APPROXIMATION OF SUBHARMONIC FUNCTIONS

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Abstract. In certain classes of subharmonic functions \(u\) on \(\mathbb{C}\) distinguished in terms of lower bounds for the Riesz measure of \(u\), a sharp estimate is obtained for the rate of approximation by functions of the form \(\log |f(z)|\), where \(f\) is an entire function. The results complement and generalize those recently obtained by Yu. Lyubarskiđ and Eu. Malinnikova.

§1. Introduction

We use the standard notions of subharmonic function theory (see [1]). We put \(D_z(t) = \{\zeta \in \mathbb{C} : |\zeta - z| < t\}, \ z \in \mathbb{C}, \ t > 0\). For a function \(u\) subharmonic in \(\mathbb{C}\), we write \(B(r,u) = \max\{u(z) : |z| = r\}, \ r > 0\), and define the order \(\rho[u]\) by the relation \(\rho[u] = \limsup r \to +\infty \log B(r,u)/\log r\). Also, let \(\mu_u\) denote the Riesz measure associated with the subharmonic function \(u\), and let \(m\) be the plane Lebesgue measure. The symbol \(C\) with indices stands for some positive constants. If \(u = \log |f|\), where \(f\) is an entire function with zeros \(\{a_k\}\), then \(\mu_u = \sum k n_k \delta(z - a_k)\), where \(n_k\) is the multiplicity of the zero \(a_k\), and \(\delta(z - a_k)\) is the Dirac function concentrated at \(a_k\). However, the class of functions subharmonic in \(\mathbb{C}\) is broader than that of functions of the form \(\log |f|\), where \(f\) is an entire function. Since it is often easier to construct a subharmonic function rather than an entire one with desired asymptotic properties, a natural problem arises of approximating subharmonic functions by the logarithms of the moduli of entire functions. Apparently, V. S. Azarin [2] was the first to investigate this problem in the general form in the class of functions subharmonic in the plane and having finite order of growth. The results cited below have numerous applications in function theory and potential theory (see, e.g., [3–6]).

In 1985 R. S. Yulmukhametov [7] obtained the following remarkable result. For any function \(u\) subharmonic in \(\mathbb{C}\) and of order \(\rho \in (0, +\infty)\), and for any \(\alpha > \rho\), there exists an entire function \(f\) and a set \(E_\alpha \subset \mathbb{C}\) such that

\[
|u(z) - \log |f(z)|| \leq C_\alpha \log |z|, \ \ z \to \infty, \ z \notin E_\alpha,
\]

and \(E_\alpha\) can be covered by a family of disks \(D_z(t_j), \ j \in \mathbb{N}\), satisfying the estimate 
\[
\sum_{|z| > R} t_j = O(R^{\rho - \alpha}), \ R \to +\infty.
\]
In the special case where the subharmonic function \(u\) is homogeneous, \(\log |z|\) in (1.1) can be replaced by \(O(1)\) (see [3, 8]). Passage to an integral metric allows us to drop an exceptional set in the case of approximation of subharmonic...
functions of finite order. Let $\| \cdot \|_q$ be the norm in the space $L^q(0, 2\pi)$, and let

$$Q(r, u) = \begin{cases} O(\log r), & r \to +\infty, \\ O(\log r + \log n(r, u)), & r \to +\infty, \\ O(\log r + \log n(r, u)) + \rho[u] < +\infty, & r \not\in E, \text{mes } E < +\infty, \\ \rho[u] = +\infty. & \end{cases}$$

In [9] A. A. Gol’dberg and M. O. Hirnyk proved that for an arbitrary subharmonic function $u$ there exists an entire function $f$ such that

$$\|u(re^{i\theta}) - \log |f(re^{i\theta})|\|_q = Q(r, u), \quad r > 0, \quad q > 0.$$

From the recent result of Yu. Lyubarski˘ı and Eu. Malinnikova [10] it follows that integration of the approximation rate against the plane measure makes it possible to lift the assumption that $u$ is of finite order and leads to sharp estimates.

