THE DIRECT LIMIT OF METRIZABLE ANR’S 
IS AN ANR FOR STRATIFIABLE SPACES

TARAS BANAKH

Abstract. It is proven that the direct limit $\lim_{\rightarrow} X_n$ of a sequence $X_1 \subset X_2 \subset \ldots$ metrizable A(N)R’s is an A(N)R for stratifiable spaces.

The theory of absolute extensors is one of the principal tools in infinite-dimensional topology. Initially the infinite-dimensional topology studied manifolds modeled on nice metrizable spaces like the separable Hilbert space $\ell_2$ or the Hilbert cube $Q = [0, 1]^\omega$. But later its methods were used for studying infinite-dimensional manifolds modeled on certain non-metrizable model spaces like $\mathbb{R}^\infty$ or $Q^\infty$, see [BZa], [Sa].

Given a pointed topological space $(M, \ast)$ by $M^\infty$ we denote the set

$\{(x_i)_{i \in \omega} \in M^\omega : x_i = \ast \text{ for all but finitely many indices } i\}$

endowed with the strongest topology inducing the original (product) topology on each $n$-power $M^n = \{(x_i)_{i \in \omega} : x_i = \ast \text{ for all } i \geq n\}$. It should be mentioned that for a homogeneous space $M$ (like $\mathbb{R}$ or $Q$) the topological type of the space $M^\infty$ is independent of the choice of the fixed point $\ast \in M$.

The topology of the space $M^\infty$ is an example of a so-called inductive topology. We shall say that a topological space $X$ carries the inductive topology with respect to a cover $C$ of $X$ if the topology of $X$ is the strongest topology inducing the original topology on each space $C \in C$. In other words, if a subset $U \subset X$ is open (closed) in $X$ if and only if for each $C \in C$ the intersection $U \cap C$ is open (closed) in $C$. For example, a space $X$ is a $k$-space if it carries the inductive topology with respect to the cover of $X$ by compacta. A topological space $X$ will be called an $M_\omega$-space (resp. $k_\omega$-space) if it carries the inductive topology with respect to some countable cover by closed metrizable (resp. compact) subspaces. It can be shown that each $M_\omega$-space carries the inductive topology with respect to an increasing cover $\{X_n\}_{n \in \omega}$ by closed metrizable subspaces of $X$. In this case we shall say that $X$ is the direct limit of the sequence $X_1 \subset X_2 \subset \ldots$ and write $X = \lim_{\rightarrow} X_n$.

In particular, the spaces $\mathbb{R}^\infty$ and $Q^\infty$ are $k_\omega$ and $M_\omega$. It is well known that $\mathbb{R}^\infty$ equipped with its natural linear structure is a locally convex liner topological space while $Q^\infty$ is homeomorphic to a locally convex space (for example, to $\ell_2$ carrying the bounded-weak topology). In fact, according to [Ba_1], [Ba_2] (see also [BZd])

1991 Mathematics Subject Classification. 54C55, 54E20, 54B99, 54H12.

Key words and phrases. direct limit, metrizable space, stratifiable space, absolute neighborhood retract, absolute neighborhood extensor.

Typeset by AMS-TeX
each infinite-dimensional locally convex $\mathcal{M}_\omega$-space is homeomorphic either to $\mathbb{R}^\infty$ or to $\mathbb{Q}^\infty$. In this situation we can apply the Dugundji Theorem [Du] and conclude that the spaces $\mathbb{R}^\infty$ and $\mathbb{Q}^\infty$ are absolute extensors for metrizable spaces.

