The groups $\mathrm{Aff}_\lambda(\mathbb{C})$ determine the spectrum of an arbitrary group G in the sense that $\sigma(G) = \{\lambda \in \mathbb{C}^* : \mathrm{Aff}_\lambda(\mathbb{C}) \text{ is isomorphic to a quotient group of some subgroup of } G\}$.

This fact allows us to make an upper estimation for the spectrum of a group pessessing certain group property $\mathcal{P}$, hereditary in the sense that each subgroup and each quotient group of a group possessing the property $\mathcal{P}$ have that property. More precisely, the spectrum $\sigma(G)$ of a group G possessing a hereditary property $\mathcal{P}$ lies in the set $\{\lambda \in \mathbb{C}^* : \mathrm{Aff}_\lambda(\mathbb{C}) \text{ has the property } \mathcal{P}\}$.

One of such hereditary properties, namely, the (n, m)-reversivity is studied in the fifth section. This property is defined as follows. Given a subset A of a group G define its n-th oscillators $(\pm A)^n$ and $(\mp A)^n$ by induction: $(\pm A)^0 = (\mp A)^0 = \{e\}$ while $(\pm A)^{n+1} =$

$A \cdot (\mp A)^n$ and $(\mp A)^{n+1} = A^{-1} \cdot (\pm A)^n$ for $n \ge 0$. Given $m \in \mathbb{N}$ we put $A^m = \{a_1 \cdots a_m : a_1, \dots, a_m \in A\} \subset G$ and $A^\infty = \bigcup_{m\in\mathbb{N}} A^m$. A group G is defined to be (n,m)-*reversive* for some $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{\infty\}$ if $(\mp A)^n \subset (\pm A^m)^n$ for any subset $A \subset G$ containing the unit of the group G. The notion of an (n,m)-reversive group, which came from topological algebra (see [BR]), generalizes the more familiar concepts of a collapsing group and a group containing no free semigroup with two generators. Namely, a group G is $(2,\infty)$-reversive (resp. $(2,m)$-reversive for some $m \in \mathbb{N}$) if and only if G contains no free semigroup with two generators (resp. is collapsing in the sense of [SS], [Sh]). The main result of the fifth section asserts that $\sigma(G) \subset \{z \in \mathbb{C} : |z| = 1\}$ for any (n,∞)-reversive group G. Moreover, if G is (n,m)-reversive for some $n,m \in \mathbb{N}$, then $\sigma(G) \subset \sqrt[k]{1}$ for some $k \in \mathbb{N}$ depending only on n and m. This result implies that for a polycyclic group G the following conditions are equivalent: (i) G is virtually nilpotent; (ii) G is collapsing; (iii) G contains no free semigroup with two generators; (iv) G has polynomial growth, (v) G is (n,∞)-reversive for some $n \in \mathbb{N}$; (vi) G is $(2,m)$-reversive for some $m \in \mathbb{N}$; (vii) $\sigma(G)$ is a bounded subset of $\mathbb{C}$; (viii) $\sigma(G) \subset \sqrt[k]{1}$ for some k. It should be mentioned that some of the above equivalence are well-known. In particular, (i)⇔(iv) follows from the famous Gromov Theorem [Gr] on the equivalence of the virtual nilpotence and the polynomial growth in the class of finitely generated groups, while (i)⇔(iii) was proven by J. Rosenblatt in [Ro].

In the final sixth section we make some comments on the nature of eigenvalues of group automorphisms and pose several open problems related to group spectra.

1. Notation and terminology

As usual, by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{N}$ we denote sets of complex, real, rational, integer, and positive integer numbers, respectively; $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ will stand for the unit circle in the complex plane, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ for the multiplicative group of non-zero complex numbers, and $\omega = \{0\} \cup \mathbb{N}$ for the first infinite ordinal. By $\mathrm{Card}(A)$ we denote the cardinality of a set A; "iff" is an abbreviation for "if and only if".

1.1. **Algebraic numbers.** A complex number $\lambda \in \mathbb{C}$ is *algebraic* if λ is a root of a polynomial $P(z)$ with rational coefficients. The smallest degree of such a polynomial is called *the degree of an algebraic number* λ and is denoted by $\deg(\lambda)$. For any algebraic number λ there is a unique polynomial with rational coefficients $P(z) = \sum_{i=0}^d a_i z^i$ of degree $d = \deg(\lambda)$ such that $a_d = 1$ and $P(\lambda) = 0$. This polynomial is called the *minimal polynomial* of λ. Two algebraic numbers are called *algebraically conjugated* if their minimal polynomials coincide. It is well known (and can be easily shown) that a complex number λ is algebraically conjugated to an algebraic number μ if and only if λ is a root of the minimal polynomial of μ.

Algebraic numbers form an algebraically closed subfield of $\mathbb{C}$ denoted by $\mathbb{A}$. Complex numbers $\lambda \in \mathbb{C} \setminus \mathbb{A}$ are called *transcendental*. By $\mathbb{A}^* = \mathbb{A} \setminus \{0\}$ we denote the multiplicative group of non-zero algebraic numbers. For a positive integer r let $\mathbb{A}(r) \subset \mathbb{A}$ be the set of algebraic numbers of degree $\le r$ and $\mathbb{A}^*(r) = \mathbb{A}^* \cap \mathbb{A}(r)$. Let also $\mathbb{A}(\infty) = \mathbb{A}$ and $\mathbb{A}^*(\infty) = \mathbb{A}^*$.

An algebraic number $\lambda \in \mathbb{A}$ is called an *integer algebraic number* if its minimal polynomial has integer coefficients. The set $\mathbb{A}_\mathbb{Z}$ of integer algebraic numbers form a subring in the field $\mathbb{A}$ of algebraic numbers. The multiplicative group of units (that is invertible elements) of this ring is denoted by $\mathbb{A}_1^*$. An algebraic number λ belongs to $\mathbb{A}_1^*$ if and only if the minimal polynomial $P(z) = \sum_{i=0}^d a_i z^i$ of λ has integer coefficients and $|a_0| = 1$. For a positive integer r let $\mathbb{A}^*(r) = \mathbb{A} \cap \mathbb{A}(r)$ and $\mathbb{A}_\mathbb{Z}^* = \mathbb{A}_\mathbb{Z} \cap \mathbb{C}^*$.

For $n \in \mathbb{N}$ let $\sqrt[n]{1} = \{z \in \mathbb{C} : z^n = 1\}$ and $\sqrt[\infty]{1} = \bigcup_{n\in\mathbb{N}} \sqrt[n]{1}$ be the multiplicative group of roots from the unit. According to a well-known Kronecher Theorem (see [**?**, 3.2], Theorem 2 of [BoS, §II.3.4] or Remark in [Ro, p.49]) an algebraic integer is a root of 1 if and only if all of its algebraically conjugates have absolute value 1.

1.2. **Group Theory.** Given a group G by e we denote the neutral element of G. For a subset A of a group G let $\langle A\rangle$ denote the subgroup of G generated by A, $\sqrt[n]{A} = \{g \in A : g^n \in A\}$ and $\sqrt[\infty]{A} = \{g \in G : g^n \in A \text{ for some } n \in \mathbb{N}\}$. The subset $\sqrt[\infty]{e}$ is called *the periodic part* of the group G while elements of $\sqrt[\infty]{e}$ are called *periodic elements* of G. A group is *periodic* if it coincides with its periodic part. A group G is called *torsion-free* if $\sqrt[\infty]{e} = \{e\}$. In general, the periodic part needs not be a subgroup of a group. However, for any abelian (more generally, nilpotent) group G the periodic part of G is a subgroup of G, see [KM, 16.2.7].

A group G is defined to have *finite rank* r if every finitely generated subgroup of G is generated by $\le r$ elements. The smallest number r with this property is called *the rank* of G and is denoted by $r(G)$. For groups G which are not of finite rank we put $r(G) = \infty$. Under a *p-rank* $r_p(G)$ of an *abelian* group G where p is a prime number we understand the rank $r(G_p)$ of the maximal p-subgroup $G_p = \{x \in G : x^{p^k} = e \text{ for some } k \in \mathbb{N}\}$ of G. The *free rank* $r_0(G)$ of an *abelian* group G is equal to the rank $r(G/\sqrt[\infty]{e})$ of the quotient group of G by its periodic part.

We write $H \le G$ (resp. $H \trianglelefteq G$) if H is a (normal) subgroup of a group G. A subgroup $H \le G$ is *characteristic* in G if $f(H) = H$ for each automorphism f of G.

A decreasing series $G_0 \ge G_1 \ge \dots$ of subgroups of a group G is called *subnormal* (resp. *normal*) if each group G_{i+1} is normal in G_i (resp. normal in G).

For two subsets A, B of a group G let $[A, B] = \langle [a, b] : a \in A, b \in B\rangle$ be the commutator group of the sets A, B, where $[a, b] = aba^{-1}b^{-1}$ is the *commutant* of elements $a, b \in G$.

Let $G^{(0)} = G$ and $G^{(n+1)} = [G^{(n)}, G^{(n)}]$ for $n \ge 0$. It is well known that for every $n \ge 0$ the subgroup $G^{(n)}$, called the *n-th commutator group* of G, is characteristic in G. A group G is defined to be *solvable* if $G^{(s)} = \{e\}$ for some $s \in \mathbb{N}$. The smallest number s with $G^{(s)} = \{e\}$ is called the *solvability degree* of G. The descreasing normal series $G = G^{(0)} \trianglerighteq G^{(1)} \trianglerighteq \dots \trianglerighteq G^{(s)} = \{e\}$ is called *the commutator series* of G while the quotient groups $G^{(k)}/G^{(k+1)}$, $k < s$, are referred to as *factors* of this series. Observe that these factors are abelian groups. For a solvable group G the number $h(G) = \sum_{k\in\mathbb{N}} r_0(G^{(k-1)}/G^{(k)})$ is called the *Hirsch rank* of G. Under a *polycyclic* group we understand a solvable group whose any subgroup is finitely generated; a *metabelian* group is a solvable group of solvability degree ≤ 2. A group G is called *locally solvable* if any finitely generated subgroup of G is solvable.

Following A.I. Malcev [Mal] (see also [Ku, §D.24.1]) under a *solvable A_1-group* we understand a solvable group G whose commutator series has factors of finite free rank. Observe that each solvable group of finite rank is a solvable A_1-group.

Next, let $\gamma_1 G = G$ and $\gamma_{n+1}G = [\gamma_n G, G]$ for $n \ge 0$. It is well-known that each subgroup $\gamma_n G$, $n \ge 0$, is characteristic in G. A group G is called *nilpotent* if $\gamma_n G = \{e\}$ for some $n \in \mathbb{N}$. A group G is called *virtually nilpotent* if G contains a nilpotent subgroup H of finite index. The subgroup H always can be chosen to be normal, so virtually nilpotent groups are referred to as *nilpotent-by-finite* groups. A group G is called *abelian-by-cyclic* if G contains a normal abelian subgroup H with cyclic quotient G/H.

According to the Gromov Theorem [Gr], a finitely generated group G is virtually nilpotent if and only if it has polynomial growth. A group G is defined to have *polynomial growth* if for every finite subset $A \subset G$ there is a real polynomial $p(x)$ such that

$\mathrm{Card}(A^n) \le p(n)$ for every $n \in \mathbb{N}$; G is *exponentially bounded* if $\lim_{n\to\infty} \sqrt[n]{\mathrm{Card}(A^n)} = 1$ for any finite subset $A \subset G$. Here $A^n = \{a_1 \cdots a_n : a_1, \dots, a_n \in A\} \subset G$ for $A \subset G$. A group is said to have *exponential growth* if it is not exponentially bounded. Each group containing a free semigroup with two generators has exponential growth. For solvable groups the converse is also true, see [Ro].

A group G is the *semidirect product* $H \rtimes K$ of a normal subgroup $H \subset G$ and a subgroup $K \subset G$ if $H \cap K = \{e\}$ and $H \cdot K = G$. If, in addition, $ab = ba$ for any $a \in H$, $b \in K$, then G is called the *direct product* of H and K and is denoted by $G = H \times K$. For a group G and a set A let $G_0^A = \{(g_a)_{a\in A} \in G^A : \text{the set } \{a \in A : g_a \neq e\} \text{ is finite}\}$ be the direct product of A copies of the group G. Under the *wreath product* $G \wr H$ of two groups G, H we understand the product $G_0^H \times H$ endowed with the group operation $((g_a)_{a\in H}, h) * ((g'_a)_{a\in H}, h') = ((g_a \cdot g'_{ha})_{a\in H}, hh')$, see [Sk, p.83].

Finally we remind some information concerning abelian groups (saying about such groups we shall use the additive form for denoting the group operation).

A subset X of an abelian group G is *linearly independent* if for any pairwise distinct elements $x_1, \dots, x_k \in X$ and any integers $n_1, \dots, n_k$ the equality $\sum_{i=1}^k n_i x_i = 0$ holds if and only if $n_1 = \dots = n_k = 0$. Under a *basis* of an abelian torsion-free group G we understand any maximal linearly independent subset of G. Let us observe that the size of any basis of G is equal to the (free) rank of G.

If $X = \{x_1, \dots, x_k\}$ is a basis of an abelian torsion-free group of finite free rank k, then each element $x \in G$ can be uniquely written as $x = \sum_{i=1}^k r_i x_i$ for some rational numbers $r_1, \dots, r_k$ (called the *coordinates* of x in the basis $x_1, \dots, x_k$). Here the equality $x = \sum_{i=1}^k r_i x_i$ is understood in the sense that $mx = \sum_{i=1}^k (mr_i)x_i$ where m is an integer such that mr_i is integer for all $i \le k$. If the basis $\{x_1, \dots, x_k\}$ generates the group G, then the coordinates $r_1, \dots, r_k$ of any element $x \in G$ are integer numbers. It is well-known that each finitely-generated abelian torsion-free group admits a basis generating the group.

Let $A : G \to G$ be an automorphism of an abelian torsion-free group G. Under the *matrix* of the automorphism A in a basis $x_1, \dots, x_k$ of G we understand the $k \times k$-matrix $M = (a_{ij})_{i,j=1}^k$ with rational coefficients such that $A(x_j) = \sum_{i=1}^k a_{ij} x_i$ for every $j \le k$. If the elements $x_1, \dots, x_k$ generate the group G, then the matrix M has integer coefficients as well as its inverse matrix M^{-1} (which is the matrix of the inverse automorphism A^{-1} of G). In this case $\det(M) = \pm 1$ and the eigenvalues of the matrix M belong to the set $\mathbb{A}_1^*$ of units of the ring of integer algebraic numbers.

As usual, under an eigenvalue of a matrix M we understand a complex root λ of the characteristic polynomial $\det(zE - M)$ of M where E stands for the identity matrix.

It is well-known that the characteristic polynomial $\det(zE - M)$ of the matrix M of an automorphism A does not depend on the choice of the basis for G, so we can say about *the characteristic polynomial* $P_A(z)$ of the automorphism A (which is equal to the characteristic polynomial $\det(zE - M)$ of the matrix M of A in any basis of G). Observe that the characteristic polynomial $P_A(z)$ of A has rational coefficients and has degree equal to the free rank $r_0(G)$ of G. Moreover, if the group G is finitely generated, then $P_A(z)$ has integer coefficients and the free member equal to ± 1.

Let $A : G \to G$ be an automorphism of an arbitrary (not necessarily abelian) group G. A subgroup $H \subset G$ is called *A-invariant* if $A(H) = H$. A subgroup H of G is called *1^A-generated* if there is an element $x \in H$ such that $H = \langle A^n(x) : n \in \mathbb{Z} \rangle$.

Finally, let us remind some information concerning finitely presented groups. A group G is *finitely presented* if it can be defined using a finite number of relations over a finite set of generators, see [Sk, p.112]. According to [BiS] a torsion-free abelian-by-cyclic group

G is finitely presented if and only if G is generated by elements $t, a_1, \dots, a_n$ such that (i) $a_1, \dots, a_n$ is a basis of a torsion-free abelian normal subgroup $A \subset G$ with cyclic quotient G/A generated by the coset tA and (ii) the matrix M of the automorphism $i^t : A \to A$, $i^g : a \mapsto tat^{-1}$, in the basis $a_1, \dots, a_n$ has integer entries. In this case G has a finite presentation

$$\langle t, a_1, \dots, a_n \mid [a_i, a_j] = e,\ ta_it^{-1} = \phi_M(a_i),\ \ i, j = 1, \dots, n\rangle$$

where $\phi_M(a_i)$ is the word $a_1^{m_1} \cdots a_n^{m_n}$ and the vector $(m_1, \dots, m_n)$ is the i^{th}-column of the matrix M.

2. Group automorphisms and their spectra

In this section we study the spectrum of a single automorphism $A : G \to G$ on a group and reduce the problem of its description to calculating the roots of characteristic polynomials of some induced automorphisms of abelian torsion-free groups of finite rank.

Let us remind that the *spectrum* $\sigma(A)$ of an automorphism $A : G \to G$ of a group G consists of all the eigenvalues of A. A complex number $\lambda \in \mathbb{C}$ is an *eigenvalue* of A if there are an A-invariant subgroup $H \subset G$ and a non-trivial homomorphism $\varphi : H \to \mathbb{C}$ into the additive group of complex numbers such that $\varphi \circ A(x) = \lambda \cdot \varphi(x)$ for all $x \in H$.

At first we remark an obvious interplay between the spectrum of an automorphism $A : G \to G$ and the spectra of some induced automorphisms. Given an A-invariant subgroup H of G and a normal A-invariant subgroup N of H let $A_H = A|H : H \to H$ be the restriction of A onto H and $A_{H/N} : H/N \to H/N$ be a unique automorphism of the quotient group H/N such that $\pi \circ A_H = A_{H/N} \circ \pi$ where $\pi : H \to H/N$ stands for the quotient homomorphism.