**Theorem A** [10]. Let $u(z)$ be a subharmonic function in $\mathbb{C}$. Then for each $q > 1/2$ there exists $R_0 > 0$ and an entire function $f$ such that

$$\frac{1}{\pi R^2} \int_{|z| < R} |u(z) - \log |f(z)|| dm(z) < q \log R, \quad R > R_0.$$  

(1.2)

An example constructed in [10] shows that we cannot take $q < 1/2$ in (1.2). In connection with Theorem A, the following M. Sodin’s question is known, which is a refinement of Question 1 in [11, p. 315].

**Question.** Given a subharmonic function $u$ on $\mathbb{C}$, does there exist an entire function $f$ and a constant $\alpha \in [0, 1)$ such that

$$\int_{|z| < R} |u(z) - \log |f(z)| - \alpha \log |z|| dm(z) = O(R^2), \quad R > R_0?$$

(1.3)

**Remark 1.** Question 1 in [11, p. 315] corresponds to the case of $\alpha = 0$. The example in [10] mentioned above implies the negative answer to this question.

On the other hand, restrictions on the Riesz measure that bound it from below make it possible to refine estimate (1.2).

**Theorem B** [10]. Let $u(z)$ be a subharmonic function in $\mathbb{C}$. If, for some $R_0 > 0$ and $q > 1$, we have

$$\mu_\alpha(\{z : R < |z| \leq qR\}) > 1, \quad R > R_0,$$

(1.4)

then there exists an entire function $f$ satisfying

$$\sup_{R > 0} R^{-2} \int_{|z| < R} |u(z) - \log |f(z)|| dm(z) < \infty.$$

Moreover, for every $\varepsilon > 0$ there exists a set $E_\varepsilon \subset \mathbb{C}$ such that

$$\limsup_{R \to \infty} m(\{z \in E : |z| < R\}) R^{-2} < \varepsilon$$

and

$$u(z) - \log |f(z)| = O(1), \quad z \not\in E_\varepsilon, \quad z \to \infty.$$  

(1.5)

A gap is seen between the statements of Theorems A and B. The following question arises: how the estimate of the left-hand side of (1.2) can be improved if (1.4) fails? The answer to this question is given by Theorem 1.

Let $\Phi$ be the class of slowly varying functions $\psi$: $[1, +\infty) \to (1, +\infty)$ (in particular, $\psi(2r) \sim \psi(r)$ as $r \to +\infty$).
Theorem 1. Let $u$ be a function subharmonic in $\mathbb{C}$, and let $\mu = \mu_u$. If for some $\psi \in \Phi$ there exists a constant $R_1$ satisfying the condition
\begin{equation}
\mu \left( \{ z : R < |z| \leq R\psi(R) \} \right) > 1, \quad R > R,
\end{equation}
then there exists an entire function $f$ such that
\begin{equation}
\int_{|z| < R} |u(z) - \log |f(z)|| \, dm(z) = O(R^2 \log \psi(R)).
\end{equation}

Corollary 1. Under the conditions of Theorem 1, for arbitrary $\varepsilon > 0$ there exists $K(\varepsilon) > 0$ and a set $E_\varepsilon \subset \mathbb{C}$ such that
\begin{equation}
\limsup_{r \to \infty} \frac{m(E \cap \overline{D}_r)}{r^2} < \varepsilon
\end{equation}
and
\begin{equation}
|u(z) - \log |f(z)|| \leq K_\varepsilon \log \psi(|z|), \quad z \notin E_\varepsilon.
\end{equation}

The following example and Theorem 2 show that estimate (1.7) is sharp in the class of subharmonic functions satisfying (1.6).

For $\varphi \in \Phi$, put
\begin{equation}
u(z) = u_{\varphi}(z) = \frac{1}{2} \sum_{k=1}^{+\infty} \log \left| 1 - \frac{1}{r_k} \right|,
\end{equation}
where $r_0 = 2$, $r_{k+1} = r_k \varphi(r_k)$, $k \in \mathbb{N} \cup \{0\}$. Then $\mu_u$ satisfies condition (1.6) with $\psi(x) = \varphi^2(x)$.

Theorem 2. Let $\psi \in \Phi$ be such that $\psi(r) \to +\infty$ ($r \to +\infty$). There exists no entire function $f$ with
\begin{equation}
\int_{|z| < R} |u_{\psi}(z) - \log |f(z)|| \, dm(z) = o(R^2 \log \psi(R)), \quad R \to \infty.
\end{equation}

From Theorem 2' below it follows that the answer to M. Sodin’s question formulated above is in the negative. It was M. Hirnyk who drew the author’s attention to that question, as well as to the fact that the negative answer follows from the above example.