We recall that a topological space $X$ is an absolute (neighborhood) extensor for a class $\mathcal{C}$ of topological spaces (briefly $X$ is an A(N)R[$\mathcal{C}$]) if each continuous map $f : B \to X$ defined on a closed subspace $B$ of a space $C \in \mathcal{C}$ admits a continuous extension on the whole $C$ (on a neighborhood of $B$ in $C$). A topological space $X$ is called an absolute (neighborhood) retract for a class $\mathcal{C}$ (briefly $X$ is an A(N)R[$\mathcal{C}$]) if $X \in \mathcal{C}$ and $X$ is a (neighborhood) retract in each space $Y \in \mathcal{C}$ containing $X$ as a closed subspace. It is clear that a space $X \in \mathcal{C}$ is an A(N)R[$\mathcal{C}$] provided $X$ is an A(N)E[$\mathcal{C}$]. The converse is true for some classes $\mathcal{C}$, in particular for the class $\mathcal{M}$ of metrizable spaces, see [Hu].

The Dugundji Theorem mentioned above asserts that each convex subset $X$ of a locally convex linear topological space is an AE($\mathcal{M}$). This theorem was generalized by C.Borges [Bo] who proved that such an $X$ is an absolute extensor for the class $\mathcal{S}$ of stratifiable spaces. The class $\mathcal{S}$ of stratifiable spaces contains the class $\mathcal{M}$ of all metrizable spaces and (unlike to $\mathcal{M}$) is closed with respect to many countable topological operations, see [Bo], [Ce], [Gru, §5]. In particular, a space is stratifiable if it carries the inductive topology with respect to a countable closed cover by stratifiable subspaces. This implies that each $\mathcal{M}_\omega$-space is stratifiable. Hence the spaces $\mathbb{R}^\infty$ and $\mathbb{Q}^\infty$ being homeomorphic to stratifiable locally convex spaces are absolute retracts in the class of stratifiable spaces.

In [Pe] E.Pentsak started studying infinite-dimensional manifolds modeled on the space $\ell^\infty_2$, the direct limit of Hilbert spaces. In spite of some similarity in definitions, there is an essential difference between the spaces $\mathbb{R}^\infty$ and $\ell^\infty_2$: the natural linear operations on $\ell^\infty_2$ fail to be continuous and thus $\ell^\infty_2$ is not a linear topological space. Moreover, according to [Ba], $\ell^\infty_2$ is homeomorphic to no topological group and no closed convex subset of a linear topological space. In this situation we cannot apply the Dugundji or Borges Theorems to conclude that $\ell^\infty_2$ is an absolute extensor for stratifiable spaces.

In this paper we develop an alternative approach allowing us to prove that (in spite of absence of a linear structure) the space $\ell^\infty_2$ is an absolute extensor for stratifiable spaces. The main result of this note is

**Theorem.** An $\mathcal{M}_\omega$-space is an A(N)E[$\mathcal{S}$] if and only if it is an A(N)E[$\mathcal{M}$].

We shall say that a topological space $X$ is a direct limit of metrizable A(N)R’s if $X$ carries the inductive topology with respect to an increasing closed cover $\{X_n\}_{n \in \mathbb{N}}$ by metrizable A(N)R[$\mathcal{M}$]’s. According to [Ko], any such a space $X$ is an A(N)E[$\mathcal{M}$]. This result together with Theorem yields

**Corollary.** The direct limit of metrizable A(N)R’s is an A(N)R for stratifiable spaces.

**Main Lemma**

Recall that an equiconnected map on a topological space $X$ is a continuous function $\lambda : X \times X \times [0,1] \to X$ such that $\lambda(x,y,0) = x$, $\lambda(x,y,1) = y$, and $\lambda(x,x,t) = x$ for every $x,y \in X$, $t \in [0,1]$. A subset $U$ of a space $X$ equipped with
an equiconnected function $\lambda$ is called $\lambda$-convex, provided $\lambda(U \times U \times [0,1]) \subset U$. An equiconnected space is a pair $(X, \lambda)$ consisting of a topological space $X$ and an equiconnected map $\lambda$ on $X$. A topological space $X$ is called topologically convex if it admits an equiconnected map $\lambda$ such that the family of open $\lambda$-convex sets forms a base of the topology of $X$.