Proposition 2.1. *Suppose $A : G \to G$ is an automorphism of a group G, H is an A-invariant subgroup of G and N is a normal A-invariant subgroup of H. Then $\sigma(A_{H/N}) \subset \sigma(A_H) \subset \sigma(A) \subset \mathbb{C}^*$.*

Proof. If $\lambda \in \sigma(A)$, then there is an A-invariant subgroup $K \subset G$ and a non-trivial homomorphism $\varphi : K \to \mathbb{C}$ such that $\varphi \circ A|K = \lambda \cdot \varphi$. Take any point $y \in K$ with $\varphi(y) \neq 0$ and find $x \in K$ such that $y = A(x)$ (such a point x exists since A_K is an automorphism of K). Then $0 \neq \varphi(y) = \varphi \circ A(x) = \lambda \cdot \varphi(x)$ which yields $\lambda \neq 0$ and hence $\sigma(A) \subset \mathbb{C}^*$.

Given $\lambda \in \sigma(A_H)$, find an A_H-invariant subgroup $K \subset H$ and a nontrivial homomorphism $\varphi : K \to \mathbb{C}$ such that $\varphi \circ A_H|K = \lambda \cdot \varphi$. Then $\varphi \circ A|K = \lambda \cdot \varphi$ which means that $\lambda \in \sigma(A)$ and hence $\sigma(A_H) \subset \sigma(A)$.

To prove that $\sigma(A_{H/N}) \subset \sigma(A_H)$, fix any $\lambda \in \sigma(A_{H/N})$ and find an $A_{H/N}$-invariant subgroup $K \subset H/N$ and a nontrivial homomorphism $\varphi : K \to \mathbb{C}$ such that $\varphi \circ A_{H/N}|K = \lambda \cdot \varphi$. Denote by $\pi : H \to H/N$ the quotient homomorphism and consider the subgroup $L = \pi^{-1}(K) \subset H$ and the non-trivial homomorphism $\psi = \varphi \circ \pi|L : L \to \mathbb{C}$. Then $\psi \circ A_H|L = \varphi \circ \pi \circ A_H|L = \varphi \circ A_{H/N} \circ \pi|L = \lambda \cdot \varphi \circ \pi|L = \lambda \cdot \psi$ which yields $\lambda \in \sigma(A_H)$. □

Now let us consider automorphisms of abelian torsion-free groups.

Theorem 2.2. *Let $A : G \to G$ be an automorphism of an abelian torsion-free group G.*

1. *If G has finite free rank r, then the spectrum $\sigma(A)$ of A consists of the roots of the characteristic polynomial $P_A(z)$ of A. Consequently,* $\text{Card}(\sigma(A)) \le r$ *and $\sigma(A) \subset \mathbb{A}^*(r)$.*
2. *$\sigma(A) = \cup\{\sigma(A_H) : H$ is a 1^A-generated subgroup of $G\}$.*

(3) $\sigma(A) \subset \mathbb{A}^*$ *if and only if* $\sigma(A) \neq \mathbb{C}^*$ *if and only if each* 1^A*-generated subgroup of* G *has finite free rank.*
(4) $\sigma(A) \subset \mathbb{A}_1^*$ *if and only if each* 1^A*-generated subgroup of* G *is finitely generated.*
(5) $\sigma(A^n) = \{\lambda^n : \lambda \in \sigma(A)\}$ *for any* $n \in \mathbb{Z}$.

Proof. (1) Suppose that G has a finite free rank r and let $M = (a_{ij})_{i,j=1}^r$ be the matrix of the automorphism A in a basis $x_1, \dots, x_r$ of G. Then $P_A(z) = \det(zE - M) = \det(zE - M^*)$, where $P_A(z)$ stands for the characteristic polynomial of A and M^* for the transposed matrix to M. If λ is a root of $P_A(z)$, then λ is an eigenvalue of the matrix M^*. Consequently, there is an eigenvector $\vec{c} = (c_1, \dots, c_r) \in \mathbb{C}^k$ of M^* corresponding to λ. The latter means that $\vec{c}$ is a non-zero vector such that $M^*\vec{c} = \lambda\vec{c}$, i.e., $\sum_{i=1}^r a_{ij}c_i = \lambda c_j$ for every $j \leq r$. Let $\varphi : G \to \mathbb{C}$ be a unique homomorphism of G into $\mathbb{C}$ such that $\varphi(x_i) = c_i$ for $i \leq r$. This homomorphism is not trivial since $\varphi(G) \supset \{c_1, \dots, c_r\} \neq \{0\}$.

Then for each element $x \in G$ with the coordinates $q_1, \dots, q_r$ in the basis $x_1, \dots, x_r$ we have $\varphi \circ A(x) = \varphi \circ A(\sum_{j=1}^r q_j x_j) = \sum_{j=1}^r q_j \cdot \varphi \circ A(x_j) = \sum_{j=1}^r q_j \cdot \varphi(\sum_{i=1}^r a_{ij}x_i) = \sum_{j=1}^r q_j(\sum_{i=1}^r a_{ij}\varphi(x_i)) = \sum_{j=1}^r q_j(\sum_{i=1}^r a_{ij}c_i) = \lambda \cdot \sum_{j=1}^r q_j c_j = \lambda \cdot \sum_{j=1}^r q_j \varphi(x_j) = \lambda \cdot \varphi(\sum_{j=1}^r q_j x_j) = \lambda \cdot \varphi(x)$. This means that λ is an eigenvalue of the automorphism A.

Assume conversely, that λ is an eigenvalue of the automorphism A. This means that for some A-invariant subgroup $H \subset G$ and a non-trivial homomorphism $\varphi : H \to \mathbb{C}$ we have $\varphi \circ A(x) = \lambda \cdot \varphi(x)$ for all $x \in H$. Pick any basis $x_1, \dots, x_r$ for the group G so that for some $k \leq r$ the elements $x_1, \dots, x_k$ form a basis of the subgroup H. Let $M = (a_{ij})_{i,j=1}^r$ be the matrix of the automorphism A in the basis $x_1, \dots, x_r$. Since $A(H) = H$, we get that $a_{ij} = 0$ for any $i, j \in \{1, \dots, r\}$ with $j \leq k < i$. Then $N = (a_{ij})_{i,j=1}^k$ is the matrix of the automorphism $A|H$ of the group H in the basis $x_1, \dots, x_k$. Let $\vec{c} = (c_1, \dots, c_k) \in \mathbb{C}^k$ be the vector with coordinates $c_i = \varphi(x_i)$ for $i \leq k$. Note that for every $i \in \{1, \dots, k\}$ we have $\sum_{i=1}^k a_{ij}c_i = \sum_{i=1}^k a_{ij}\varphi(x_i) = \varphi(\sum_{i=1}^k a_{ij}x_i) = \varphi \circ A(x_j) = \lambda\varphi(x_j) = \lambda \cdot c_j$. In the matrix form this means that $N^*\vec{c} = \lambda\vec{c}$, i.e., λ is the eigenvalue of the transposed matrix N^* to the matrix N. This yields that λ is a root of the characteristic polynomial $P_{A|H}(z) = \det(zE - N^*) = \det(zE - N)$ of the matrices N and N^*. Since N is a submatrix of the matrix M and $a_{ij} = 0$ for any $j \leq k < i$, we get that the characteristic polynomial $\det(zE - N)$ of the matrix N divides the characteristic polynomial $\det(zE - M)$ of M. This yields that $\det(\lambda E - M) = 0$ and thus λ is a root of the characteristic polynomial $P_A(z) = \det(zE - M)$ of the automorphism A.

(2) It follows from Proposition 2.1 that $\sigma(A_H) \subset \sigma(A)$ for any A-invariant subgroup H of G. Hence, to prove the second statement of Theorem 2.2 it suffices for each eigenvalue of A to find a 1^A-generated subgroup $H \subset G$ with $\lambda \in \sigma(A_H)$. Given an eigenvalue $\lambda \in \sigma(A)$, find an A-invariant subgroup $K \subset G$ and a nontrivial homomorphism $\varphi : K \to \mathbb{C}$ such that $\varphi \circ A(x) = \lambda \cdot \varphi(x)$ for all $x \in K$. Fix any $x_0 \in K$ with $\varphi(x_0) \neq 0$ and consider the 1^A-generated subgroup $H = \langle A^n(x_0) : n \in \mathbb{Z}\rangle$ of K. It is clear that H is an A-invariant subgroup of G and $\varphi|H : H \to \mathbb{C}$ is a nontrivial homomorphism such that $\varphi \circ A(x) = \lambda \cdot \varphi(x)$ for any $x \in H$. This means that λ is an eigenvalue of the automorphism $A_H : H \to H$.

(3) It follows from the previous two statements that $\sigma(A) \subset \mathbb{A}^*$ and thus $\sigma(A) \neq \mathbb{C}^*$ provided each 1^A-generated subgroup of G has finite free rank. It rests to show that $\sigma(A) = \mathbb{C}^*$ provided G contains a 1^A-generated subgroup $H = \langle A^n(x) : n \in \mathbb{Z}\rangle$ of infinite free rank. It follows that the system $A^n(x)$, $n \in \mathbb{Z}$, of generators of H is linearly independent (otherwise the group H would have finite rank). Consequently, for each non-zero complex number $\lambda \in \mathbb{C}^*$ we can define a non-trivial homomorphism $\varphi : H \to \mathbb{C}$

letting $\varphi(A^n(x)) = \lambda^n$ for $n \in \mathbb{Z}$. It can be easily shown that $\varphi \circ A(y) = \lambda \cdot \varphi(y)$ for each $y \in H$. This yields that $\lambda \in \sigma(A)$ and hence $\sigma(A) = \mathbb{C}^*$.

(4) Assume that each 1^A-generated subgroup of G is finitely generated and fix any $\lambda \in \sigma(A)$. Find an A-invariant subgroup $H \subset G$ and a non-trivial homomorphism $\varphi : H \to \mathbb{C}$ such that $\varphi \circ A|H = \lambda \cdot \varphi$. Without loss of generality, the subgroup H is 1^A-generated and thus finitely generated. Then H, being an abelian finitely generated torsion-free group, admits a basis $x_1, \dots, x_r$ generating H. The matrix M of A in the basis $x_1, \dots, x_r$ is an invertible element in the ring of $r \times r$-matrices with integer coefficients. Consequently, $\det(M) = \pm 1$ and the characteristic polynomial $\det(zE - M) = \sum_{i=0}^r a_i z^i$ of M has integer coefficients with $|a_0| = a_r = 1$. By (1), the eigenvalue λ is a root of this characteristic polynomial, which means that $\lambda \in \mathbb{A}_1^*$.

Now assume conversely, that G contains a 1^A-generated subgroup $H = \langle A^n(x) : n \in \mathbb{Z}\rangle$ which is not finitely-generated. We have to verify that $\sigma(A) \not\subset \mathbb{A}_1^*$. If the group H has infinite free rank, then $\sigma(A) = \mathbb{C}^* \not\subset \mathbb{A}_1^*$ and we are done. So assume that the group H has finite free rank r.

We claim that $\sigma(A_H) \not\subset \mathbb{A}_1^*$. Assuming the converse and applying the statement (1) of this theorem, we get that the roots of the characteristic polynomial of the automorphism A_H belong to the set $\mathbb{A}_1^*$. We can think of H as a subgroup of the linear r-dimensional complex space $\mathbb{C}^r$ such that the automorphism A_H extends to a linear automorphism B of $\mathbb{C}^r$. We can assume that the matrix of the automorphism B has a normal Jordan form and thus the space $\mathbb{C}^r$ is decomposed into the direct sum $\mathbb{C}^r = \mathbb{C}^{r_1} \oplus \dots \oplus C^{r_l}$ of B-invariant linear subspaces corresponding to the Jordan cells. Let $\mathrm{pr}_i : \mathbb{C}^r \to \mathbb{C}^{r_i}$, $i \le l$, denote the natural projections. Since the group $H \subset \mathbb{C}^r$ is infinitely generated, the projection $H_s = \mathrm{pr}_s(H) \subset \mathbb{C}^{r_s}$ is infinitely generated for some $s \le l$. Observe that the subgroup H_s is 1^B-generated: it is generated by the vectors $B^n(y)$, $n \in \mathbb{Z}$, where $y = \mathrm{pr}_s(x)$. Since the matrix of the automorphism $B|\mathbb{C}^{r_s}$ is a Jordan cell, $B|\mathbb{C}^{r_s} = \lambda \cdot E + N$ where $\lambda \in \mathbb{A}_1^*$ is the eigenvalue of $B|\mathbb{C}^{r_s}$, E is the identity operator on $\mathbb{C}^{r_s}$ and $N : \mathbb{C}^{r_s} \to \mathbb{C}^{r_s}$ is a nilpotent linear operator, which means that $N^{r_s} = 0$.

Analogously, $B^{-1}|\mathbb{C}^{r_s} = \frac{1}{\lambda}E + \tilde{N}$ where $\tilde{N}$ is another nilpotent linear operator on $\mathbb{C}^{r_s}$. It follows from $\lambda \in \mathbb{A}_1^*$ that the complex numbers $\lambda^0, \dots, \lambda^{d-1}$, where d is the degree of the algebraic number λ, generate the group $\langle \lambda^n : n \in \mathbb{Z}\rangle \subset \mathbb{C}$. Then the group H_s lies in the finitely generated subgroup $\langle \lambda^i \cdot y, N^j(y), \tilde{N}^j(y) : 0 \le i < d,\ 0 < j < r_s\rangle$ of $\mathbb{C}^{r_s}$. Consequently, H_s is finitely generated, which is a contradiction.

(5) Finally, given any $n \in \mathbb{Z}$, we prove the equality $\sigma(A^n) = \sigma(A)^n$ were $\sigma(A)^n = \{\lambda^n : \lambda \in \sigma(A)\}$. In fact, the inclusion $\sigma(A^n) \supset \sigma(A)^n$ is easy: take any $\lambda \in \sigma(A)$ and find an A-invariant subgroup $H \subset G$ and a non-trivial homomorphism $\varphi : H \to \mathbb{C}$ such that $\varphi \circ A(x) = \lambda \cdot \varphi(x)$ for any $x \in H$. In particular, for any $y \in H$ and $x = A^{-1}(y)$ we get $\varphi(y) = \varphi \circ A \circ A^{-1}(y) = \lambda \circ \varphi \circ A^{-1}(y)$ and thus $\varphi \circ A^{-1}(y) = \lambda^{-1} \cdot \varphi(y)$ for any $y \in H$. By induction, it can be shown that $\varphi \circ A^n(x) = \lambda^n \cdot \varphi(x)$ for every $n \in \mathbb{Z}$ and $x \in H$. Thus $\lambda^n \in \sigma(A^n)$ for any $n \in \mathbb{Z}$.

To prove the inclusion $\sigma(A^n) \subset \sigma(A)^n$, fix any $\lambda \in \sigma(A^n)$ and using the statement (2) of this theorem, find a 1^{A^n}-generated subgroup $H \subset G$ with $\lambda \in \sigma(A^n_H)$. Write $H = \langle A^{ni}(x) : i \in \mathbb{Z}\rangle$ for some $x \in G$ and consider the 1^A-generated subgroup $\bar{H} = \langle A^i(x) : i \in \mathbb{Z}\rangle$. If the group $\bar{H}$ has infinite free rank, then by the third statement, $\sigma(A) = \mathbb{C}^*$ and hence $\lambda \in \sigma(A)^n = \mathbb{C}^*$. So we assume that $\bar{H}$ has finite free rank r. Let $x_1, \dots, x_r$ by any basis for the group $\bar{H}$ and let M be the matrix of the automorphism $A_{\bar{H}}$ in the basis $x_1, \dots, x_r$. Since M^n is the matrix of the automorphism $A^n_{\bar{H}}$ in the basis $x_1, \dots, x_r$, we conclude that $\sigma(A_{\bar{H}})^n = \sigma(A_{\bar{H}}^n)$. Then $\lambda \in \sigma(A_H^n) \subset \sigma(A_{\bar{H}}^n) = \sigma(A_{\bar{H}})^n \subset \sigma(A)^n$ and hence $\sigma(A^n) \subset \sigma(A)^n$. $\square$

Next, we study the spectrum of an automorphism of an arbitrary abelian group. We remind that $\sqrt[\infty]{e}$ stands for the periodic part of a group G. It is known that for an abelian group G the periodic part $\sqrt[\infty]{e}$ of G is a characteristic subgroup in G. Observe that the quotient group $G/\sqrt[\infty]{e}$ is abelian and torsion-free. Moreover, $r_0(G/\sqrt[\infty]{e}) = r_0(G)$.

Given an automorphism $A : G \to G$ of G let $A_{G/\sqrt[\infty]{e}} : G/\sqrt[\infty]{e} \to G/\sqrt[\infty]{e}$ denote the induced automorphism of the quotient group $G/\sqrt[\infty]{e}$ such that $A_{G/\sqrt[\infty]{e}} \circ \pi = \pi \circ A$ where $\pi : G \to G/\sqrt[\infty]{e}$ is the quotient homomorphism.

Under the *characteristic polynomial* $P_A(z)$ of an automorphism $A : G \to G$ of an abelian group G of finite free rank we shall understand the characteristic polynomial of the automorphism $A_{G/\sqrt[\infty]{e}}$ of the abelian torsion-free group $G/\sqrt[\infty]{e}$ (which has rational coefficients and has degree equal to the free rank $r_0(G)$ of G).

Given a homomorphism $\varphi : G \to \mathbb{C}$, observe that $\sqrt[\infty]{e} \subset \mathrm{Ker}(\varphi)$ and thus $\varphi = \psi \circ \pi$ for some homomorphism $\psi : G/\sqrt[\infty]{e} \to \mathbb{C}$. This simple observation allows us to reduce the study of the spectrum of an automorphism $A : G \to G$ of an abelian group G to studying the spectrum of the automorphism $h_{G/\sqrt[\infty]{e}} : G/\sqrt[\infty]{e} \to G/\sqrt[\infty]{e}$ of the torsion-free abelian group $G/\sqrt[\infty]{e}$. More precisely, the following corollary of Theorem 2.2 holds.