Theorem 2’. Let $\sigma$ be an arbitrary positive continuous function defined on $[1, +\infty)$, and let $\sigma(t) \to 0$ ($t \to +\infty$). No entire function $f$ and no constant $\alpha \in [0, 1)$ can satisfy
\begin{equation}
\int_{|z| < R} |u_{\psi}(z) - \log |f(z)|| - \alpha \log |z| \, dm(z) = o \left( R^2 \int_1^R \frac{\sigma(t)}{t} \, dt \right), \quad R \to \infty.
\end{equation}

Remark 2. The growth of the quantity $\int_1^R \frac{\sigma(t)}{t} \, dt$ as $R \to +\infty$ is restricted only by the condition $\int_1^R \frac{\sigma(t)}{t} \, dt = o(\log R)$.

Remark 3. The author does not know whether it is possible to refine estimate (1.8) of the exceptional set for (1.9). In [12], sharp estimates of the exceptional set outside of which (1.9) is true were obtained for a class of subharmonic functions subject to some additional restriction on the Riesz measure.
§2. Proof of Theorem 1

2.1. Partition of measures. It has turned out that, in order to have a “good” approximation of a subharmonic function by the logarithm of the modulus of an entire function, we need a “good” approximation of the corresponding Riesz measure by a discrete measure. The Riesz measure defined on the Borel sets in the plane is subject to the only requirement that it should be finite on the compact sets. The following theorem on partition of measures is the principal step in the proofs of Theorems 1, 2, and 2′.

Theorem C. Let \( \mu \) be a measure in \( \mathbb{R}^2 \) with compact support, and let \( \text{supp} \mu \subset \Pi \) and \( \mu(\Pi) \in \mathbb{N} \), where \( \Pi \) is a square. Suppose, moreover, that for any line \( L \) parallel to a side of the square \( \Pi \), there is at most one point \( p \in L \) such that

\[
0 < \mu(\{p\}) < 1 \quad \text{and} \quad \mu(L \setminus \{p\}) = 0.
\]

Then there exists a system of rectangles \( \Pi_k \subset \Pi \) with sides parallel to the sides of \( \Pi \), and a system of measures \( \mu_k \) with the following properties:

1. \( \text{supp} \mu_k \subset \Pi_k \);
2. \( \mu_k(\Pi_k) = 1 \), \( \sum_k \mu_k = \mu \);
3. the interiors of the convex hulls of the supports of \( \mu_k \) are pairwise disjoint;
4. the ratio of the side lengths of rectangles \( \Pi_k \) lies in the interval \([1/3, 3]\);
5. each point of the plane belongs to the interiors of at most 4 rectangles \( \Pi_k \).

Theorem C was proved by R. S. Yulmukhametov (see [7, Theorem 1]) for absolutely continuous measures (i.e., for \( \nu \) such that \( m(E) = 0 \Leftrightarrow \nu(E) = 0 \)). In this case condition (2.1) is fulfilled automatically. In [4, Theorem 2.1], D. Drasin showed that Yulmukhametov’s proof works if the condition of continuity is replaced by condition (2.1). In its turn, condition (2.1) can be achieved by rotating the initial square (see [4]). Though in [10] it was noted that Theorem C remains valid even without condition (2.1), the author does not know any proof of this fact. Moreover, in the proof of Theorem 2.1 in [4] (this is a version of Theorem C), condition (2.1) was used substantially. In this connection, the paper [13] by A.F. Grishin and S.V. Makarenko [13] should be mentioned, where a two-dimensional version of Theorem C was proved under the condition that the measure loads no line parallel to a coordinate axis.

Remark 4. In the proof of Theorem C given in [4], the rectangles \( \Pi_k \) are obtained by splitting the given rectangles, starting with \( \Pi \), into smaller rectangles in the following way. The length of the smaller side of an initial rectangle coincides with that of a side of the resulting rectangle, and the length of the other side of the resulting rectangle is between one third and two third of the length of the other side of the initial rectangle. Thus, the following form of Theorem C, which will be used in the sequel, is true.