Extending the Borges Theorem [Bo] (asserting that each convex subset of a locally convex space is an AE$[S]$) R.Cauty proved that each topologically convex space is an AE$[S]$. This result of R.Cauty will be our main instrument in the proof of the AE-property of direct limits.

For a topological space $X$ by $\exp(X)$ the set of all closed subsets of $X$ is denoted. A map $\mathcal{F} : Z \rightarrow \exp(X)$ of a topological space $Z$ is called upper semi-continuous if for every open set $U \subset X$ the set $\mathcal{F}^{-1}(U) = \{z \in Z \mid \mathcal{F}(z) \subset U\}$ is open in $Z$.

We will use the following well known fact: if $X$ is a normal topological space and $\mathcal{F} : Z \rightarrow \exp(X)$ an upper semicontinuous map then for every closed subset $F \subset X$ the map $F \cap \mathcal{F} : Z \rightarrow \exp(F) \subset \exp(X)$ is upper semi-continuous, see [En].

For a map $\mathcal{F} : Z \rightarrow \exp(X)$ and a subset $A \subset Z$ let $\mathcal{F}(A) = \bigcup_{a \in A} \mathcal{F}(a)$. A map $\mathcal{F} : X \rightarrow \exp(X)$ is called monotone if $\mathcal{F}(\mathcal{F}(x)) \subset \mathcal{F}(x)$ for every $x \in X$.

**Main Lemma.** Suppose $X$ is a normal topological space, $\lambda : X \times X \times [0,1] \rightarrow X$ an equiconnected function, and $\mathcal{F} : X \rightarrow \exp(X)$ an upper semi-continuous map such that $\lambda(x,y,t) \in \mathcal{F}(x) \cup \mathcal{F}(y)$ for every $(x,y,t) \in X \times X \times [0,1]$. Suppose $X_1 \subset X_2 \subset \ldots$ is a sequence of subsets of $X$ such that for every $n \in \mathbb{N}$ there is $m \in \mathbb{N}$ with $\mathcal{F}(X_n) \subset X_m$. Then

1. the set $\bigcup_{n=1}^{\infty} X_n$ is $\lambda$-convex;
2. the function $\lambda : \lim X_n \times \lim X_n \times [0,1] \rightarrow \lim X_n$ is continuous;
3. if $\mathcal{F}$ is monotone and $X$ admits a base of open $\lambda$-convex subsets, then $\lim X_n$ has a base of open $\lambda$-convex sets too.

**Proof.** The first statement follows trivially from $\lambda(x,y,t) \in \mathcal{F}(x) \cup \mathcal{F}(y)$ and $\mathcal{F}(\bigcup_{n=1}^{\infty} X_n) \subset \bigcup_{m=1}^{\infty} X_m$.

The proof of the rest two statements are a bit more complex. At first let us make a remark.

Since for every $x \in \lim X_n$ we have $\mathcal{F}(x) \subset X_m$ for some $m$, it is legal to write $\mathcal{F} : \lim X_n \rightarrow \exp(\lim X_n)$. Let us show that this map is upper semicontinuous. Fix an open set $U \subset \lim X_n$. Because $\lim X_n$ has the direct limit topology, to show that $\mathcal{F}^{-1}(U)$ is open in $\lim X_n$ it suffices to verify that the intersection $\mathcal{F}^{-1}(U) \cap X_n$ is open in $X_n$ for every $n$. So, fix $n$, and find $m$ with $\mathcal{F}(X_n) \subset X_m$. Then $\mathcal{F}^{-1}(U) \cap X_n = \{x \in X_n \mid \mathcal{F}(x) \subset U \cap X_m\}$ is open in $X_n$ because the set $U \cap X_m$ is open in $X_m$ and the restriction $\mathcal{F}|_{X_n} : X_n \rightarrow \exp(X_m)$ is upper semicontinuous.