Corollary 2.3. *Let $A : G \to G$ be an automorphism of an abelian group G, $\sqrt[\infty]{e}$ be the periodic part of G and $A_{G/\sqrt[\infty]{e}} : G/\sqrt[\infty]{e} \to G/\sqrt[\infty]{e}$ be the induced automorphism of the quotient torsion-free group $G/\sqrt[\infty]{e}$. Then*

(1) $\sigma(A) = \sigma(A_{G/\sqrt[\infty]{e}})$.
(2) *If G has finite free rank r, then the spectrum $\sigma(A)$ of A consists of the roots of the characteristic polynomial $P_A(z)$ of A which yields* $\mathrm{Card}(\sigma(A)) \le r$ *and* $\sigma(A) \subset \mathbb{A}^*(r)$.
(3) $\sigma(A) = \cup\{\sigma(A_H) : H$ *is a 1^A-generated subgroup of G*$\}$.
(4) $\sigma(A) \subset \mathbb{A}^*$ *if and only if $\sigma(A) \neq \mathbb{C}^*$ if and only if each 1^A-generated subgroup of G has finite free rank.*
(5) $\sigma(A) \subset \mathbb{A}_1^*$ *if and only if for each 1^A-generated subgroup H of G the group $H/\sqrt[\infty]{e}$ is finitely generated.*
(6) $\sigma(A^n) = \{\lambda^n : \lambda \in \sigma(A)\}$ *for any $n \in \mathbb{Z}$.*

Using Corollary 2.3 we are able to describe the spectrum of an automorphism of arbitrary group.

Theorem 2.4. *Let $A : G \to G$ be an automorphism of a group G. Then*

(1) $\sigma(A) = \cup\{\sigma(A_{H/H^{(1)}}) : H$ *is a 1^A-generated subgroup of G*$\}$;
(2) $\sigma(A) \subset \mathbb{A}^*$ *if and only if $\sigma(A) \neq \mathbb{C}^*$ if and only if for each 1^A-generated subgroup $H \subset G$ the abelian group $H/H^{(1)}$ has finite free rank;*
(3) $\sigma(A) \subset \mathbb{A}_1^*$ *if and only if for each 1^A-generated subgroup H of G the group $H/\sqrt[\infty]{H^{(1)}}$ is finitely generated;*
(4) $\sigma(A) \subset \mathbb{A}_1^*$ *if each subgroup of G is finitely generated;*
(5) $\sigma(A^n) \supset \{\lambda^n : \lambda \in \sigma(A)\}$ *for every $n \in \mathbb{Z}$.*

Proof. (1) It follows from Proposition 2.1 that $\sigma(A) \supset \cup\{\sigma(A_{H/H^{(1)}}) : H$ is a 1^A-generated subgroup of $G\}$. To prove the inverse inclusion, assume that $\lambda \in \sigma(A)$ and find an A-invariant subgroup $H \subset G$ and a non-trivial homomorphism $\varphi : H \to \mathbb{C}$ with $\varphi \circ A|H = \lambda \cdot \varphi$. Fix any $x \in H$ with $\varphi(x) \neq 0$. Without loss of generality, $H = \langle A^n(x) : n \in \mathbb{Z}\rangle$ and hence H is 1^A-generated. Since $\mathbb{C}$ is abelian, the kernel $\mathrm{Ker}(\varphi) = \{y \in H : \varphi(y) = 0\}$ of φ contains the commutator subgroup $H^{(1)}$ of H. Consequently, there is a homomorphism $\psi : H/H^{(1)} \to \mathbb{C}$ such that $\varphi = \psi \circ \pi$ where $\pi : H \to H/H^{(1)}$ stands for the quotient homomorphism. By $A_{H/H^{(1)}} : H/H^{(1)} \to H/H^{(1)}$ we denote the automorphism of the quotient group $H/H^{(1)}$ induced by the automorphism A (let us note that the subgroup $H^{(1)}$, being characteristic in H, is A-invariant). Then $\varphi \circ A|H = \lambda \cdot \varphi$ implies $\psi \circ A_{H/H^{(1)}} = \lambda \cdot \psi$ which means that $\lambda \in \sigma(A_{H/H^{(1)}})$.

Two next statements of Theorem 2.4 follows from the first statement of this theorem and the corresponding statements of Corollary 2.3. The forth statement of Theorem 2.4 follows from the third one, while the final fifth statement can be proven by the argument of Theorem 2.2(5). □

Next, we show that the spectrum of an automorphism A of a solvable group G is determined by the action of A on the factors $G^{(k)}/G^{(k+1)}$ of the commutator series of G.

Theorem 2.5. *Suppose $A : G \to G$ is an automorphism of a group G and $\{e\} = G_0 \subset G_1 \subset \cdots \subset G_\alpha = G$ is an increasing transfinite sequence of A-invariant subgroups of G such that $G_\gamma = \bigcup_{\beta<\gamma} G_\beta$ if $\gamma \le \alpha$ is a limit ordinal and G_β is a normal subgroup of $G_{\beta+1}$ for any ordinal $\beta < \alpha$. Then*

(1) *$\sigma(A) = \bigcup_{\beta<\alpha} \sigma(A_{G_{\beta+1}/G_\beta})$.*
(2) *If $\sigma(G) \neq \mathbb{C}^*$ and the factors $G_{\beta+1}/G_\beta$ are abelian for each $\beta < \alpha$, then $\mathrm{Card}(\sigma(A)) \le \sum\limits_{\beta<\alpha} r_0(G_{\beta+1}/G_\beta)$ and $\sigma(A) \subset \mathbb{A}^*(r)$ where $r = \sup\limits_{\beta<\alpha} r_0(G_{\beta+1}/G_\beta)$.*
(3) *If for every ordinal $\beta < \alpha$ the factor $G_{\beta+1}/G_\beta$ is abelian and the group $G_{\beta+1}/\sqrt[\infty]{G_\beta}$ is finitely generated, then $\sigma(A) \subset \mathbb{A}_1^*$.*

Proof. The inclusion $\sigma(A) \supset \bigcup_{\beta<\alpha} \sigma(A_{G_{\beta+1}/G_\beta})$ follows from Proposition 2.1. To prove the reverse inclusion, fix any $\lambda \in \sigma(A)$ and find an A-invariant subgroup $H \subset G$ and a non-trivial homomorphism $\varphi : H \to \mathbb{C}$ with $\varphi \circ A|H = \lambda \cdot \varphi$. Let $\mathrm{Ker}(\varphi) = \{x \in H : \varphi(x) = 0\}$ and let $\beta \le \alpha$ be the smallest ordinal such that $H \cap G_\beta \not\subset \mathrm{Ker}(\varphi)$. It follows that β is not limit, i.e., $\beta = \gamma + 1$ for some ordinal γ with $H \cap G_\gamma \subset \mathrm{Ker}(\varphi)$. Let $\pi : G_{\gamma+1} \to G_{\gamma+1}/G_\gamma$ be the quotient homomorphism and $A_{G_{\gamma+1}/G_\gamma} : G_{\gamma+1}/G_\gamma \to G_{\gamma+1}/G_\gamma$ be the homomorphism induced by $A_{G_{\gamma+1}}$ (that is, $A_{G_{\gamma+1}/G_\gamma} \circ \pi = \pi \circ A_{G_{\gamma+1}}$). Fix any point $x \in (H \cap G_\beta) \setminus \mathrm{Ker}(\varphi)$ and consider the 1^A-generated subgroup $H_x = \langle A^n(x) : n \in \mathbb{Z} \rangle \subset H \cap G_{\gamma+1}$ and its image $\pi(H_x) \subset G_{\gamma+1}/G_\gamma$. Since $H_x \cap G_\gamma \subset \mathrm{Ker}(\varphi)$, there is a homomorphism $\psi : \pi(H_x) \to \mathbb{C}$ such that $\psi \circ \pi|H_x = \varphi|H_x$.

We claim that $\psi \circ A_{G_{\gamma+1}/G_\gamma}|\pi(H_x) = \lambda \cdot \psi$. Indeed, take any $y \in \pi(H_x)$ and find $z \in H_x$ with $\pi(z) = y$. Then $\psi \circ A_{G_{\gamma+1}/G_\gamma}(y) = \psi \circ A_{G_{\gamma+1}/G_\gamma} \circ \pi(z) = \psi \circ \pi \circ A(z) = \varphi \circ A(z) = \lambda \cdot \varphi(z) = \lambda \cdot \psi \circ \pi(z) = \lambda \cdot \psi(y)$.

(2) Suppose that $\sigma(A) \neq \mathbb{C}^*$ and the factors $G_{\beta+1}/G_\beta$ are abelian for all ordinals $\beta < \alpha$. Let $r = \sup_{\beta<\alpha} r_0(G_{\beta+1}/G_\beta)$. Fix arbitrary ordinal $\beta < \alpha$. By the first statement of this theorem, $\sigma(A_{G_{\beta+1}/G_\beta}) \neq \mathbb{C}^*$. Applying Corollary 2.3(2), we conclude that $\mathrm{Card}\big(\sigma(A_{G_{\beta+1}/G_\beta})\big) \le r_0(G_{\beta+1}/G_\beta)$ and $\sigma(A_{G_{\beta+1}/G_\beta}) \subset \mathbb{A}^*(r_0(G_{\beta+1}/G_\beta)) \subset \mathbb{A}^*(r)$.

Then $\sigma(A) = \bigcup_{\beta<\alpha} \sigma(A_{G_{\beta+1}/G_\beta}) \subset \mathbb{A}^*(r)$ and $\mathrm{Card}(\sigma(A)) = \mathrm{Card}\big(\bigcup_{\beta<\alpha} \sigma(A_{G_{\beta+1}/G_\beta})\big) \le \sum_{\beta<\alpha} \mathrm{Card}(\sigma(A_{G_{\beta+1}/G_\beta})) \le \sum_{\beta<\alpha} r_0(G_{\beta+1}/G_\beta)$.

(3) Assume that for each ordinal $\beta < \alpha$ the factor $G_{\beta+1}/G_\beta$ is abelian and the group $G_{\beta+1}/\sqrt[\infty]{G_\beta}$ is finitely generated. It follows from Corollary 2.3(5) that $\sigma(A_{G_{\beta+1}/G_\beta}) \subset \mathbb{A}_1^*$ for each $\beta < \alpha$ and thus $\sigma(A) = \bigcup_{\beta<\alpha} \sigma(A_{G_{\beta+1}/G_\beta}) \subset \mathbb{A}_1^*$. □

Corollary 2.6. *If $A : G \to G$ is an automorphism of a group G and H is a normal A-invariant subgroup of G, then $\sigma(A) = \sigma(A_H) \cup \sigma(A_{G/H})$.*

Under *the characteristic polynomial* of an automorphism $A : G \to G$ of a solvable A_1-group G we understand the product $P_A(z) = \prod_{k=1}^{s} P_k(z)$, where s is the solvability degree of G and $P_k(z)$ is the characteristic polynomial of the induced automorphism $A_{G^{(k-1)}/G^{(k)}}$ of the abelian group $G^{(k-1)}/G^{(k)}$ for $k \le s$. Observe that the characteristic polynomial $P_A(z)$ has rational coefficients and degree equal to the Hirsch rank of G. Moreover, if the group G is polycyclic, then $P_A(z)$ has integer coefficients and the free member equal to ± 1.

Corollary 2.7. *Let $A : G \to G$ be an automorphism of a solvable group G. Then*

(1) $\sigma(A) = \bigcup_{k=0}^{\infty} \sigma(A_{G^{(k)}/G^{(k+1)}})$.
(2) *If G is a solvable A_1-group, then $\sigma(A)$ consists of roots of the characteristic polynomial $P_A(z)$ of A. Consequently,* $\mathrm{Card}(\sigma(A)) \le h(G)$ *and* $\sigma(A) \subset \mathbb{A}^*(r)$ *where* $r = \sup_{k\ge 0} r_0(G^{(k)}/G^{(k+1)})$.
(3) *If for every $k \ge 0$ the group $G^{(k)}/\sqrt[\infty]{G^{(k+1)}}$ is finitely generated, then $\sigma(A) \subset \mathbb{A}_1^*$.*
(4) *$\sigma(A) \subset \mathbb{A}_1^*$ if G is a polycyclic group.*

3. The spectrum of a group

For an element g of a group G let $i^g : G \to G$, $i^g : x \mapsto gxg^{-1}$, denote the inner automorphism generated by the element g. We recall the definition of our principal concept in this paper – the spectrum $\sigma(G)$ of a group G.

Definition 1. *The* spectrum *of a group G is the subset $\sigma(G) = \bigcup_{g\in G} \sigma(i^g)$ of the complex plane $\mathbb{C}$ consisting of the eigenvalues of all inner automorphisms of G.*

The spectrum of a group G carries a non-trivial information only in the non-commutative case: for any abelian group G we get $\sigma(G) \subset \{1\}$. As we shall see later the same is true for a much wider class of all periodically Engelian groups.

We remind that a group G is *Engelian* if for any elements $a, b \in G$ there is $n \in \mathbb{N}$ such that $[a, nb] = e$ where the multiple commutators $[a, nb]$ are defined inductively: $[a, 1b] = [a, b]$ and $[a, (k+1)b] = [[a, kb], b]$ for $k \ge 1$. It is known that the class of Engelian groups contains all nilpotent groups while latter class contains all abelian groups, see [Ku, §D.26].

We define a group G to be *periodically Engelian* if for any elements $a, b \in G$ there is a finite sequence $n_1, \dots, n_k \in \mathbb{N}$ such that $a_{k+1} = e$ where the elements $a_0, \dots, a_{k+1}$ are defined recursively: $a_0 = e$ and $a_{i+1} = [a_i, b]^{n_i}$ for $i \le k$. Observe that the class of periodically Engelian groups contains all Engelian groups and all periodic groups.

The following theorem describes elementary properties of group spectra.

Theorem 3.1. *Let G be a group. Then*

(1) $\sigma(G) \subset \mathbb{C}^*$;
(2) *$\sigma(G) = \emptyset$ if and only if $1 \notin \sigma(G)$ if and only if G is periodic;*
(3) *$\sigma(G) \subset \{1\}$ if G is periodically Engelian;*
(4) *$\sigma(H) \subset \sigma(G)$ for any subgroup H of G;*
(5) *$\sigma(G) = \bigcup\{\sigma(H) : H$ is a 2-generated subgroup of $G\}$;*
(6) *$\sigma(G) = \sigma(G/H) \cup \bigcup_{g\in G} \sigma(i^g_H)$ for any normal subgroup H of G;*
(7) *$\sigma(G) = \sigma(K) \cup \bigcup_{g\in K} \sigma(i^g_H)$ if $G = H \rtimes K$ is the semidirect product of a normal abelian subgroup $H \subset G$ and a subgroup $K \subset G$;*

(8) *$\sigma(G)=\sigma(H)\cup\sigma(K)$ if $G=H\times K$ is the direct product of subgroups $H,K\subset G$;*

(9) *$\sigma(G)\subset \bigcup\limits_{k=1}^{[G:H]}\sqrt[k]{\sigma(H)}$ for any subgroup H of finite index $[G:H]$ in G;*

(10) *$\sigma(G)\subset\mathbb{A}_1^*$ if G is a polycyclic group.*

(11) *If G is a solvable A_1-group, then $\sigma(G)\subset\mathbb{A}^*(r)$ where $r=\sup_{k\in\mathbb{N}} r_0(G^{(k-1)}/G^{(k)})$.*

Proof. (1) The inclusion $\sigma(G)\subset\mathbb{C}^*$ follows from Proposition 2.1.

(2) If G periodic, then any homomorphism $\varphi:H\to\mathbb{C}$ from any subgroup H of G is trivial. This yields $\sigma(G)=\emptyset$. Now assume that G is not periodic. Then G contains an infinite cyclic subgroup $H=\langle g\rangle$ generated by some element $g\in G$. Let $\varphi:H\to\mathbb{C}$ be the homomorphism such that $\varphi(g)=1$. Then the restriction $i^g|\langle g\rangle$ of the inner automorphism i^g onto $\langle g\rangle$ is the identity automorphism with $\varphi\circ i^g|\langle g\rangle=1\cdot\varphi$, i.e., $1\in\sigma(i^g)\subset\sigma(G)$.

(3) Suppose the group G is periodically Engelian, but $\sigma(G)$ contains a complex number $\lambda\ne 1$. Then there are an element $g\in G$, a subgroup $H\subset G$ with $gHg^{-1}=H$, and a non-trivial homomorphism $\varphi:H\to\mathbb{C}$ such that $\varphi(gxg^{-1})=\lambda\cdot\varphi(x)$ for all $x\in H$. Fix any point $x_0\in H$ with $\varphi(x_0)\ne 0$. Since G is periodically Engelian, there is a finite number sequence $n_1,\dots,n_k\in\mathbb{N}$ such that $x_{k+1}=e$ where $x_{i+1}=[x_i,g^{-1}]^{n_i}$ for $i\le k$. Observe that for every $i\le k$ we get $\varphi(x_{i+1})=\varphi([x_i,g^{-1}]^{n_i})=n_i\varphi(x_ig^{-1}x_i^{-1}g)=n_i(\varphi(x_i)-\varphi(gx_ig^{-1}))=n_i(\varphi(x_i)-\lambda\varphi(x_i))=n_i(1-\lambda)\varphi(x_i)$. Since $\varphi(x_{k+1})=\varphi(e)=0$, by finite induction, we get $\varphi(x_0)=0$, which is a contradiction.

(4) The inclusion $\sigma(H)\subset\sigma(G)$ for any subgroup H of G follows from Proposition 2.1.