Theorem 3. Let \( \mu \) be a measure in \( \mathbb{R}^2 \) with compact support, and let \( \text{supp} \mu \subset \Pi \) and \( \mu(\Pi) \in 2\mathbb{N} \), where \( \Pi \) is a rectangle with the ratio \( b_0/a_0 = l_0 \in [1, +\infty) \) of the side lengths \( a_0, b_0 \) \( (a_0 \leq b_0) \). If, moreover, condition (2.1) is fulfilled, then there exists a system of rectangles \( \Pi_k \subset \Pi \) with sides parallel to the sides of \( \Pi \), and a system of measures \( \mu_k \) with the following properties:

1. \( \text{supp} \mu_k \subset \Pi_k \);
2. \( \mu_k(\Pi_k) = 2 \), \( \sum_k \mu_k = \mu \);
3. the interiors of the convex hulls of the supports of \( \mu_k \) are pairwise disjoint;
4. the ratio \( b_k/a_k \) of the lengths \( a_k, b_k \) \( (a_k \leq b_k) \) of the sides of \( \Pi_k \) lies in the interval \([1, l_0]\), and, moreover, if \( l_k > 3 \), then \( a_k = a_0 \);
5. each point of the plane belongs to the interiors of at most 4 rectangles \( \Pi_k \).
As was noted in [3], the idea of partition into rectangles of mass 2 is due to A. F. Gri- 
shin. In order to apply Theorem 3, we need the following lemma (see also [4, Lemma 2.4]):

**Lemma 1.** Let \( \nu \) be a locally finite measure in \( \mathbb{C} \). Then in any neighborhood of the origin there exists a point \( z' \) with the following properties:

a) on each line \( L_\alpha \) through \( z' \), there is at most one point \( \zeta_\alpha \) such that \( \nu(\{\zeta_\alpha\}) > 0 \), and, moreover, \( \nu(L_\alpha \setminus \{\zeta_\alpha\}) = 0 \);

b) on each circle \( C_\rho \) with center \( z' \), there exists at most one point \( \zeta_\rho \) such that \( \nu(\{\zeta_\rho\}) > 0 \), and, moreover, \( \nu(C_\rho \setminus \{\zeta_\rho\}) = 0 \).

We give a simple example for illustration. Let \( \nu(z) = \sum_{n \in \mathbb{N}} \delta(z - n) \). Then \( \nu(\mathbb{R}) = +\infty \). As \( z' \) we can take any point of the disk \( \{z : |z| < 1\} \) with nonzero imaginary part.

**Proof of Lemma 1.** We put \( B_n = \{z : 2^n < |z| \leq 2^{n+1}\} \), \( n \in \mathbb{N} \), and \( B_0 = \{z : |z| \leq 2\} \), \( \nu_n = \nu|_{B_n} \). Since \( \nu \) is a locally finite measure, \( \nu_n(\mathbb{C}) = \nu_n(B_n) < +\infty \). There is an at most countable set \( \zeta_{nk} \) of points such that \( \nu_n(\{\zeta_{nk}\}) > 0 \). Therefore, the set \( E_1 = \bigcup_n \bigcup_k (\zeta_{nk}) \) is at most countable. Given a pair of points in \( E_1 \), we consider the straight line through these points and the middle perpendicular to the segment connecting these points. All these lines cover some set \( A \subset \mathbb{C} \) with \( m(A) = 0 \). Let \( z_1 \in \mathbb{C} \setminus A \). By construction, an arbitrary straight line through the point \( z_1 \) contains at most one point with positive mass. The same is true for an arbitrary circle centered at \( z_1 \). We define \( \nu'_n = \nu_n - \sum_{\zeta \in (\zeta_{nk})} \nu_n(\{\zeta\}) \delta_n \), \( n \in \mathbb{Z}_+ \), and \( \delta_n(z) = \delta(z - \zeta) \). Then, for any \( z \in \mathbb{C} \), we have \( \nu'_n(\{z\}) = 0 \), \( n \in \mathbb{Z}_+ \). Since the intersection of two different circles (straight lines) is either the empty set, or a point, or two points, and \( \nu_n(\mathbb{C}) < +\infty \), the countable additivity of \( \nu_n \) shows that there exists at most countable set of circles and straight lines with positive \( \nu_n \)-measure. The union \( F_n \) of all these straight lines and all centers of these circles has zero area. Now, if \( z' \in \mathbb{C} \setminus (A \cup \bigcup_{n \in \mathbb{Z}_+} F_n) \), then for any \( n \in \mathbb{Z}_+ \) the measure \( \nu_n \) of any circle with center \( z' \) as well as that of a straight line through \( z' \) is equal to zero. Consequently, their \( \nu \)-measure is also equal to zero. Finally, by the countable additivity of the plane measure, we have \( m(A \cup \bigcup_{n \in \mathbb{Z}_+} F_n) = m(A) + \sum_n m(F_n) = 0 \). Thus, any point \( z' \in (\mathbb{C} \setminus (A \cup \bigcup_{n \in \mathbb{Z}_+} F_n)) \cap U \), where \( U \) is an arbitrary neighborhood of the origin, possesses the required properties. \( \square \)