To show that the map $\lambda : \lim X_n \times \lim X_n \times [0,1] \rightarrow \lim X_n$ is continuous, fix a triple $(x_0, y_0, t_0) \in \lim X_n \times \lim X_n \times [0,1]$ and a neighborhood $U \subset \lim X_n$ of $\lambda(x_0, y_0, t_0)$. Find $n$ such that $x_0, y_0 \in X_n$ and $m$ such that $\mathcal{F}(X_n) \subset X_m$. Let $W \subset X$ be a neighborhood of $\lambda(x_0, y_0, t_0)$ such that $\overline{W} \cap X_m \subset U \cap X_m$ (here $\overline{W}$ is the closure of $W$ in $X$). Observe that the set $\overline{W} \cap (\bigcup_{n=1}^{\infty} X_n)$ is closed in $\lim X_n$. Then the map $\overline{W} \cap \mathcal{F} : \lim X_n \rightarrow \exp(\lim X_n)$ is upper semicontinuous. Consequently, the set $V = \{x \in \lim X_n \mid \overline{W} \cap \mathcal{F}(x) \subset U\}$ is an open neighborhood of $x_0$ and $y_0$ in $\lim X_n$. Using the continuity of the map $\lambda : X \times X \times [0,1] \rightarrow X$, we
may find neighborhoods $W(x_0), W(y_0) \subset X$, $V(t_0) \subset [0,1]$ of the points $x_0, y_0, t_0$, respectively, such that $\lambda(W(x_0) \times W(Y_0) \times V(t_0)) \subset W$. Let $V(x_0) = V \cap W(x_0)$, $V(y_0) = V \cap W(y_0)$. Evidently, $V(x_0), V(y_0)$ are open neighborhoods of $x_0, y_0$ in $\lim X_n$. We claim that $\lambda(V(x_0) \times V(y_0) \times V(t_0)) \subset U$. Indeed, fix any $(x, y, t) \in V(x_0) \times V(y_0) \times V(t_0)$ and remark that because of $x, y \in V$ we have $\overline{W} \cap F(x) \subset U$ and $\overline{W} \cap F(y) \subset U$. On the other hand, because $x \in W(x_0), y \in W(y_0), t \in V(t_0)$, we have $\lambda(x, y, t) \in \overline{W}$. Then, using the fact that $\lambda(x, y, t) \in F(x) \cup F(y)$ we get $\lambda(x, y, t) \in \overline{W} \cap (F(x) \cup F(y)) = (\overline{W} \cap F(x)) \cup (\overline{W} \cap F(y)) \subset U$. Therefore, the map $\lambda : \lim X_n \times \lim X_n \times [0,1] \to \lim X_n$ is continuous.

Now suppose $F$ is monotone and $X$ has a basis of $\lambda$-convex open subsets. Fix any $x_0 \in \bigcup_{n=1}^{\infty} X_n$ and a neighborhood $U \subset \lim X_n$ of $x_0$. Find $n$ such that $x_0 \in X_n$ and $m$ such that $F(X_n) \subset X_m$. Let $W$ be an open $\lambda$-convex neighborhood of $x_0$ in $X$ such that $\overline{W} \cap X_m \subset U \cap X_m$. Because of the upper semi-continuity of the map $\overline{W} \cap F : \lim X_n \to \text{exp}(\lim X_n)$ the set $V = \{x \in \lim X_n \mid \overline{W} \cap F(x) \subset U\}$ is open in $\lim X_n$. We claim that the set $O = W \cap V$ is a $\lambda$-convex open neighborhood of $x_0$ in $\lim X_n$ such that $O \subset U$. Since $x = \lambda(x, x, t) \in F(x)$, we get $x \in \overline{W} \cap F(x) \subset U$ for every $x \in V \cap W$, and thus $O \subset U$. To prove that $O$ is $\lambda$-convex, fix a triple $(x, y, t) \in O \times O \times [0,1]$. Since the set $W \ni x, y$ is $\lambda$-convex, $\lambda(x, y, t) \in W$. To see that $\lambda(x, y, t) \in V$, observe that $F(\lambda(x, y, t)) \subset F(F(x) \cup F(y)) = F(F(x)) \cup F(F(y)) \subset F(x) \cup F(y)$. Then $\overline{W} \cap F(\lambda(x, y, t)) \subset \overline{W} \cap (F(x) \cup F(y)) = (\overline{W} \cap F(x)) \cup (\overline{W} \cap F(y)) \subset U$ and thus $\lambda(x, y, t) \in V$ (by the definition of $V$). Therefore $O = W \cap V$, being an intersection of two $\lambda$-convex sets, is $\lambda$-convex. \hfill \Box