(5) The inclusion $\sigma(G)\supset\bigcup\{\sigma(H):H$ is a 2-generated subgroup of $G\}$ follows from the previous item. To prove the inverse inclusion, fix any eigenvalue $\lambda\in\sigma(G)$ and find an element $g\in G$, a subgroup $K\subset G$ with $gKg^{-1}=K$ and a non-trivial homomorphism $\varphi:H\to\mathbb{C}$ such that $\varphi(gxg^{-1})=\lambda\varphi(x)$ for each $x\in K$. Fix any element $x\in K$ with $\varphi(x)\ne 0$ and consider the 2-generated subgroup $H=\langle x,g\rangle\subset G$. Let $L=K\cap H$. It follows that $gLg^{-1}=L$ and $\varphi(gyg^{-1})=\lambda\varphi(y)$ for each $y\in L$. This means that $\lambda\in\sigma(H)$.

(6) Suppose H is a normal subgroup of G. The inclusion $\sigma(G)\supset\sigma(G/H)\cup\bigcup_{g\in G}\sigma(i^g_H)$ follows from Proposition 2.1. To prove the inverse inclusion, fix any eigenvalue $\lambda\in\sigma(G)$ and find an element $g\in G$ with $\lambda\in\sigma(i^g)$. According to Corollary 2.6, $\lambda\in\sigma(i^g)=\sigma(i^g_H)\cup\sigma(i^g_{G/H})=\sigma(i^g_H)\cup\sigma(i^{gH}_{G/H})\subset\sigma(i^g_H)\cup\sigma(G/H)$.

(7) Suppose $G=H\rtimes K$ is the semidirect product of a normal abelian subgroup $H\subset G$ and a subgroup $K\subset G$. It follows that the quotient group G/H is isomorphic to K and thus $\sigma(G/H)=\sigma(K)$. Observe that each element $g\in G$ can be uniquely written as $g=kh$ for some $k\in K$ and $h\in H$. Then for each $x\in H$ we get $i^g_H(x)=gxg^{-1}=khxh^{-1}k^{-1}=kxk^{-1}=i^k_H(x)$. By the preceding item, $\sigma(K)\cup\bigcup_{g\in K}\sigma(i^g_H)\subset\sigma(G)=\sigma(G/H)\cup\bigcup_{g\in G}\sigma(i^g_H)=\sigma(K)\cup\bigcup_{g\in K}\sigma(i^g_H)$ which just yields $\sigma(G)=\sigma(K)\cup\bigcup_{g\in K}\sigma(i^g_H)$.

(8) Suppose $G=H\times K$ is a direct product of normal subgroups H and K. Note that for any element $g=hk=kh$ where $k\in K$, $h\in H$ we get $i^g_H=i^h_H$. Then by the item (6), we get $\sigma(G)=\sigma(G/H)\cup\bigcup_{g\in G}\sigma(i^g_H)=\sigma(K)\cup\bigcup_{h\in H}\sigma(i^h_H)=\sigma(K)\cup\sigma(H)$.

(9) Let H be a subgroup of finite index $[G:H]$ in G. Fix any eigenvalue $\lambda\in\sigma(G)$ and find an element $g\in G$, a subgroup $K\subset G$ with $gKg^{-1}=K$, and a non-trivial homomorphism $\varphi:K\to\mathbb{C}$ such that $\varphi(gxg^{-1})=\lambda\varphi(x)$ for each $x\in K$. Fix any $x\in K$ with $\varphi(x)\ne 0$. It follows that $g^r,x^q\in H$ for some $1\le r,q\le[G:H]$. Let $a=g^r$ and $y=x^q$ and consider the subgroup $L=\langle a^nya^{-n}:n\in\mathbb{Z}\rangle\subset K\cap H$. It is easy to verify that $aLa^{-1}=L$, $\varphi(y)=q\varphi(x)\ne 0$ and $\varphi(aza^{-1})=\varphi(g^rzg^{-r})=\lambda^r\varphi(z)$ for each $z\in L$. This means that $\lambda^r\in\sigma(H)$ and hence $\lambda\in\sqrt[r]{\sigma(H)}\subset\bigcup_{k=1}^{[G:H]}\sqrt[k]{\sigma(H)}$.

(10,11) The last two statements of Theorem 3.1 follow from Corollary 2.7. □

According to Theorem 3.1, $\sigma(G) \subset \{1\}$ for any periodically Engelian group G. For locally solvable groups the inverse statement is also true.

Theorem 3.2. *A locally solvable group G is periodically Engelian if and only if $\sigma(G) \subset \{1\}$.*

Proof. In light of Theorem 3.1(3) it suffices to prove that a locally solvable group G is periodically Engelian provided $\sigma(G) \subset \{1\}$. For this we need an auxiliary result.

We define an endomorphism $A : G \to G$ of an abelian group G to be *periodically nilpotent* if for any $x \in G$ there is $n \in \mathbb{N}$ such that the element $A^n(x)$ is periodic.

Claim. For any automorphism $A : G \to G$ of an abelian group G with $\sigma(A) \subset \{1\}$ the endomorphism $A - \mathrm{Id} : x \mapsto A(x) - x$ of G is periodically nilpotent.

Proof. Fix an arbitrary $x \in G$. Since $\sigma(A) \subset \{1\} \subset \mathbb{A}^*$, the 1^A-generated subgroup $H = \langle A^n(x) : n \in \mathbb{Z}\rangle$ of G has finite free rank r, see Corolary 2.3(4). Let P be the periodic part of H and $b_1, \dots, b_r$ be a basis of the torsion-free group H/P. Let M be the matrix of the automorphism $A_{H/P}$ in the basis $b_1, \dots, b_r$. Since $\sigma(A_{H/P}) \subset \sigma(A) \subset \{1\}$, all the eigenvalues of the matrix M are equal to 1. This implies that the matrix $M - \mathrm{Id}$ of the endomorphism $(A - \mathrm{Id})_{H/P}$ is nilpotent in the sense that $(M - \mathrm{Id})^r = 0$. This yields that $(A - \mathrm{Id})^r(x) \in P$ which just means that the endomorphism $A - \mathrm{Id}$ is periodically nilpotent. □

Now we are able to finish the proof of the theorem. Fix any elements $x_0, g \in G$. Without loss of generality we can assume that the group G is generated by x_0 and g and thus is solvable. Consider the commutator series $G = G^{(0)} \trianglerighteq G^{(1)} \trianglerighteq \cdots \trianglerighteq G^{(s+1)} = \{e\}$ of G and the automorphisms $h_k = i^g_{G^{(k-1)}/G^{(k)}}$ of the abelian groups $G^{(k-1)}/G^{(k)}$ induced by the inner automorphism $i^g : x \mapsto gxg^{-1}$ of G. Since $\sigma(h_k) \subset \sigma(i^g) \subset \sigma(G) \subset \{1\}$, it is legal to apply the above Claim to conclude that for every $k \le s+1$ the endomorphism $\mathrm{Id} - h_k$ of the group $G^{(k-1)}/G^{(k)}$ is periodically nilpotent. This allows us to construct inductively two sequences $m_1, \dots, m_s$ and $p_1, \dots, p_s$ of positive integers such that $x_{s+1} = e$ where $x_{k+1} = [x_k, m_k g]^{p_k}$ for $0 \le k \le s$. Let $l = s + \sum_{i=1}^{s} m_i$ and define a number sequence $n_1, \dots, n_l$ letting $n_i = p_j$ if $i = j + \sum_{k=1}^{j} m_k$ for some $j \le s$ and $n_i = 1$ otherwise. Observe that $y_{l+1} = x_{s+1} = e$ where $y_0 = x_0$ and $y_{k+1} = [y_k, g]^{n_k}$ for $0 \le k \le l$. This means that the group G is periodically Engelian. □

Next, we consider some group properties which can be characterized by the spectrum.

Theorem 3.3. *Let G be a finitely-generated solvable group such that for every $k \ge 0$ the the factor $G^{(k)}/G^{(k+1)}$ has finite p-rank for every prime p. Then*

1. $\sigma(G) \subset \mathbb{A}^*$ *if G has finite rank.*
2. $\sigma(G) \subset \mathbb{A}^*_1$ *if and only if G is polycyclic;*
3. $\sigma(G) \subset \sqrt[\infty]{1}$ *if and only if G is virtually nilpotent if and only if G contains no free semigroup with two generators;*
4. $\sigma(G) \subset \{1\}$ *if and only if G is periodically Engelian;*
5. $\sigma(G) = \emptyset$ *if and only if G is finite;*

Proof. (1) If G has finite rank, then for every $k \ge 0$ the group $G^{(k)}/G^{(k+1)}$ has finite free rank and hence $\sigma(G) \subset \mathbb{A}^*$ according to Theorem 3.1(11).

(2) If G is polycyclic, then $\sigma(G) \subset \mathbb{A}^*_1$ according to Theorem 3.1(10). The proof of the inverse statement will be done by induction on the solvability degree of G. To fulfill the inductive step we need the following fact proven in [Ro, Lemma 4.10].

Lemma 3.1. *Let A be an abelian normal subgroup of a finitely generated group B with polycyclic quotient B/A. If for any $a \in A$ and $b \in B$ the group $\langle b^n a b^{-n} : n \in \mathbb{Z}\rangle$ is finitely generated, then the group B is polycyclic.*

Now we are able to finish the proof of the second statement of Theorem 3.3. For abelian groups (i.e., groups of solvability degree 1) this statement is trivial. Assume that it has been proved for groups G whose solvability degree does not exceed some number $s \geq 1$. Suppose that the group G has the solvability degree $s+1$. Consider the commutator series $G = G^{(0)} \trianglerighteq G^{(1)} \trianglerighteq \cdots \trianglerighteq G^{(s+1)} = \{e\}$ of G. Assuming that $\sigma(G) \subset \mathbb{A}_1^*$ and applying Corollary 2.3(5), we get that for any $a \in G^{(s)}$ and $b \in G$ the quotient group H/T of the subgroup $H = \langle b^n a b^{-n} : n \in \mathbb{Z}\rangle \subset G^{(s)}$ by its periodic part T is finitely generated.

We claim that the group T is finite. Observe that H/T is a free abelian group of finite rank. This implies that H is a direct product $T \times F$ of its periodic part T and some subgroup $F \subset H$ isomorphic to H/T. Write $a = t \cdot v$ where $t \in T$ and $v \in F$. Let m be the exponent of t (i.e., the minimal m such that $t^m = e$). The periodic part T of H is a characteristic subgroup of H which yields that $bTb^{-1} = T$. Moreover, it can be shown that $T = \langle b^n t b^{-n} : n \in \mathbb{Z}\rangle$. Consequently, the subgroup T has finite exponent m. By Prüfer-Baer Theorem (see [Fu, 17.2]), T is a direct product of cyclic groups. Taking into account that T has finite exponent and the p-rank $r_p(T)$ of T is finite for every prime p, we conclude that the group T is finite. Then the group $H = \langle b^n a b^{-n} : n \in \mathbb{Z}\rangle$, being a direct sum of a finite group T and a finitely generated group F, is finitely generated.

The quotient group $G/G^{(s)}$ has solvability rank $\leq s$ and satisfies $\sigma(G/G^{(s)}) \subset \sigma(G) \subset \mathbb{A}_1^*$ according to Theorem 3.1(6). Then the inductive hypothesis implies that the group $G/G^{(s)}$ is polycyclic. Applying finally Lemma 3.1, we conclude that the group G is polycyclic too.

(3) The second equivalence of the third statement follows from Theorems 4.7 and 4.12 of [Ro] (stating that a finitely generated solvable group is virtually nilpotent if and only if it contains no free semigroup with two generators).

If the group G is virtually nilpotent, then by Theorem 3.1(3 and 9), $\sigma(G) \subset \sqrt[n]{1}$ for some $n \in \mathbb{N}$.

If, conversely, $\sigma(G) \subset \sqrt[\infty]{1}$, then G is polycyclic according to the previous item. Now we can apply Theorem 4.12 of [Ro] or Proposition 4.2 of [Alp] (asserting that a polycyclic group H with $\sigma(H) \subset \mathbb{T}$ is virtually nilpotent) to conclude that the group G is virtually nilpotent.

(4) The fourth statement follows from Theorem 3.2.

(5) Finally, the last statement follows from Theorem 3.1(2) and the well-known fact asserting that a periodic polycyclic group is finite. □

Remark 3.1. The restriction on the p-ranks of the factors of the commutator series in Theorem 3.3 is essential. Indeed, given any positive integer p consider the wreath product $(\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z}$. It is a 2-generated metabelian group which is not polycyclic. By [Ro] it contains a free semigroup with two generators. On the other hand, by Theorem 3.1(6), $\sigma((\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z}) = \sigma(\mathbb{Z}) = \{1\}$ which implies that the group $(\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z}$ is periodically Engelian.

Next, we study the algebraic structure of the spectrum $\sigma(G)$ of a group G. Observe that together with each complex number $\lambda \in \sigma(G)$ (which is an eigenvalue of some inner automorphism i^g) the set $\sigma(G)$ contains all its integer powers λ^n (which are eigenvalues of the inner automorphisms i^{g^n}). This yields that $\sigma(G)$ is a union of subgroups of $\mathbb{C}^*$. In some cases, $\sigma(G)$ is a *finite* union of subgroups of $\mathbb{C}^*$.

We shall say that a group G has *(finitely) splittable spectrum* if $\sigma(G) = \bigcup_{\varphi\in\Phi}\varphi(G)$ for some (finite) family Φ of homomorphisms $\varphi : G \to \mathbb{C}^*$ of G into the multiplicative group $\mathbb{C}^*$ of complex numbers.

Theorem 3.4. *Each solvable A_1-group G contains a subgroup H of finite index $[G : H]$ depending only on the ranks $r_0(G^{(k)}/G^{(k+1)})$, $k \geq 0$, whose spectrum $\sigma(H)$ is finitely splittable. More precisely, $\sigma(H) = \bigcup_{\varphi\in\Phi}\varphi(H)$ for some family Φ of homomorphisms $\varphi : H \to \mathbb{C}^*$ of size* $\mathrm{Card}(\Phi) = h(G)$ *such that for every $g \in H$ the product $\prod_{\varphi\in\Phi}(z-\varphi(g))$ coincides with the characteristic polynomial of the inner automorphism i^g of H.*

Proof. In the proof we shall use the Kolchinov-Malcev Theorem [KM, 21.1.5] concerning the structure of solvable subgroups of the group $GL(n,\mathbb{C})$ of invertible $n \times n$-matrices over the field $\mathbb{C}$. This theorem states that each solvable subgroup of $GL(n,\mathbb{C})$ contains a triangulable subgroup of finite index depending only on n. A subgroup $\mathcal{G} \subset GL(n,\mathbb{C})$ is *triangulable* if $W\mathcal{G}W^{-1} \subset T(n,\mathbb{C})$ for some $W \in GL(n,\mathbb{C})$ where $T(n,\mathbb{C})$ is the subgroup of $GL(n,\mathbb{C})$ consisting of upper-triangular matrices. Observe that the function $\varphi_i : T(n,\mathbb{C}) \to \mathbb{C}^*$, $i \in \{1,\dots,n\}$, assigning to an upper-triangular matrix $A = (a_{ij})_{i,j=1}^n$ its i-th diagonal element a_{ii} is a group homomorphism. Moreover, for each $A \in T(n,\mathbb{C})$ the product $\prod_{i=1}^n(z-\varphi_i(A))$ coincides with the characteristic polynomial of the matrix A, which yields that $\{\varphi_i(A) : 1 \leq i \leq n\}$ is the set of eigenvalues of the matrix A. This observation implies that for a triangulable subgroup $\mathcal{G} \subset GL(n,\mathbb{C})$ with $W\mathcal{G}W^{-1} \subset T(n,\mathbb{C})$ there are homomorphisms $\psi_1,\dots,\psi_n : \mathcal{G} \to \mathbb{C}^*$ defined by $\psi_i(A) = \varphi_i(WAW^{-1})$ for $A \in \mathcal{G}$, such that for each $A \in \mathcal{G}$ the product $\prod_{i=1}^n(z-\psi_i(A))$ coincides with the characteristic polynomial of A and $\{\psi_i(A) : 1 \leq i \leq n\}$ with the set of eigenvalues of the matrix A.

Given a solvable A_1-group G, for every $k \in \mathbb{N}$ consider the abelian torsion-free group $G^{(k-1)}/\sqrt[\infty]{G^{(k)}}$ where $\sqrt[\infty]{G^{(k)}} = \{x \in G^{(k-1)} : x^n \in G^{(k)} \text{ for some } n \in \mathbb{N}\}$. Let $Aut\big(G^{(k-1)}/\sqrt[\infty]{G^{(k)}}\big)$ be the automorphism group of $G^{(k-1)}/\sqrt[\infty]{G^{(k)}}$ and $\pi_k : G \to Aut\big(G^{(k-1)}/\sqrt[\infty]{G^{(k)}}\big)$ be the homomorphism assigning to each element $g \in G$ the automorphism $i^g_{G^{(k-1)}/\sqrt[\infty]{G^{(k)}}}$ of $G^{(k-1)}/\sqrt[\infty]{G^{(k)}}$ induced by the inner automorphism i^g of G. Let $n_k = r_0(G^{(k-1)}/G^{(k)})$. Fixing any basis in the abelian torsion-free group $G^{(k-1)}/\sqrt[\infty]{G^{(k)}}$ we identify the group $Aut(G^{(k-1)}/\sqrt[\infty]{G^{(k)}})$ with a subgroup of the matrix group $GL(n_k,\mathbb{C})$. According to the Kolchinov-Malcev Theorem, the group $\pi_k(G)$, being a solvable subgroup of $GL(n_k,\mathbb{C})$, contains a triangulable subgroup $H_k \subset \pi_k(G)$ of finite index depending only on n_k. The subgroup $H_k \subset GL(n_k,\mathbb{C})$, being triangulable, admits homomorphisms $\varphi_i^{(k)} : H_k \to \mathbb{C}^*$ for $i \in \{1,\dots,n_k\}$ such that for each $A \in H_k$ the product $\prod_{i=1}^{n_k}(z-\varphi_i^{(k)}(A))$ coincides with the characteristic polynomial of A and hence $\{\varphi_i^{(k)}(A) : 1 \leq k \leq n_k\}$ coincides with the set of eigenvalues of the matrix A.

The subgroup $H = \bigcap_{k=1}^{\infty}\pi_k^{-1}(H_k)$ has finite index in G (depending only on the indices of H_k in $\pi_k(G)$). Let $\Phi = \{\varphi_i^{(k)} \circ \pi_k|H : H \to \mathbb{C}^* : k \geq 0, 1 \leq i \leq n_k\}$ and observe that $\mathrm{Card}(\Phi) = \sum_{k\in\mathbb{N}} n_k = h(G)$. It follows from Corollary 2.7(2) that for any $g \in H$ the product $\prod_{\varphi\in\Phi}(z-\varphi(g))$ coincides with the characteristic polynomial of the inner automorphism i^g_H of H and hence $\sigma(i^g_H) = \{\varphi(i^g_H) : \varphi \in \Phi\}$. Consequently, $\sigma(H) = \bigcup_{\varphi\in\Phi}\varphi(H)$. □

4. The test groups $\mathrm{Aff}_\lambda(\mathbb{C})$

In this section we introduce and study so-called test groups $\mathrm{Aff}_\lambda(\mathbb{C})$, $\lambda \in \mathbb{C}^*$, which are of crucial importance for determining the spectrum $\sigma(G)$ of a group G. We shall show that a complex number $\lambda \in \mathbb{C}^*$ belongs to the spectrum $\sigma(G)$ if and only if the group $\mathrm{Aff}_\lambda(\mathbb{C})$ is a quotient group of some subgroup of $\mathbb{C}$.

As expected, $\mathrm{Aff}(\mathbb{C})$ is the group of affine transformations of $\mathbb{C}$ of the form $w = az + b$ where $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$. The group $\mathrm{Aff}(\mathbb{C})$ is isomorphic to the multiplicative matrix group $\big\{\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) : a \in \mathbb{C}^*,\ b \in \mathbb{C}\big\}$.

The group $\mathrm{Aff}(\mathbb{C})$ has the structure of the semi-direct product $\mathbb{C} \rtimes \mathbb{C}^*$ with the group operation defined by $(b,a) * (b',a') = (b + ab', aa')$. The group $\mathbb{C}$ can be identified with the normal subgroup $T(\mathbb{C}) = \{(b,1) \in \mathrm{Aff}(\mathbb{C}) : b \in \mathbb{C}\}$ consisting all translations of $\mathbb{C}$ while $\mathbb{C}^*$ is isomorphic to the subgroup $GL(\mathbb{C}) = \{(0,a) : a \in \mathbb{C}^*\}$ consisting of linear tranformations of $\mathbb{C}$. By $\chi : \mathrm{Aff}(\mathbb{C}) \to \mathbb{C}^*$, $(b,a) \mapsto a$, we denote the natural homomorphism whose kernel coincides with the subgroup $T(\mathbb{C})$.

Observe that an element $(0,\lambda) \in GL(\mathbb{C}) \subset \mathrm{Aff}(\mathbb{C})$ acts on the subgroup $T(\mathbb{C}) \subset \mathrm{Aff}(\mathbb{C})$ by conjugations as $(0,\lambda)(b,1)(0,\lambda)^{-1} = (\lambda b, 1)$, i.e., as the linear operator multiplying each $b \in \mathbb{C}$ by λ.

Given a complex number $\lambda \in \mathbb{C}^*$ let $\mathrm{Aff}_\lambda(\mathbb{C})$ be the subgroup of $\mathrm{Aff}(\mathbb{C})$ generated by two elements: $(1,1) \in T(\mathbb{C})$ and $(0,\lambda) \in GL(\mathbb{C})$. Note that $\chi(\mathrm{Aff}_\lambda(\mathbb{C})) = \{\lambda^n : n \in \mathbb{Z}\} \subset \mathbb{C}^*$ is a cyclic subgroup of $\mathbb{C}^*$ while $T_\lambda(\mathbb{C}) = T(\mathbb{C}) \cap \mathrm{Aff}_\lambda(\mathbb{C})$ is the additive subgroup of $T(\mathbb{C}) \cong \mathbb{C}$ generated by the set $\{(\lambda^n, 1) : n \in \mathbb{Z}\}$. Thus each element $(z,1) \in T_\lambda(\mathbb{C})$ can be written as $z = \sum_{i=-n}^{n} z_i \lambda^i$ for some $n \in \mathbb{N}$ and integers z_i, $|i| \le n$. Observe that the group $\mathrm{Aff}_\lambda(\mathbb{C})$ is a semi-direct product of $T_\lambda(\mathbb{C})$ and the cyclic group $\{(0,\lambda^n) : n \in \mathbb{Z}\} \subset GL(\mathbb{C})$. The algebraic structure of the group $\mathrm{Aff}_\lambda(\mathbb{C})$ depends essentially on the arithmetic properties of the number λ.

Theorem 4.1. *For a non-zero complex number λ the group $\mathrm{Aff}_\lambda(\mathbb{C})$ has the following properties:*

1. $\mathrm{Aff}_\lambda(\mathbb{C})$ *is metabelian (more precisely, abelian-by-cyclic).*
2. λ *is an algebraic number of degree r if and only if the group $T_\lambda(\mathbb{C})$ has the free rank r.*
3. $\sigma(\mathrm{Aff}_\lambda(\mathbb{C})) = \{z^n : n \in \mathbb{Z},\ P_\lambda(z) = 0\}$ *if λ is an algebraic number with minimal polynomial $P_\lambda(z)$.*
4. *The number λ is transcendental if and only if $\sigma(\mathrm{Aff}_\lambda(\mathbb{C})) = \mathbb{C}^*$ if and only if the group $T_\lambda(\mathbb{C})$ has infinite free rank if and only if $T_\lambda(\mathbb{C})$ is a free abelian group of infinite rank.*
5. $\lambda \in \mathbb{A}_1^*$ *if and only if the group $T_\lambda(\mathbb{C})$ is finitely generated if and only if* $\mathrm{Aff}_\lambda(\mathbb{C})$ *is polycyclic.*
6. $\lambda \in \sqrt[\infty]{1}$ *if and only if $\sigma(\mathrm{Aff}_\lambda(\mathbb{C})) = \sqrt[k]{1}$ for some k if and only if* $\mathrm{Aff}_\lambda(\mathbb{C})$ *is virtually nilpotent if and only if* $\mathrm{Aff}_\lambda(\mathbb{C})$ *contains no free semigroup with two generators.*
7. $\lambda \in \mathbb{A}_\mathbb{Z}^* \cup (\mathbb{A}_\mathbb{Z}^*)^{-1}$ *if and only if the group* $\mathrm{Aff}_\lambda(\mathbb{C})$ *is finitely presented.*

Proof. (1) The first statement of Theorem 4.1 is obvious and follows from the definition of $\mathrm{Aff}_\lambda(\mathbb{C})$.

(2) To prove the second statement, assume that λ is an algebraic number of degree r. It follows that the numbers $1, \lambda, \dots, \lambda^{r-1} \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$ while λ^r is a $\mathbb{Q}$-linear combination of $\lambda^0, \dots, \lambda^{r-1}$ and so does any power λ^n, $n \in \mathbb{Z}$, Consequently, the group $T_\lambda(\mathbb{C}) = \langle (\lambda^n, 1) : n \in \mathbb{Z} \rangle$ has free rank r.

Assume conversely that the group $T_\lambda(\mathbb{C})$ has finite free rank r. Then the elements $(\lambda^0, 1), \dots, (\lambda^{r-1}, 1)$ form a basis of the group $T_\lambda(\mathbb{C})$ which implies that λ is an algebraic number of degree r.

(3) Now assume that λ is an algebraic number with the minimal polynomial $P_\lambda(z) = \sum_{i=0}^{d} a_i z^i$ where $a_d = 1$. By $i^\lambda : \mathrm{Aff}_\lambda(\mathbb{C}) \to \mathrm{Aff}_\lambda(\mathbb{C})$ we denote the inner automorphism generated by the element $(0,\lambda) \in \mathrm{Aff}_\lambda(\mathbb{C})$. For any $(b,1) \in T_\lambda(\mathbb{C})$ we get $i^\lambda(b,1) = (0,\lambda)(b,1)(0,\lambda^{-1}) = (\lambda b, 1)$. It follows from the previous assertion that the group $T_\lambda(\mathbb{C})$ has finite free rank equal d and the elements $(\lambda^0,1),\dots,(\lambda^{d-1},1)$ form a basis for $T_\lambda(\mathbb{C})$. For this basis we get $i^\lambda(\lambda^i,1) = (\lambda^{i+1},1)$ if $i < d-1$ and $i^\lambda(\lambda^{d-1},1) = (-\sum_{i=0}^{d-1} a_i\lambda_i, 1)$. Thus the matrix of the automorphism $i^\lambda|T_\lambda(\mathbb{C})$ in this basis has the Frobenius normal form, while the characteristic polynomial of this automorphism is equal to $P_\lambda(z)$, see [War, §38]. By Theorem 2.2(1), $\sigma(i^\lambda|T_\lambda(\mathbb{C})) = \{\lambda_1,\dots,\lambda_d\}$ where $\lambda_1,\dots,\lambda_d$ are the roots of the polynomial $P_\lambda(z)$. By Theorem 2.2(5) for every $n \in \mathbb{Z}$, we get $\sigma(i^{\lambda^n}|T_\lambda(\mathbb{C})) = \{\lambda_1^n,\dots,\lambda_d^n\}$. Applying Theorem 3.1(7), we conclude that $\sigma(\mathrm{Aff}_\lambda(\mathbb{C})) = \sigma(\mathrm{Aff}_\lambda(\mathbb{C})/T_\lambda(\mathbb{C})) \cup \bigcup_{n\in\mathbb{Z}} \sigma(i^{\lambda^n}|T_\lambda(\mathbb{C})) = \sigma(\{\lambda^n : n\in\mathbb{Z}\}) \cup \{\lambda_1^n,\dots,\lambda_d^n : n \in \mathbb{Z}\} = \{z^n : n\in\mathbb{Z},\ P_\lambda(z) = 0\}$.

(4) If λ is transcendental, then the numbers λ^n, $n \in \mathbb{Z}$, are linearly independent over $\mathbb{Q}$. Consequently, $T_\lambda(\mathbb{C}) = \langle(\lambda^n,1) : n\in\mathbb{Z}\rangle$ is a free abelian group of infinite rank. Observe that the subgroup $T_\lambda(\mathbb{C})$ of $\mathrm{Aff}_\lambda(\mathbb{C})$ is 1^{i^λ}-generated. Applying Theorem 2.2(3), we get $\mathbb{C}^* = \sigma(i^\lambda) \subset \sigma(\mathrm{Aff}_\lambda(\mathbb{C})) \subset \mathbb{C}^*$.

If λ is an algebraic number, then by the previous item we get $\sigma(\mathrm{Aff}_\lambda(\mathbb{C})) \subset \mathbb{A}^* \neq \mathbb{C}^*$ and $T_\lambda(\mathbb{C})$ has finite free rank. This proves the fourth statement of Theorem 4.1.

(5) The fifth statement follows from the third statement of this theorem and Theorem 3.3(2).

(6) The sixth statement follows from the third statement of this theorem and Theorem 3.3(3).

(7) Finally, the seventh statement follows from the characterization of finitely presented torsion-free abelian-by-cyclic groups given in [BiS] (and mentioned in the first section). □

It is interesting to notice that the algebraic structure of the groups $\mathrm{Aff}_\lambda(\mathbb{C})$ is completely determined by their spectra.

Theorem 4.2. *Let $\lambda,\mu \in \mathbb{C}^*$. The groups $\mathrm{Aff}_\lambda(\mathbb{C})$ and $\mathrm{Aff}_\mu(\mathbb{C})$ are isomorphic if and only if $\sigma(\mathrm{Aff}_\lambda(\mathbb{C})) = \sigma(\mathrm{Aff}_\mu(\mathbb{C}))$.*

Proof. The "only if" part is trivial. To prove the "if" part, suppose that $\sigma(\mathrm{Aff}_\lambda(\mathbb{C})) = \sigma(\mathrm{Aff}_\mu(\mathbb{C}))$. Depending on the arithmetic properties of the number λ we consider three cases.

(1) λ is transcendental. Then $\sigma(\mathrm{Aff}_\lambda(\mathbb{C})) = \mathbb{C}^*$ and hence $\sigma(\mathrm{Aff}_\mu(\mathbb{C})) = \mathbb{C}^*$ which yields that μ is transcendental too, see Theorem 4.1(4). In this case the groups $\mathrm{Aff}_\lambda(\mathbb{C})$ and $\mathrm{Aff}_\mu(\mathbb{C})$ are isomorphic to the wreath product $\mathbb{Z} \wr \mathbb{Z}$.

(2) λ is a root of the unit. Then $\sigma(\mathrm{Aff}_\lambda(\mathbb{C})) = \sqrt[k]{1}$ where $k \in \mathbb{N}$ is the minimal number such that $\lambda^k = 1$. It follows that $\sigma(\mathrm{Aff}_\mu(\mathbb{C})) = \sqrt[k]{1}$ and hence μ is a root of the unit with $\mu^k = 1$ and $\mu^i \neq 1$ for $0 < i < k$, see Theorem 4.1(3). Observe that $\{\lambda^i : i \in\mathbb{Z}\} = \sqrt[k]{1} = \{\mu^i : i\in\mathbb{Z}\}$ which yields $T_\lambda(\mathbb{C}) = T_\mu(\mathbb{C})$. Since the numbers λ and μ are generators of the cyclic group $\sqrt[k]{1}$, there are numbers $1 \le p, q < k$ such that $\lambda^p = \mu$ and $\mu^q = \lambda$.

Then the groups $\mathrm{Aff}_\lambda(\mathbb{C})$ and $\mathrm{Aff}_\mu(\mathbb{C})$ are isomorphic via the isomorphism $h : (b,\lambda^i) \mapsto (b,\mu^{qi})$ with inverse $h^{-1} : (b,\mu^i) \mapsto (b,\lambda^{pi})$.

(3) $\lambda \in \mathbb{A}^* \setminus \sqrt[\infty]{1}$. Then the spectrum $\sigma(\mathrm{Aff}_\lambda(\mathbb{C})) \subset \mathbb{A}^*$ is countable and the same is true for the spectrum $\sigma(\mathrm{Aff}_\mu(\mathbb{C}))$ which yields that $\mu \in \mathbb{A}^* \setminus \sqrt[\infty]{1}$. Let $P_\lambda(z)$ and $P_\mu(z)$ be the minimal polynomials of the algebraic numbers λ and μ, respectively.

To detect the number λ from the spectrum $\sigma(\mathrm{Aff}_\lambda(\mathbb{C}))$, we introduce a notion of an extremal point. Given a subset $A \subset \mathbb{C}$, define a point $a \in A$ to be *extremal* if $a \neq b^n$ for any $b \in A$ and $n \geq 2$. By $\mathrm{ext}(A)$ we denote the set of all extremal points of $A \subset \mathbb{C}$.

Now consider the set $\mathrm{ext}(\sigma(\mathrm{Aff}_\lambda(\mathbb{C}))) = \mathrm{ext}(\sigma(\mathrm{Aff}_\mu(\mathbb{C})))$ and observe that it is not empty and lies in the intersection $\{z, z^{-1} : P_\lambda(z) = 0\} \cap \{z, z^{-1} : P_\mu(z) = 0\}$. Pick any extremal point ν of the spectrum $\sigma(\mathrm{Aff}_\lambda(\mathbb{C}))$. Then $P_\mu(\nu^\varepsilon) = 0$ and $P_\lambda(\nu^{\varepsilon'}) = 0$ for some $\varepsilon, \varepsilon' \in \{-1, 1\}$. Replacing, if necessary, ν by ν^{-1} we can assume that $\varepsilon' = 1$, i.e., $P_\lambda(\nu) = 0$. This means that ν is algebraically conjugated to λ. Similarly, $P_\mu(\nu) = 0$ or $P_\mu(\nu^{-1}) = 0$ implies that ν is algebraically conjugated to μ or μ^{-1}. Therefore λ is algebraically conjugated to μ or μ^{-1}.

In the first case the groups $\mathrm{Aff}_\lambda(\mathbb{C})$ and $\mathrm{Aff}_\mu(\mathbb{C})$ are isomorphic via the isomorphism $h : \mathrm{Aff}_\lambda(\mathbb{C}) \to \mathrm{Aff}_\mu(\mathbb{C})$ sending $(\lambda^i, 1)$ onto $(\mu^i, 1)$ and $(0, \lambda^i)$ onto $(0, \mu^i)$ for $i \in \mathbb{Z}$. In the second case, these groups are isomorphic via the isomorphism sending $(\lambda^i, 1)$ onto $(\mu^{-i}, 1)$ and $(0, \lambda^i)$ onto $(0, \mu^{-i})$ for $i \in \mathbb{Z}$. □

The following theorem which is the main result of this section displays the role of the groups $\mathrm{Aff}_\lambda(\mathbb{C})$ for determining the spectrum of a group.

Theorem 4.3. *A complex number $\lambda \in \mathbb{C}^*$ belongs to the spectrum $\sigma(G)$ of a group G if and only if the group $\mathrm{Aff}_\lambda(\mathbb{C})$ is isomorphic to a quotient group of some subgroup of G. Consequently, $\sigma(G) = \bigcup_{\lambda \in \sigma(G)} \sigma(\mathrm{Aff}_\lambda(\mathbb{C}))$.*

Proof. If $\mathrm{Aff}_\lambda(\mathbb{C})$ is isomorphic to a qutient group of some subgroup of G, then $\lambda \in \sigma(\mathrm{Aff}_\lambda(\mathbb{C})) \subset \sigma(G)$ according to Proposition 2.1 and Theorem 4.1(3). This proves the "if" part of the theorem.