**Proof of Theorem**

For an infinite set $A$ by $l^2(A) = \{(x_a)_{a \in A} \in \mathbb{R}^A \mid \sum_{a \in A} x_a^2 < \infty\}$ we denote the standard Hilbert space of density $A$, endowed with the norm $\|x_a\|_{l^2(A)} = (\sum_{a \in A} x_a^2)^{1/2}$. For two vectors $x = (x_a)_{a \in A}$ and $y = (y_a)_{a \in A}$ of $l^2(A)$ we write $x \leq y$ if $x_a \leq y_a$ for every $a \in A$, and set $\min\{x, y\} = (\min\{x_a, y_a\})_{a \in A} \in l^2(A)$.

Let $l^2_+(A) = \{x \in l^2(A) \mid x \geq 0\}$ denote the positive cone of $l^2(A)$ and $S_+(A) = \{x \in l^2(A) \mid x \geq 0, \|x\| = 1\}$ its positive unit sphere.

For a subset $A' \subset A$ we identify $l^2_+(A')$ with the subspace $\{(x_a)_{a \in A} \in l^2_+(A) \mid x_a = 0$ for $a \notin A'\}$ of $l^2_+(A)$.

Let $X \in \mathcal{M}_\omega$ be an $\mathcal{N}[\mathcal{M}]$-space. Write $X = \lim X_n$, where $X_0 \subset X_1 \subset X_2 \subset \ldots$ is a sequence of closed metrizable subsets of $X$. Applying Hausdorff’s Theorem on extending metrics [Ha], we may find a continuous metric $\rho \leq 1$ on $X$ which generates the topology of each $X_n$. According to Proposition 7.1 of [BP, Ch.VI], there is an embedding $g : (X, \rho) \to S_+(A_0)$ of the metric space $(X, \rho)$ into the positive unit sphere $S_+(A_0)$ of $l^2(A_0)$ for some set $A_0$. Let $A = A_0 \cup \{0\} \cup \mathbb{N}$ and $A_n = A_0 \cup \{0, \ldots, n\}$ for $n \in \mathbb{N}$. Consider the embedding $f : (X, \rho) \to S_+(A)$ defined for $x \in X$ by $f(x) = (f(x)_a)_{a \in A}$, where

$$f(x)_a = \begin{cases} \frac{1}{\sqrt{2}} g(x)_a, & \text{if } a \in A_0; \\ 2^{-n} \rho(x, X_{n-1}), & \text{if } a = n \in \mathbb{N}; \\ \left(\frac{1}{2} - \sum_{n=1}^{\infty} (2^{-n} \rho(x, X_{n-1}))^2\right)^{1/2}, & \text{if } a = 0. \end{cases}$$
Observe that
(1) \[ f(X) \cap l^2_+(A_n) = f(X_n) \text{ for every } n \in \mathbb{N}. \]
Let \( Y = \{ y \in l^2_+(A) \mid \exists x \in X \text{ with } 0 \leq y \leq f(x) \} \). Remark that \( Y \cap S_+(A) = f(X) \), thus \( f(X) \) is a closed subset in \( Y \). This and (1) imply
(2) \[ f(X_n) \text{ is a closed subset in } Y \cap l^2_+(A_n) \text{ for every } n. \]