To prove the "only if" part, fix any $\lambda \in \sigma(G)$ and find an element $g \in G$, a subgroup $H \subset G$ with $gHg^{-1} = H$, and a non-trivial homomorphism $\varphi : H \to \mathbb{C}$ such that $\varphi(gxg^{-1}) = \lambda \cdot \varphi(x)$ for each $x \in H$. Fix any $x_0 \in H$ with $\varphi(x_0) \neq 0$. Multiplying φ by a suitable constant, we may assume that $\varphi(x_0) = 1$. Without loss of generality, $H = \langle g^n x_0 g^{-n} : n \in \mathbb{Z}\rangle$.

If $\lambda = 1$, then the element x_0 generates an infinite cyclic subgroup of G isomorphic to $\mathbb{Z} \cong \mathrm{Aff}_1(\mathbb{C})$. So we can assume that $\lambda \neq 1$. Consider the 2-generated subgroup $\langle g, x_0\rangle$ and observe that each element $x \in \langle g, x_0\rangle$ can be written as $x = hg^n$ for some $h \in H$ and $n \in \mathbb{Z}$.

We claim that the map $\pi : \langle g, x_0\rangle \to \mathrm{Aff}_\lambda(\mathbb{C})$ defined by $\pi(hg^n) = (\varphi(h), \lambda^n)$ for $h \in H$, $n \in \mathbb{Z}$, is a homomorphism of $\langle g, x_0\rangle$ onto $\mathrm{Aff}_\lambda(\mathbb{C})$. If $\langle g\rangle \cap H = \{e\}$, then h is well-defined since each element $x \in \langle g, x_0\rangle$ can be uniquely written as $x = hg^n$ for $h \in H$, $n \in \mathbb{Z}$. If $\langle g\rangle \cap H \neq \{e\}$, find the smallest positive integer k with $g^k \in H \setminus \{e\}$. We claim that $\lambda^k = 1$ and $\varphi(g^k) = 0$. Indeed, $\varphi(x_0) = \varphi(g^k) + \varphi(x_0) - \varphi(g^k) = \varphi(g^k x_0 g^{-k}) = \lambda^k \varphi(x_0)$ which just yields $\lambda^k = 1$. To see that $\varphi(g^k) = 0$, observe that $\varphi(g^k) = \varphi(g \cdot g^k \cdot g^{-1}) = \lambda\varphi(g^k)$ and use the fact that $\lambda \neq 1$.

Now we are ready to show that the map π is well defined. Assuming that $h_1 g^n = h_2 g^m$ for some $h_1, h_2 \in H$ and integer $n \leq m$ we get $g^{m-n} = h_2^{-1} h_1 \in H$ and thus $m - n = k \cdot l$ for some $l \in \mathbb{Z}$. Then $h_1 = h_2 g^{m-n} = h_2 g^{kl}$ and $\pi(h_1 g^n) = (\varphi(h_1), \lambda^n) = (\varphi(h_2 g^{kl}), \lambda^{m-kl}) = (\varphi(h_2) + l \cdot \varphi(g^k), \lambda^m (\lambda^k)^{-l}) = (\varphi(h_2) + l \cdot 0, \lambda^m \cdot 1) = (\varphi(h_2), \lambda^m) = \pi(h_2 g^m)$ which shows that π is well defined.

To see that π is a group homomorphism, observe that for any $h_1, h_2 \in H$ and $n, m \in \mathbb{Z}$ we have $\pi(h_1 g^n h_2 g^m) = \pi(h_1 g^n h_2 g^{-n} \cdot g^{n+m}) = (\varphi(h_1 g^n h_2 g^{-n}), \lambda^{m+n}) = (\varphi(h_1) + \varphi(g^n h_2 g^{-n}), \lambda^{n+m}) = (\varphi(h_1) + \lambda^n \varphi(h_2), \lambda^{n+m}) = (\varphi(h_1), \lambda^n) * (\varphi(h_2), \lambda^m) = \pi(h_1 g^n) * \pi(h_2 g^m)$. □

Theorem 4.3 allows us to estimate the spectrum of groups possessing some hereditary property $\mathcal{P}$. We shall say that a group property $\mathcal{P}$ is *hereditary* if for any group G possessing the property $\mathcal{P}$ each isomorphic copy of G, each subgroup of G, and each quotient group of G has that property.

Corollary 4.4. *If a group G possesses a hereditary property $\mathcal{P}$, then $\sigma(G) \subset \{\lambda \in \mathbb{C}^* : \mathrm{Aff}_\lambda(\mathbb{C})$ has the property $\mathcal{P}\}$.*

5. Interplay between reversive properties of a group and its spectrum

In this section we study so-called reversive properties of groups. The motivation to such kind of research came from the theory of paratopological groups, see [BR].

Given a subset A of a group G define its n-th oscillators $(\pm A)^n$ and $(\mp A)^n$ by induction: let $(\pm A)^0 = (\mp A)^0 = \{e\}$ and $(\pm A)^{n+1} = A\cdot(\mp A)^n$, $(\mp A)^{n+1} = A^{-1}\cdot(\pm A)^n$ for $n \geq 0$. For $m \in \mathbb{N}$ by A^m we denote the m-fold product of A in the group G and let $A^\infty = \bigcup_{m\in\mathbb{N}} A^m$ be the semigroup generated by the subset A in G.

Let $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{\infty\}$. A group G is defined to be (n, m)-*reversive* if for any subset $A \subset G$ containing the neutral element e of G we have $(\mp A)^n \subset (\pm A^m)^n$. It is easy to show that each (n, m)-reversive group G is $(n + 1, q)$-reversive for any $n \in \mathbb{N}$ and $m < q \leq \infty$.

Observe that a group G is $(1, \infty)$-reversive (resp. $(1, m)$-reversive for some $m \in \mathbb{N}$) if and only if G is periodic (G is of finite exponent).

Let us note that the property of a group to be (n, m)-reversive is hereditary, which allows us to apply Theorem 4.3 to studying the spectrum of (n, m)-reversive groups. Our principal result in this direction is

Theorem 5.1. *Let G be a group.*

(1) *If G is (n, ∞)-reversive for some $n \in \mathbb{N}$, then $\sigma(G) \subset \mathbb{A} \cap \mathbb{T}$.*
(2) *If G is (n, m)-reversive for some $n, m \in \mathbb{N}$, then $\sigma(G) \subset \sqrt[k]{1}$ for some $k \in \mathbb{N}$ depending only on n and m.*

Proof. 1) Assume that a group G is (n, ∞)-reversive for some $n \in \mathbb{N}$ but $\sigma(G) \not\subset \mathbb{A} \cap \mathbb{T}$. Replacing n by $n + 1$, if necessary, we can assume that n is an odd number. Applying Theorems 4.1(3) and 4.3 we conclude that the spectrum $\sigma(G)$ of G is unbounded and thus contains an eigenvalue $\lambda \in \sigma(G)$ with $|\lambda| > n + 1$. By Theorem 4.3, $\mathrm{Aff}_\lambda(\mathbb{C})$ is a quotient group of a subgroup of G. Taking into account that the (n, ∞)-reversibility is a hereditary property, we conclude that the group $\mathrm{Aff}_\lambda(\mathbb{C})$ is (n, ∞)-reversive.

To derive a contradiction, let us make some remarks concerning the group operation in $\mathrm{Aff}_\lambda(\mathbb{C})$.

Given an element $(b, \lambda^k) \in \mathrm{Aff}_\lambda(\mathbb{C})$ observe that $(b, \lambda^k)^{-1} = (-b\lambda^{-k}, \lambda^{-k})$ which can be written as $(b, \lambda^k)^\varepsilon = (\varepsilon b\lambda^{\frac{k}{2}(\varepsilon-1)}, \lambda^{\varepsilon k})$ for $\varepsilon \in \{-1, 1\}$. Next, if $(b_1, \lambda^{m_1}), \dots, (b_k, \lambda^{m_k}) \in \mathrm{Aff}_\lambda(\mathbb{C})$ and $\varepsilon_1, \dots, \varepsilon_k \in \{-1, 1\}$, then

$$\prod_{i=1}^{k}(b_i, \lambda^{m_i}) = \Big(\sum_{i=1}^{k} b_i\lambda^{\sum_{j=1}^{i-1} m_j}, \lambda^{\sum_{i=1}^{k} m_i}\Big)$$

and

$$\prod_{i=1}^{k}(b_i, \lambda^{m_i})^{\varepsilon_i} = \prod_{i=1}^{k}(\varepsilon_i b_i\lambda^{\frac{m_i}{2}(\varepsilon_i-1)}, \lambda^{\varepsilon_i m_i}) = \Big(\sum_{i=1}^{k} \varepsilon_i b_i\lambda^{\frac{m_i}{2}(\varepsilon_i-1)+\sum_{j=1}^{i-1}\varepsilon_j m_j}, \lambda^{\sum_{i=1}^{k}\varepsilon_i m_i}\Big).$$

To prove that $\mathrm{Aff}_\lambda(\mathbb{C})$ is not (n,∞)-reversive, consider the elements $g_i=(\delta_i,\lambda^i)\in\mathrm{Aff}_\lambda(\mathbb{C})$, $i\ge 0$, where $\delta_i=(i\mod 2)$. Note that g_0 is the neutral element of $\mathrm{Aff}_\lambda(\mathbb{C})$.

We claim that $\prod_{i=n}^{2n-1}g_i^{(-1)^i}\notin(\pm S)^n$ where $S\subset\mathrm{Aff}_\lambda(\mathbb{C})$ is the semigroup generated by the elements $g_0,g_n,g_{n+1},\dots,g_{2n-1}$. Assuming that this is not so we would find finite sequences $m_1,\dots,m_l\in\{n,\dots,2n-1\}$ and $\varepsilon_1,\dots,\varepsilon_l\in\{-1,1\}$ for some $l\in\mathbb{N}$ such that

$$\prod_{i=n}^{2n-1}g_i^{(-1)^i}=\prod_{i=1}^{l}g_{m_i}^{\varepsilon_i}$$

and the sequence $\varepsilon_1,\dots,\varepsilon_l$ is n-oscillating in the sense that there are numbers $0=l_0\le l_1\le\dots\le l_n=l$ such that for any $i\in\{1,\dots,n\}$ and $j\in(l_{i-1},l_i]$ we get $\varepsilon_j=1$ if i is odd and $\varepsilon_j=-1$ if i is even.

Observe that

$$\prod_{i=n}^{2n-1}g_i^{(-1)^i}=\prod_{i=n}^{2n-1}(\delta_i,\lambda^i)^{(-1)^i}=\Big(\sum_{i=n}^{2n-1}(-1)^i\delta_i\lambda^{\frac{i}{2}((-1)^i-1)+\sum_{j=n}^{i-1}(-1)^j j},\lambda^{\sum_{i=n}^{2n-1}(-1)^i i}\Big)$$

and

$$\prod_{i=1}^{l}g_{m_i}^{\varepsilon_i}=\prod_{i=1}^{l}(\delta_{m_i},\lambda^{m_i})^{\varepsilon_i}=\Big(\sum_{i=1}^{l}\varepsilon_i\delta_{m_i}\lambda^{\frac{m_i}{2}(\varepsilon_i-1)+\sum_{j=1}^{i-1}\varepsilon_j m_j},\lambda^{\sum_{i=1}^{l}\varepsilon_i m_i}\Big).$$

The equality $\prod_{i=n}^{2n-1}g_i^{(-1)^i}=\prod_{i=1}^{l}g_{m_i}^{\varepsilon_i}$ implies $\sum_{i=1}^{l}\varepsilon_i m_i=\sum_{i=n}^{2n-1}(-1)^i i$ and $P(\lambda)=0$ where

$$P(\lambda)=-\sum_{i=n}^{2n-1}(-1)^i\delta_i\lambda^{\frac{i}{2}((-1)^i-1)+\sum_{j=n}^{i-1}(-1)^j j}+\sum_{i=1}^{l}\varepsilon_i\delta_{m_i}\lambda^{\frac{m_i}{2}(\varepsilon_i-1)+\sum_{j=1}^{i-1}\varepsilon_j m_j}=\sum_{i=p}^{q}a_i\lambda^i$$

for some integers $p<q$ and some integer coefficients a_i with $|a_i|\le n$ for $p\le i\le q$ (the upper bound $|a_i|\le n$ follows from the n-oscillation nature of the sequence (ε_i) and the strict positivity of the numbers m_i). We claim that all the coefficients a_i are zero. Assuming the converse we could suppose that $a_q\ne 0$. Multiplying $P(\lambda)$ by λ^{-p} we would get $0=\lambda^{-p}P(\lambda)=\sum_{i=p}^{q}a_i\lambda^{i-p}=\sum_{i=0}^{q-p}a_{i+p}\lambda^i$ and thus

$$|\lambda|^{q-p}=\Big|\sum_{i=0}^{q-p-1}\frac{a_{i+p}}{a_q}\lambda^i\Big|\le\sum_{i=0}^{q-p-1}\Big|\frac{a_{i+p}}{a_q}\Big|\cdot|\lambda|^i\le n\sum_{i=0}^{q-p-1}|\lambda|^i=n\frac{|\lambda|^{q-p}-1}{|\lambda|-1}\le|\lambda|^{q-p}-1<|\lambda|^{q-p}$$

which is a contradiction. Thus all the coefficients a_i are zero.

This implies that for every odd number $r\in\{n,\dots,2n-1\}$ there is a number $i_r\in\{1,\dots,l\}$ such that $\varepsilon_{i_r}=-1$ and $-r+\sum_{j=n}^{r-1}(-1)^j j=-m_{i_r}+\sum_{j=1}^{i_r-1}\varepsilon_j m_j$ which yields

$$\sum_{j=1}^{i_r}\varepsilon_j m_j=\sum_{j=n}^{r}(-1)^j j=\frac{r-n}{2}-r=-\frac{n+r}{2}.$$

It follows that $i_r\ne i_s$ if $r,s\in\{n,\dots,2n-1\}$ are two distinct odd numbers. Since n is odd, be the Dirichlet principle there are two distinct odd numbers $r,s\in\{n,\dots,2n-1\}$ such that $i_r<i_s$ and $i_r,i_s\in(l_{p-1},l_p]$ for some even $p\in\{1,\dots,n\}$. Then $\varepsilon_j=-1$ for any $j\in[i_r,i_s]$ and thus

$$-\sum_{i_r<j\le i_s}m_i=\sum_{j=1}^{i_s}\varepsilon_j m_j-\sum_{j=1}^{i_r}\varepsilon_j m_j=\sum_{j=1}^{r}(-1)^j j-\sum_{j=1}^{s}(-1)^j j=-\frac{n+r}{2}+\frac{n+s}{2}=\frac{s-r}{2}.$$

Now the absurd conclusion $n \le \big|\sum_{i_r<j\le i_s} m_j\big| = \big|\frac{s-r}{2}\big| < n$ finishes the proof of the first statement of Theorem 5.1.

2) Now assume that a group G is (n,m)-reversive for some $n,m\in\mathbb{N}$. Without loss of generality we can assume that n is an odd number. Consider the number $l = 2mn(2mn+1)^{(2mn+1)^n}$. We claim that $\sigma(G)\subset\sqrt[l!]{1}$. Assuming the converse we would find an eigenvalue $\lambda\in\sigma(G)$ such that $\lambda^{l!}\ne 1$. By Theorem 4.3, the group $\mathrm{Aff}_\lambda(\mathbb{C})$ is isomorphic to a quotient group of some subgroup of G. Since the (n,m)-reversivity is a hereditary property, we conclude that the group $\mathrm{Aff}_\lambda(\mathbb{C})$ is (n,m)-reversive. We shall show that this is not so.

Let $M=\{i\in\mathbb{Z}:|i|\le mn\}$. Given $k\in\mathbb{N}$ and two vectors $\vec{x},\vec{y}\in\mathbb{Z}^k$ let $(\vec{x}\cdot\vec{y})=\sum_{i=1}^k x_iy_i$ denote their inner product and let $\|\vec{x}\|=\max_{1\le i\le k}|x_i|$.

Claim. *For every $k\le n$ there is a vector $\vec{x}\in\mathbb{Z}^k$ with $\|\vec{x}\|\le l$ such that for any vector $\vec{a}=(a_{\vec{v}})_{\vec{v}\in M^k}\in M^{M^k}$ the equality $\sum_{\vec{v}\in M^k} a_{\vec{v}}\lambda^{(\vec{v}\cdot\vec{x})}=0$ holds if and only if $\vec{a}=\vec{0}$.*

Proof. This claim will be proven by finite induction on $k\le n$. For $k=0$ the Claim is trivial. Assume that for some $k<n$ a number vector $\vec{x}\in\mathbb{Z}^k$ with $\|\vec{x}\|\le l$ is constructed so that $\sum_{\vec{v}\in M^k} a_{\vec{v}}\lambda^{(\vec{v}\cdot\vec{x})}\ne 0$ for any non-zero vector $(a_{\vec{v}})_{\vec{v}\in M^k}\in M^{M^k}$. For any non-zero vector $\vec{a}=(a_{(\vec{v},v)})_{(\vec{v},v)\in M^k\times M}\in M^{(M^k\times M)}$ consider the polynomial

$$P_{\vec{a}}(z)=\sum_{v=-mn}^{mn}\Big(\sum_{\vec{v}\in M^k} a_{(\vec{v},v)}\lambda^{(\vec{v}\cdot\vec{x})}\Big)z^{v+mn}$$

of degree at most $2mn$. By the choice of the vector $\vec{x}$, $\sum_{\vec{v}\in M^k} a_{(\vec{v},v)}\lambda^{(\vec{v}\cdot\vec{x})}\ne 0$ for some $v\in M$ and thus the polynomial $P_{\vec{a}}(z)$ is non-trivial and has at most $2mn$ roots. Since the numbers $\lambda^0,\lambda^1,\dots,\lambda^l$ are pairwise distinct and

$$l\ge 2mn|M^{M^{k+1}}| = 2mn(2mn+1)^{(2mn+1)^{k+1}},$$

there is an integer $x\in\{0,\dots,l\}$ such that $P_{\vec{a}}(\lambda^x)\ne 0$ for any non-zero vector $\vec{a}\in M^{M^{k+1}}$. Put $\vec{y}=(\vec{x},x)\in\mathbb{Z}^k\times\mathbb{Z}=\mathbb{Z}^{k+1}$ and observe that for any non-zero vector $\vec{a}=(a_{(\vec{v},v)})_{(\vec{v},v)\in M^k\times M}\in M^{M^{k+1}}$ we have

$$\sum_{(\vec{v},v)\in M^k\times M} a_{(\vec{v},v)}\lambda^{((\vec{v},v)\cdot\vec{y})}=\sum_{v\in M}\sum_{\vec{v}\in M^k} a_{(\vec{v},v)}\lambda^{(\vec{v},\vec{x})+vx}=$$
$$=\sum_{v\in M}\Big(\sum_{\vec{v}\in M^k} a_{(\vec{v},v)}\lambda^{(\vec{v},\vec{x})}\Big)\lambda^{xv}=\lambda^{-mnx}P_{\vec{a}}(\lambda^x)\ne 0.$$

This completes the inductive step as well as the proof of Claim. □

Using the above Claim, fix a number vector $\vec{k}=(k_1,\dots,k_n)\in\mathbb{Z}^n$ with $\|\vec{k}\|\le l$ such that $\sum_{\vec{v}\in M^n} a_{\vec{v}}\lambda^{(\vec{v}\cdot\vec{k})}\ne 0$ for any non-zero vector $(a_{\vec{v}})_{\vec{v}\in M^n}\in M^{M^n}$. Observe that the numbers $k_1,\dots,k_n$ are pairwise distinct. Let $K=\{k_1,\dots,k_n\}$ and $\mathcal{K}=\{\sum_{\vec{v}\in M^n} a_{\vec{v}}\lambda^{(\vec{v}\cdot\vec{k})}:(a_{\vec{v}})_{\vec{v}\in M^n}\in M^{M^n}\}$.

Consider the subgroups $H=\{z\in\mathbb{C}:(z,1)\in T_\lambda(\mathbb{C})\}$ of $\mathbb{C}$ and its power H^K. The group H^K, being abelian, is amenable and thus admits an invariant probability finitely-additive measure μ. It is easy to see that for any distinct vectors $\vec{a},\vec{b}\in\mathbb{C}^K$ the "hyperplane" $\Gamma(\vec{a},\vec{b})=\{\vec{x}\in H^K:(\vec{x}\cdot\vec{a})=(\vec{x}\cdot\vec{b})\}$ is a subgroup of infinite index in H^K. Then $\mu(\Gamma(\vec{a},\vec{b}))=0$ and consequently, $H^K\ne\cup\{\Gamma(\vec{a},\vec{b}):\vec{a}\ne\vec{b}$ and $\vec{a},\vec{b}\in\mathcal{K}^K\}$. Pick any vector $\vec{x}\in H^K$ such that $(\vec{x}\cdot\vec{a})\ne(\vec{x}\cdot\vec{b})$ for any distinct vectors $\vec{a},\vec{b}\in\mathcal{K}^K$. Write $\vec{x}=(x_{k_1},\dots,x_{k_n})\in H^K$.

To prove that $\mathrm{Aff}_\lambda(\mathbb{C})$ fails to be (n,m)-reversive, we shall show that $(\mp A)^n \not\subset (\pm A^m)^n$ where $A = \{(0,1),(x_{k_i},\lambda^{k_i}) : 1 \le i \le n\}$. Namely, $\prod_{i=1}^n (x_{k_i},\lambda^{k_i})^{(-1)^i} \not\subset (\pm A^m)^n$. Assuming the converse, we would find number sequences $m_1,\dots,m_q \in \{k_1,\dots,k_n\}$ and $\varepsilon_1,\dots,\varepsilon_q \in \{-1,1\}$ for some $q \le nm$ such that

$$\prod_{i=1}^n (x_{k_i},\lambda^{k_i})^{(-1)^i} = \prod_{i=1}^q (x_{m_i},\lambda^{m_i})^{\varepsilon_i}$$

and the sequence $\varepsilon_1,\dots,\varepsilon_q$ is n-oscillating in the sense that there are numbers $0 = q_0 \le q_1 \le \dots \le q_n = l$ such that for any $i \in \{1,\dots,n\}$ and $j \in (q_{i-1},q_i]$ we get $\varepsilon_j = 1$ if i is odd and $\varepsilon_j = -1$ if i is even.

Observe that

$$\prod_{i=1}^n (x_{k_i},\lambda^{k_i})^{(-1)^i} = \Big(\sum_{i=1}^n (-1)^i x_{k_i}\lambda^{\frac{k_i}{2}((-1)^i-1)+\sum_{j=1}^{i-1}(-1)^j k_j}, \lambda^{\sum_{i=1}^n (-1)^i k_i}\Big)$$

while

$$\prod_{i=1}^q (x_{m_i},\lambda^{m_i})^{\varepsilon_i} = \Big(\sum_{i=1}^q \varepsilon_i x_{m_i}\lambda^{\frac{m_i}{2}(\varepsilon_i-1)+\sum_{j=1}^{i-1}\varepsilon_j m_j}, \lambda^{\sum_{i=1}^q \varepsilon_i m_i}\Big) = \Big(\sum_{r=1}^n f_r x_{k_r}, \lambda^{\sum_{i=1}^q \varepsilon_i m_i}\Big)$$

where

$$f_r = \sum_{m_i = k_r} \varepsilon_i \lambda^{\frac{m_i}{2}(\varepsilon_i-1)+\sum_{j=1}^{i-1}\varepsilon_j m_j} \in \mathcal{K} \quad \text{for } 1 \le i \le n.$$

It follows that $\sum_{i=1}^n f_i x_{k_i} = \sum_{i=1}^n (-1)^i x_{k_i}\lambda^{\frac{k_i}{2}((-1)^i-1)+\sum_{j=1}^{i-1}(-1)^j k_j}$. By the choice of the vector $\vec{x}$, we get $f_i = (-1)^i \cdot \lambda^{\frac{k_i}{2}((-1)^i-1)+\sum_{j=1}^{i-1}(-1)^j k_j}$ for every $i \le n$.

The last equality and the choice of the vector $\vec{k}$ imply that for every odd number $r \in \{1,\dots,n\}$ there is a number $i_r \in \{1,\dots,q\}$ such that $m_{i_r} = k_r$, $\varepsilon_{i_r} = -1$, and $-m_{i_r} + \sum_{j=1}^{i_r-1}\varepsilon_j m_j = -k_r + \sum_{j=1}^{r-1}(-1)^j k_j$ which yields $\sum_{j=1}^{i_r}\varepsilon_j m_j = \sum_{j=1}^r (-1)^j k_j$. It follows from the choice of the vector $\vec{k}$ that $\sum_{j=1}^r (-1)^j k_j \ne \sum_{j=1}^s (-1)^j k_j$ for distinct numbers $r,s \in \{1,\dots,n\}$. Since n is odd by the Dirichlet principle there are two distinct odd numbers $r,s \in \{1,\dots,n\}$ such that $i_r < i_s$ and $i_r, i_s \in (q_{p-1},q_p]$ for some even $p \in \{1,\dots,n\}$. Then $\varepsilon_j = -1$ for any $j \in [i_r,i_s]$ and thus

$$-\sum_{i_r<j\le i_s} m_j = \sum_{j=1}^{i_s}\varepsilon_j m_j - \sum_{j=1}^{i_r}\varepsilon_j m_j = \sum_{j=1}^r (-1)^j k_j - \sum_{j=1}^s (-1)^j k_j.$$

This implies that $\sum\limits_{\min\{r,s\}<j\le\max\{r,s\}} (-1)^j k_j = \pm\sum_{i_r<j\le i_s} m_j$ which is not possible according to the choice of the vector $\vec{k}$. This contradiction finishes the proof of the second statement of Theorem 5.1. □

It is interesting to notice that $(3,\infty)$- and $(2,\infty)$-reversive groups can be characterized as groups containing no free semigroup with two generators.

Theorem 5.2. *A group G is $(3,\infty)$-reversive if and only if G is $(2,\infty)$-reversive if and only if G contains no free semigroup with two generators.*

Proof. 1) Suppose that G contains no free semigroup with two generators. To show that G is $(2,\infty)$-reversive we have to verify that $A^{-1}A \subset A^\infty \cdot (A^\infty)^{-1}$ for any subset $A \subset G$ containing the unit of G. Fix any two distinct elements $x,y \in A$. Since the semigroup $S \subset A^\infty$ generated by $\{x,y\}$ is not free, there are two different words w,v in the alphabet

$\{x, y\}$ such that $w = v$ in G. After the left cancellation, we can assume that w and v begin with different letters, say $w = xw_1$ and $v = yv_1$ (where the words w_1, v_1 can be empty). Then $xw_1 = yv_1$ and thus $x^{-1}y = w_1v_1^{-1} \in SS^{-1} \subset A^\infty \cdot (A^\infty)^{-1}$. It follows that $A^{-1}A \subset A^\infty \cdot (A^\infty)^{-1}$, i.e., G is $(2, \infty)$-reversive.

2) Next, assume that the group G is $(2, \infty)$-reversive. To show that G is $(3, \infty)$-reversive, fix any three points $x, y, z \in G$. We have to verify that $x^{-1}yz^{-1} \in SS^{-1}S$ where S is the semigroup generated by the set $\{e, x, y, z\}$. Since the group G is $(2, \infty)$-reversive, $x^{-1}y = wv^{-1} \in SS^{-1}$, where w, v are two words in the alphabet $\{x, y\}$. Then $x^{-1}yz^{-1} = wv^{-1}z^{-1} \in SS^{-1} \subset SS^{-1}S$.

3) Assume finally that a group G is $(3, \infty)$-reversive. We shall show that G contains no free semigroup with two generators. Fix arbitrary two points $x, z \in G$. Without loss of generality, the elements x and z are distinct and differ from the unit e of G. Take any element $y \in G \setminus \{e, x, z\}$ and consider the set $A = \{e, x, y, z\}$. It follows from the $(3, \infty)$-reversivity of G that $x^{-1}yz^{-1} = u(x, y, z)v^{-1}(x, y, z)w(x, y, z)$ in G for some (possibly empty) words $u(x, y, z)$, $v(x, y, z)$, $w(x, y, z)$ in the alphabet $\{x, y, z\}$. After inversion, we get $zy^{-1}x = w^{-1}(x, y, z)v(x, y, z)u^{-1}(x, y, z)$ and $w(x, y, z)zy^{-1}xu(x, y, z) = v(x, y, z)$. It can be shown that either the word $w(x, x, z)zu(x, x, z)$ differs from $v(x, x, z)$ or the word $w(x, z, z)xu(x, z, z)$ differs from $v(x, z, z)$. In any case we find two different words in the alphabet $\{x, z\}$ which yield equal elements in G. This implies that the semigroup generated by the elements x, z in G is not free. □

Using the previous characterization we shall show that for $(3, \infty)$-reversive groups Theorem 5.1(1) holds in a more strong form, which is close to Theorem 5.1(2).

Corollary 5.3. *$\sigma(G) \subset \sqrt[\infty]{1}$ for any $(3, \infty)$-reversive group G.*

Proof. Assume that G is a $(3, \infty)$-reversive group and fix any $\lambda \in \sigma(G)$. By Theorem 4.3, the group $\mathrm{Aff}_\lambda(\mathbb{C})$ is isomorphisc to a quotient group of a subgroup of G. Taking into account that the $(3, \infty)$-reversivity is a hereditary property, we conclude that the group $\mathrm{Aff}_\lambda(\mathbb{C})$ is $(3, \infty)$-reversive and thus contains no free semigroup with two generators according to Theorem 5.2. Applying finally Theorem 4.1(6), we get $\lambda \in \sigma(\mathrm{Aff}_\lambda(\mathbb{C})) \subset \sqrt[\infty]{1}$. □

Next, we characterize $(3, m)$- and $(2, m)$-reversive groups. Following [Sh], [SS] we define a group G to be *(n, m)-collapsing* if $\mathrm{Card}(A^m) < (\mathrm{Card}(A))^m$ for any n-element subset $A \subset G$. A group G is *collapsing* if it is (n, m)-collapsing for some $n, m \in \mathbb{N}$. It is easy to see that each (n, m)-collapsing group is (p, q)-collapsing for each $n \le p < \infty$, $m \le q < \infty$. Conversely, for every $n, m \in \mathbb{N}$ there is $l \in \mathbb{N}$ such that each (n, m)-collapsing group is $(2, l)$-collapsing, see [Mac]. The class of collapsing groups is quite wide: it contains all groups with positive laws, in particular, all virtually nilpotent groups, see [SS], [Sh] or [Mac].

Theorem 5.4. *A group G is collapsing if and only if it is $(2, m)$-reversive for some $m \in \mathbb{N}$ if and only if G is $(3, m)$-reversive for some $m \in \mathbb{N}$. More precisely, for every $m \in \mathbb{N}$ each $(2, m)$-collapsing group is $(2, m)$-reversive, each $(2, m)$-reversive group is $(3, m + 1)$-reversive, and each $(3, m)$-reversive group is $(2, 4m + 2)$-collapsing.*

Proof. 1) First, assume that G is a $(2, m)$-collapsing group for some $m \in \mathbb{N}$. To prove that G is $(2, m)$-reversive, it suffices to show that $x^{-1}y \in A^m \cdot A^{-m}$ for each three element subset $A = \{x, y, e\}$ of G. Since the group G is $(2, m)$-collapsing, $\mathrm{Card}(\{x, y\}^m) < 2^m$ which yields that there are two different m-letter words w, v in the alphabet $\{x, y\}$ such that $w = v$ in G. After left cancellation, we can assume that w and v begin with different letters, say $w = xw_1$, $v = yv_1$ and have length $\le m$. Then $xw_1 = yv_1$ and thus $x^{-1}y = w_1v_1^{-1} \in A^mA^{-m}$.

2) Next, assume that a group G is $(2, m)$-reversive. To prove that G is $(3, m+1)$-reversive, it suffices given three elements $x, y, z \in G$ to find three words u, v, w of length $\le m+1$ in the alphabet $\{x, y, z\}$ such that $x^{-1}yz^{-1} = uv^{-1}w$ in G. Since G is $(2, m)$-reversive there are two words a, b of length $\le m$ in the alphabet $\{x, y\}$ such that $x^{-1}y = ab^{-1}$ in G. Let $u = a$, $v = zb$ and $w = \emptyset$. Then u, w, v are words of length $\le m+1$ in the alphabet $\{x, y, z\}$ such that $x^{-1}yz^{-1} = ab^{-1}z^{-1} = uv^{-1}w$ which yields that the group G is $(3, m+1)$-reversive.

3) Finally, assume that the group G is $(3, m)$-reversive. We shall show that G is $(2, 4m+2)$-collapsing. Fix arbitrary two points $x, z \in G$. Using the $(3, m)$-reversivity of G and repeating the argument of Theorem 5.2, we could find two diferent words w, v of lenght $\le 2m+1$ in the alphabet $\{x, z\}$ such that $w = v$ in G. After the left cancellation we can assume that the words w, v begin with different letters. Then wv and vw are two different word of the same lenght $\le 4m+2$ which are equal in G. This implies that the group G is $(2, 4m+2)$-reversive. □

Theorems 5.1 and 5.4 imply that collapsing groups have finite spectrum.

Corollary 5.5. *If G is a $(2, m)$-collapsing group for some $m \in \mathbb{N}$, then $\sigma(G) \subset \sqrt[l]{1}$ for some $l \in \mathbb{N}$ depending only on m.*

Finally, using Theorem 5.1 we characterize (n, m)-reversive polycyclic groups.

Corollary 5.6. *For a polycyclic group G the following conditions are equivalent:*

(1) *G is virtually nilpotent;*
(2) *$\sigma(G) \subset \sqrt[n]{1}$ for some n;*
(3) *$\sigma(G) \subset \mathbb{T}$;*
(4) *G is (n, ∞)-reversive for some $n \in \mathbb{N}$;*
(5) *G is $(2, \infty)$-reversive;*
(6) *G is $(2, m)$-reversive for some $m \in \mathbb{N}$;*
(7) *G is collapsing;*
(8) *G contains no free semigroup with two generators;*
(9) *G has polynomial growth.*

Proof. The equivalences (5) $\Leftrightarrow$ (8) and (6) $\Leftrightarrow$ (7) follow from Theorems 5.2 and 5.4, respectively. The equivalence (1) $\Leftrightarrow$ (9) is a partial case of Gromov Theorem [Gr] while the implication (1) $\Rightarrow$ (7) is proven in [SS] (see also [Sh] and [Mac]).

So it rests to prove the implications (6) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1). The first two implications are trivial, the third one follows from Theorem 5.1(1) and the last one from Theorem 3.3(3).