For \( n \in \mathbb{N} \) let \( Y_n = Y \cap l^2_+(A_n) \) and remark that \( Y = \bigcup_{n=1}^{\infty} Y_n \) (because of (1)). Denote by \( \tau \) the direct limit topology \( \lim_{n \to \infty} Y_n \) on \( Y \). We will apply Main Lemma and [Ca1, 1.4] in order to show that the space \((Y, \tau)\) is an AE[S]. For every \( y \in Y \) let \( \mathcal{F}(y) = \{ x \in l^2_+(A) \mid 0 \leq x \leq y \} \). One can easily check that each set \( \mathcal{F}(y) \) is a compactum lying in \( Y \), and the map \( \mathcal{F} : Y \to \exp(Y) \) is upper semi-continuous and monotone.

On \( Y \) let us consider the equiconnected function \( \lambda : Y \times Y \times [0, 1] \to Y \) defined for \( (x, y, t) \in Y \times Y \times [0, 1] \) by the formula
\[
\lambda(x, y, t) = \begin{cases} 
(1 - 2t)x + 2t \min\{x, y\}, & \text{if } t \leq \frac{1}{2}; \\
(2 - 2t) \min\{x, y\} + (2t - 1)y, & \text{if } t \geq \frac{1}{2}.
\end{cases}
\]
Evidently, \( \lambda(x, y, t) \in \mathcal{F}(x) \cup \mathcal{F}(y) \) for every \( (x, y, t) \). Observe also that \( \mathcal{F}(y_n) \subset Y_n \) for every \( n \) and that \( Y \) admits a base of open \( \lambda \)-convex sets. Thus it is legal to apply Main Lemma to conclude that the equiconnected function \( \lambda : (Y, \tau) \times (Y, \tau) \times [0, 1] \to (Y, \tau) \) is continuous and the space \((Y, \tau)\) admits a base of open \( \lambda \)-convex subsets. Finally, applying Theorem 1.4 of [Ca1], we get \((Y, \tau)\) is an AE[S].

Now let us return to our initial space \( X \). It follows from (1) and (2) that the map \( f : X = \lim_{n \to \infty} X_n \to (Y, \tau) = \lim_{n \to \infty} Y_n \) is a closed embedding. Using the fact that \( X \) is an AN(E[M], we may easily prove that \( X \) is an AN(E[Mω]). Since \((Y, \tau) \in \mathcal{M}_\omega\), this yields \( f(X) \) is a (neighborhood) retract of \((Y, \tau) \in AE[S]\), and consequently, \( X \) is an AN(E[S]. \] □

**Concluding remarks and open questions**

1. For stratifiable \( k_\omega \)-spaces our Theorem was known since R.Cauty [Ca1, 4.2].
2. In light of Theorem, one could ask: Is every stratifiable ANE(M)-space an ANE(S)? The answer is “no”: any countable space with a unique non-isolated point and without non-trivial convergent sequences can serve as a suitable counterexample.
3. Suppose \( A_1 \subset A_2 \subset \ldots \) is a sequence of countable sets with \( A_n \neq A_{n+1} \) for every \( n \). It follows from the proof of Lemma that the semilattice operation \( \min \) is continuous on the direct limit \( \lim_{n \to \infty} l^2_+(A_n) \). Applying [DT] we conclude that each space \( l^2_+(A_n) \) is homeomorphic to the separable Hilbert space \( l_2 \). Using this fact, one can easily show that the direct limit \( \lim_{n \to \infty} l^2_+(A_n) \) is homeomorphic to the direct limit \( l_\infty^2 \), considered by Pentak [Pe]. Therefore we conclude that the space \( l_\infty^2 \) supports the structure of a topological semilattice (even Lawson semilattice). This fact is of interest because the topology of the space \( l_\infty^2 \) is not compatible with many other algebraical structures. For example, \( l_\infty^2 \) is homeomorphic to no closed convex set in a linear topological space and no topological group (more generally, no closed multiplicative subset (or multiplicative subset with unity) in a topological group), see [Ba].
**Question 1.** Is the space $l_2^\infty$ homeomorphic to a topological lattice? to a convex set in a linear topological space?