To prove the implication (3) $\Rightarrow$ (2), assume that $\sigma(G) \subset \mathbb{T}$. Taking into account that G is polycyclic we get the inclusion $\sigma(G) \subset \mathbb{A} \cap \mathbb{T}$. Then for any eigenvalue $\lambda \in \sigma(G)$ all of its algebraically conjugated belong to $\sigma(G)$ and thus have absolute value 1. By the Kronecker Theorem (see [**?**, 3.2] or [Ro, p.49]), λ is a root of 1. Thus $\sigma(G) \subset \sqrt[\infty]{1}$. Next, by Theorem 3.4, G contains a subgroup H of finite index whose spectrum $\sigma(H)$ is finitely splittable. Since H is finitely generated, $\sigma(H) \subset \sqrt[\infty]{1}$ is a finite union of finitely generated subgroups of the multiplicative group $\sqrt[\infty]{1}$. Consequently, the spectrum $\sigma(H)$ of H is finite and $\sigma(H) \subset \sqrt[n]{1}$ for some $n \in \mathbb{N}$. Finally, applying Theorem 3.1(9), we get $\sigma(G) \subset \sqrt[m]{\sigma(H)} \subset \sqrt[nm]{1}$ for some $m \in \mathbb{N}$. □

6. Some comments and open problems

Now it is time to reveal the genuine nature of eigenvalues of a group automorphism $A : G \to G$. In fact, they are the usual eigenvalues of the conjugated linear operator $A^\sharp$ acting on the linear inverse semigroup $\mathrm{PHom}(G, \mathbb{C})$ of partial homomorphisms from G into the field $\mathbb{C}$. Under a *partial homomorphism* from a group G into a group K we understand a homomorphism $\varphi : D(\varphi) \to K$ defined on some subgroup $D(\varphi)$ of G. If K is a commutative group, then the set $\mathrm{PHom}(G, K)$ of partial homomorphisms from G into K carries the structure of a commutative inverse semigroup.

We remind that a semigroup S is called an *inverse semigroup* if for each $x \in S$ there is a unique element x^{-1}, called the *inverse element* to x, such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$, see [CP]. Each group is an inverse semigroup; conversely, an inverse semigroup is a group if and only if it has a unique idempotent. We remind that an element $x \in S$ is an *idempotent* if $xx = x$.

Suppose G is a group and $(K, +)$ is an abelian group. For two partial homomorphisms $\varphi, \psi \in \mathrm{PHom}(G, K)$ let $\varphi+\psi$ be the partial homomorphism with $D(\varphi+\psi) = D(\varphi)\cap D(\psi)$ such that $(\varphi + \psi)(x) = \varphi(x) + \psi(x)$ for each $x \in D(\varphi + \psi)$. It is easy to verify that this operation "+" turns $\mathrm{PHom}(G, K)$ into a commutative inverse semigroup. The inverse element to a partial homomorphism $\varphi : D(\varphi) \to K$ is the homomorphism $-\varphi : D(\varphi) \to K$. A partial homomorphism $\varphi \in \mathrm{PHom}(G, K)$ is an idempotent in $\mathrm{PHom}(G, K)$ if and only if $\varphi(D(\varphi)) = \{0\} \subset K$. Thus the semigroup of idempotents of $\mathrm{PHom}(G, K)$ can be identified with the semilattice of subgroups of G endowed with the operation of intersection (this semilattice was intensively studied in Lattice Theory, see [Bi] or [Schm]).

Problem 6.1. *To which extent the properties of the inverse semigroup* $\mathrm{PHom}(G, K)$ *determine the structure of the group G? In particular, are two groups G, H isomorphic if the inverse semigroups* $\mathrm{PHom}(G, K)$ *and* $\mathrm{PHom}(H, K)$ *are isomorphic for every abelian group K?*

Let us remark that even for the trivial group $K = \{0\}$ the inverse semigroup $\mathrm{PHom}(G, \{0\})$ (which can be identified with the lattice of subgroups of G) carries a non-trivial information on the structure of a group G, see [Bi] or [Schm]. For example, a group G is solvable if $\mathrm{PHom}(G, \{0\})$ is isomorphic to $\mathrm{PHom}(H, \{0\})$ for some solvable group H, see [Ya].

If K is a ring, then besides the additive operation, the semigroup $\mathrm{PHom}(G, K)$ possesses an external operation of multiplication by scalars $(\cdot) : K \times \mathrm{PHom}(G, K) \to \mathrm{PHom}(G, K)$, $(\cdot) : (\lambda, \varphi) \mapsto \lambda \cdot \varphi$. In this case $\mathrm{PHom}(G, K)$ carries an algebraic structure which can be called the structure of an *inverse K-module*.

Under a *linear operator* on $\mathrm{PHom}(G, K)$ we shall understand a semigroup homomorphism $A : \mathrm{PHom}(G, K) \to \mathrm{PHom}(G, K)$ such that $A(\lambda \cdot \varphi) = \lambda \cdot A(\varphi)$ for any $\lambda \in K$ and $\varphi \in \mathrm{PHom}(G, K)$. A partial homomorphism $\varphi \in \mathrm{PHom}(G, K)$ is called an *eigenvector* of a linear operator $A : \mathrm{PHom}(G, K) \to \mathrm{PHom}(G, K)$ if $\varphi + \varphi \neq \varphi$ and $A(\varphi) = \lambda \cdot \varphi$ for some scalar $\lambda \in K$ called the *eigenvalue* corresponding to the eigenvector φ.

Each automorphism $A : G \to G$ of the group G induces a linear operator $A^\sharp : \mathrm{PHom}(G, K) \to \mathrm{PHom}(G, K)$ assigning to a partial homomorphism $\varphi \in \mathrm{PHom}(G, K)$ the partial homomorphism $\varphi \circ A : A^{-1}(D(\varphi)) \to K$ defined on the subgroup $A^{-1}(D(\varphi))$ of G.

Observe that a partial homomorphism $\varphi \in \mathrm{PHom}(G, K)$ is an eigenvector of the operator $A^\sharp$ if and only if $A(D(\varphi)) = D(\varphi)$ and $A^\sharp(\varphi) = \varphi \circ A|D(\varphi) = \lambda \cdot \varphi$ for some $\lambda \in K$.

Now we see that a complex number $\lambda \in \mathbb{C}$ is an eigenvalue of a group automorphism $A : G \to G$ if and only if λ is an eigenvalue of the induced linear operator $A^\sharp : \mathrm{PHom}(G, K) \to \mathrm{PHom}(G, K)$. This observation allows us to generalize the notion of an eigenvalue of a group automorphism as follows.

An element λ of a ring K is called a *K-eigenvalue* of a group automorphism $A : G \to G$ if λ is an eigenvalue of the induced linear operator $A^\sharp : \mathrm{PHom}(G, K) \to \mathrm{PHom}(G, K)$ (the latter means that there is a non-zero homomorphism $\varphi : H \to K$ defined an A-invariant subgroup $H \subset G$ such that $\varphi \circ A|H = \lambda \cdot \varphi$). The set of all K-eigenvalues of A is called the *K-spectrum* of A and is denoted by $\sigma_K(A)$.

Under the *K-spectrum* of a group G we understand the union $\sigma_K(G) = \bigcup_{g\in G} \sigma_K(i^g)$ of the K-spectra of all the inner automorphisms of G. It is clear that $\sigma_K(G) = \sigma_K(G')$ for any isomorphic groups G, G'.

In this terminology the spectra considered in the previous sections are nothing else but $\mathbb{C}$-spectra. As we saw, the $\mathbb{C}$-spectrum carries no information on the structure of a periodic group. It may happen that the situation in more fortunate for $\mathbb{K}$-spectra where $\mathbb{K}$ is a field of finite characteristic.

Problem 6.2. *What can be said about the $\mathbb{K}$-spectrum of a (periodic) group for a field $\mathbb{K}$ of finite characteristic.*

According to Theorem 4.2 the spectrum completely determines the algebraic structure of the groups $\mathrm{Aff}_\lambda(\mathbb{C})$. It may happen that the spectrum characterizes these groups in the class of torsion-free solvable groups with spectrally minimal Hirsch rank. We shall say that a solvable group G has *spectrally minimal Hirsch rank* if $h(G) = \min\{h(H) : H$ is a solvable group with $\sigma(H) = \sigma(G)\}$.

Question 6.3. *Suppose that G is a solvable torsion-free group with spectrally minimal Hirsch rank. Is G isomorphic to $\mathrm{Aff}_\lambda(\mathbb{C})$ if $\sigma(G) = \sigma(\mathrm{Aff}_\lambda(\mathbb{C}))$?*

As we saw in Theorem 4.1, the condition $\lambda \in \mathbb{A}^*_\mathbb{Z} \cup (\mathbb{A}^*_\mathbb{Z})^{-1}$ is responsible for the finite presentability of the group $\mathrm{Aff}_\lambda(\mathbb{C})$.

Problem 6.4. *What can be said about the spectrum of a finitely presented group? In particular, is a torsion-free finitely generated solvable group G with $\sigma(G) \subset \mathbb{A}^*_\mathbb{Z} \cup (\mathbb{A}^*_\mathbb{Z})^{-1}$ finitely presented?*

Another open problem concerns the interplay between the spectrum and the asymptotic geometry of a finitely generated group G. The asymptotic geometry studies the properties of groups preserved by quasi-isometries, see [Gro] and [Har]. Under a *quasi-isometry* between metric spaces (X, d_X) and (Y, d_Y) we understand a map $f : X \to Y$ satisfying the following two conditions for some constant $C \ge 1$: (i) $\frac{1}{C} d_X(x, x') - C \le d_Y(f(x), f(x')) \le C d_X(x, x') + C$ for any $x, x' \in X$ and (ii) $d_Y(y, f(X)) < C$ for any $y \in Y$.

Any group G generated by a finite set X is endowed with the word metric $d_X(a, b) = \min\{n \in \omega : a^{-1}b \in (X \cup \{e\} \cup X^{-1})^n\}$. It is known that for any finite generating subsets $X, Y \subset G$ the identity map $(G, d_X) \to (G, d_Y)$ is a quasi-isometry. Hence we can say about asymptotic properties of a group with no referring to a particular finite generating set of G. In [FM] the reader can find conditions on integer algebraic numbers $\lambda, \mu \in \mathbb{A}^*_\mathbb{Z}$ under which the (finitely presented) test groups $\mathrm{Aff}_\lambda(\mathbb{C})$ and $\mathrm{Aff}_\mu(\mathbb{C})$ are quasi-isometric.

Problem 6.5. *To which extent the spectrum determines the asymptotic geometry of a finitely generated group G? In particular, are two polycyclic groups G and H quasi-isometric if they have spectrally minimal Hirsch ranks and $\sigma(G) = \sigma(H)$?*

Another open problem concerns the algebraic structure of the spectrum of a solvable group. According to Theorem 3.4, each (finitely generated) solvable A_1-group G contains a subgroup H of finite index whose spectrum $\sigma(H)$ is a finite union of (finitely generated) subgroup of $\mathbb{C}^*$.

Question 6.6. *Is the spectrum $\sigma(G)$ of a (finitely generated) solvable A_1-group G a finite union of (finitely generated) subgroups of $\mathbb{C}^*$?*

Finally let us discuss (n,∞)-reversive groups. Theorem 5.1 shows that $\sigma(G) \subset \mathbb{A} \cap \mathbb{T}$ for any such a group G. On the other hand, for any $(3,\infty)$-reversive group G a stronger inclusion holds: $\sigma(G) \subset \sqrt[\infty]{1}$, see Corollary 5.3.

Problem 6.7. *Is $\sigma(G) \subset \sqrt[\infty]{1}$ for any (n,∞)-reversive group G, where $n \in \mathbb{N}$?*

In fact, except for the case $n = 1$ we know no example distinguishing between the (n,∞)-reversive and $(n+1,\infty)$-reversive groups. Theorem 5.2 shows that no such an example exists for $n = 2$.

Problem 6.8. *Is there an (m,∞)-reversive group which is not (n,∞)-reversive for some $m > n \geq 2$. In particular, is the wreath product $(\mathbb{Z}/p\mathbb{Z}) \wr \mathbb{Z}$ (n,∞)-reversive for some $n, p \geq 2$? Is a test group $\mathrm{Aff}_\lambda(\mathbb{C})$ (n,∞)-reversive for some $n \in \mathbb{N}$ and $\lambda \notin \sqrt[\infty]{1}$?*

In fact the very term "(n,m)-reversive group" was suggested by the notion of a left (right) reversive semigroup, well-known in the Theory of Semigroups, see [CP]. We remind that a semigroup S is *left* (resp. *right*) *reversive* if for any elements $a, b \in S$ the intersection $aS \cap bS$ (resp. $Sa \cap Sb$) is not empty. If S is a subsemigroup of a group, then the latter condition is equivalent to $S^{-1}S \subset SS^{-1}$ (resp. $SS^{-1} \subset S^{-1}S$). Hence a group G is $(2,\infty)$-reversive if and only if any subsemigroup S of G is left reversive. An example of a left reversive semigroup which is not right reversive can be found in [CP, Ex.1 to §1.10]. This is the semigroup generated by the transformations $w = 2z$ and $w = z + 1$ in the test group $\mathrm{Aff}_2(\mathbb{C})$.

Problem 6.9. *Is it true that for every $n \in \mathbb{N}$ there is a subsemigroup S of a group G such that $(\mp S)^n \subset (\pm S)^n$ but $(\pm S)^n \not\subset (\mp S)^n$?*

A positive answer to this problem would imply a positive solution of Problem 1(4) from [BR] concerning n-oscillating paratopological groups.

The spectrum $\sigma(G)$ of a group G consists of eigenvalues of all the inner automorphisms of G and thus can be referred to as the *inner spectrum* of G. Besides this inner spectrum it is reasonable to consider also the *full spectrum* $\Sigma(G)$ consisting of eigenvalues of all the automorphisms of G. The inner and full spectra are related as follows:

$$\sigma(G) \subset \Sigma(G) \subset \sigma(\mathrm{Hol}(G))$$

where $\mathrm{Hol}(G)$ is the holomorph of the group G. Observe that the sets $\sigma(G)$, $\Sigma(G)$ and $\sigma(\mathrm{Hol}(G))$ can differ substantially. For example, $\sigma(\mathbb{Z}^2) = \{1\}$, $\Sigma(\mathbb{Z}^2) = \mathbb{A}_1^* \cap \mathbb{A}(2)$, while $\sigma(\mathrm{Hol}(\mathbb{Z}^2)) = \mathbb{C}^*$ (because the automorphism group of $\mathbb{Z}^2$ contains a free group with two generators, see [Sk, p.98]).

Problem 6.10. *Invesigate the interplay between properties of a group G and properties of its full spectrum $\Sigma(G)$.*

It seems that unlike to the inner spectrum the full spectrum can shed some light on the structure of abelian or nilpotent groups.

7. Acknowledgement

The author express his thanks to Orest Artemovych, Rostyslav Hryniv, Rostyslav Grigorchuk, Mykola Komarnytskyi, Igor Protasov, and Sasha Ravsky for valuable and stimulating discussions concerning the subject of the paper.

References

[Alp] R. Alperin, *Uniform growth of polycyclic groups*, Geometriae Dedicata 92 (2002), 1–9.

[BR] T. Banakh, O. Ravsky, *Oscillator topologies on a paratopological group and related number invariants*, In: Algebraic structures and their Applications, Institute of Math., Kyiv, 2002.

[BiS] R. Bieri, R. Strebel, *Almost finitely presented soluble groups*, Comment. Math. Helv. 53 (1978), 258–278.

[Bi] G. Birkhoff, *Lattice Theory*, Coll. Publ. **25**, Amer. Math. Soc., 1979.

[BoS] Z.I. Borevich, I.R. Shavarevich, *Theory of Numbers*, Nauka, Moskva, 1985.

[CP] A.H. Clifford, G.B. Preston, *The Algebraic Theory of Semigroups. I*, Amer. Math. Soc., 1961.

[FM] B. Farb, L. Mosher, *On the asymptotic geometry of abelian-by-cyclic groups*, Acta Math. **184** (2000), 145-202.

[Fu] L. Fuchs, *Infinite Abelian Groups*, Academic Press, NY, 1970.

[Gr] M. Gromov, *Groups of polynomial growth and expanding maps*, Publ. Math. I.H.E.S., No. 53 (1981), 53–78 (with appendix by J. Tits).

[Gro] M. Gromov, *Asymptotic invariants of infinite groups*, Geometric Group Theory (G.Niblo and M. Roller, eds.), LMS Lecture Notes, vol.182, Cambdidge Univ. Press, 1993.

[Har] P. de la Harpe, *Topics in geometric group theory*, The University of Chicago Press, 2000.

[KM] M.I.Kargapolov, Yu.I. Merzliakov, *Fundamentals of the theory of groups*, Nauka, Moskva, 1977 (in Russian).

[Ku] A.G. Kurosh, *Theory of Groups*, Moskva, Nauka, 1967 (in Russian).

[Mac] O. Macedońska, *Collapsing groups and positive laws*, Commun. Algebra, **8** (2000), 3661–3666.

[MKS] W. Magnus, A. Karras, D. Solitar, *Combinatorial Group Theory*, Intersci. Publ., NY, 1966.

[Mal] A.I. Malcev, *On certain classes of infinite solvable groups*, Mat. Sb. 28 (1951), 567–588.

[Ro] J. Rosenblatt. *Invariant measures and growth conditions*, Trans. Amer. Math. Soc. 193 (1974), 33–53.

[SS] J. Semple, A. Shalev, *Combinatorial conditions in residually finite groups, I*, J. Algebra **151** (1993), 43–50.

[Schm] R. Schmidt, *Subgroup Lattices of Groups*, De Gruyter Expositions in Math. **14**, de Gruyter, Berlin, 1994.

[Sh] A. Shalev, *Combinatorial conditions in residually finite groups, II*, J. Algebra. **151** (1993), 51–62.

[Sk] L. Skorniakov et al. *General Algebra*, Nauka, Nauka, 1990 (in Russian).

[War] B.L. Van der Waerden, *Algebra*, Nauka, Moskva, 1976 (in Russian).

[Ya] B.G. Yakovlev, *Lattice isomorphisms of solvable groups*, Algebra i Logika **9** (1970) 349–369 (in Russian).

E-mail address: `tbanakh@franko.lviv.ua`

Department of Mathematics, Ivan Franko Lviv National University, Universytetska 1, Lviv, 79000, Ukraine