In connection with the second part of this question let us make the following observation. Denote by $Q = [-1,1]^{\omega}$ the Hilbert cube and by $s = (-1,1)^{\omega}$ its pseudo-interior. For $n \in \mathbb{N}$ let $Q_n = (1 - \frac{1}{n})Q = [-1 + \frac{1}{n}, 1 - \frac{1}{n}]^{\omega} \subset Q$. Let $\lambda$ denote the equiconnected function on $s \times Q$ generated by the convex structure of $s \times Q$, i.e. $\lambda(x,y,t) = (1-t)x + ty$. Evidently, the set $\bigcup_{n=1}^{\infty} s \times Q_n \subset s \times Q$ is convex. It can be easily shown that the direct limit $\lim_{\rightarrow} s \times Q_n$ is homeomorphic to $l_2^\infty$. We claim that the equiconnected function $\lambda$ is continuous on $\lim_{\rightarrow} s \times Q_n$. To see this, apply Main Lemma with the map $F: s \times Q \to \exp(s \times Q)$ defined by $F(x) = \frac{1}{2}x + \frac{1}{2}Q \times Q$. Therefore, $l_2^\infty$ admits a kind of a convex structure, but that, of course, does not answer Question 1.

4. Corollary rises the following

**Question 2.** Suppose a stratifiable space $X$ is a direct limit $\lim_{\rightarrow} X_n$ of a sequence $X_1 \subset X_2 \subset \ldots$ of closed subsets of $X$ such that each $X_n$ is an $A(N)R[S]$. Is the space $X$ equiconnected? an $A(N)R[S]$?

5. Having in mind Question 2 and Main Lemma let us introduce a definition. For a topological space $X$ let $2^X$ denote the set of all compact subsets of $X$. We call a space $X$ **compactly equiconnected** if $Y$ admits an equiconnected function $\lambda: X \times X \times [0,1] \to X$ and an upper semi-continuous map $F: X \to 2^X$ such that $\lambda(x,y,t) \in F(x) \cup F(y)$ for every $(x,y,t) \in X \times X \times [0,1]$. Analysing the proof of Theorem, we see that every metrizable space embeds as a closed subset into a compactly equiconnected metrizable AR.

**Question 3.** Does every stratifiable space embeds as a closed subset into a compactly equiconnected stratifiable space? Equivalently, is every absolute retract for stratifiable spaces compactly equiconnected?

6. In connection with Theorem and Questions 2 and 3, let us consider the following surprising example. According to [Ca2] there is a strongly countable-dimensional metrizable compactum $K$ such that the free convex set $P_\infty(K)$ over $K$ is not an AE for the class of all metrizable compacta and thus $P_\infty(K)$ is not an $AE(S)$. Writing $K = \bigcup_{n=1}^{\infty} K_n$, where $K_1 \subset K_2 \subset \ldots$ is a tower of finite-dimensional compacta, we see that the stratifiable convex (and thus equiconnected) space $P_\infty(K)$ which is not an $AE(S)$ can be written as $P_\infty(K) = \bigcup_{n=1}^{\infty} P_\infty(K_n)$, where each $P_\infty(K_n) \subset P_\infty(K_{n+1})$ is a closed convex $AE(S)$-subspace of $P_\infty(K)$.

7. The author would like to express his sincere thanks to Robert Cauty for fruitful and stimulating discussions on the subject of the paper.
ON DIRECT LIMITS OF METRIZABLE ANR'S

References


[Sa] K. Sakai, On $\mathbb{R}^\infty$-manifolds and $Q^\infty$-manifolds, Topology Appl. 18 (1984), 69–79.

Department of Mathematics, Lviv University, Universytetska 1, Lviv, 290602, Ukraine and Instytut Matematyki, Akademia Świętokrzyska, Kielce, Poland