Taking the above lemma into account, we may assume that properties a) and b) in Lemma 1 are fulfilled at the origin. We follow the method of proof used in [10], assuming that \( \psi(x) \to +\infty \) as \( x \to +\infty \), because otherwise Theorem 1 is equivalent to Theorem B. Without loss of generality, we can assume that \( u(z) \) is harmonic in a neighborhood of \( z = 0 \). Otherwise, we choose an arbitrary \( a > 0 \) such that \( n(a) \leq N \leq n(a + 0) \) for some \( N \in \mathbb{N} \) and introduce the measure \( \nu \) that is equal to \( \mu \) in \( D_0(a) \) and contains the part \( \mu_{\{|z|=a\}} \) so that \( \nu(D_0(a)) = N \). Then \( \mu - \nu \equiv 0 \) in \( D_0(a) \) and, in place of \( u \), we consider the function \( \bar{u}(z) = u(z) - \int_C \log|z - \zeta| d\nu(\zeta) \). The quantity \( \int_C \log|z - \zeta| d\nu(\zeta) - N \log|z| \) is bounded as \( z \to +\infty \). Therefore, without loss of generality we assume that \( R_0 = \sup\{|r > 0 : \sup \mu \cap D_0(r) = \emptyset\} > 1 \). Put \( \Psi_1(R) = R\psi(R) \), \( \Psi_n(R) = \Psi_1(\Psi_{n-1}(R)) \) for \( n \in \mathbb{N} \), \( \Psi_0(R) \equiv R \), \( R > 1 \). By induction, we define measures \( \mu^{(j)}_l \), \( j \in \{1, 2, 3\} \), and a sequence \( (R_k) \), \( k \in \mathbb{N} \). Suppose that the \( \mu^{(j)}_l \) have already been defined for \( l < k \), and the \( R_l \) are defined for \( l \leq k \). Let

\[
Q_k = \{\zeta \in \mathbb{C} : R_k \leq |\zeta| \leq \Psi_1(R_k)\}, \quad \mu^-_k = \left(\mu - \sum_{j=1}^{k-1} (\mu^{(1)}_j + \mu^{(2)}_j + \mu^{(3)}_j)\right)\bigg|_{Q_k}.
\]
If \( \mu_k(Q_k) < 2 \), we put \( \mu_k^{(1)}(Q_k) \equiv 0 \), \( \mu_k^{(2)} = \mu_k^{(3)} \equiv 0 \). If \( \mu_k^{-1}(Q_k) \geq 2 \), we write \( \mu_k = \mu_k^{(1)} + \mu_k^{(2)} \) so that \( \mu_k^{(1)}(Q_k) = 2 |\mu_k^{-1}(Q_k)/2| \), where \([a]\) denotes the integral part of \( a \). If \( \mu_k^{(2)}(Q_k) \geq 1 \), we put \( \mu_k^{(3)}(Q_k) = 0 \) and \( R_{k+1} = \Psi_1(R_k) \). Otherwise, we define

\[
R_{k+1} = \inf \{ R > R_k : \mu(\{ \Psi_1(R_k) < |z| \leq R \}) \geq 1 \},
\]

and by \( \mu_k^{(3)} \) we mean the sum of the restriction of \( \mu \) to \( \{ \zeta : \Psi_1(R_k) < |\zeta| < R_{k+1} \} \) and \( \tilde{\mu} \), where \( \tilde{\mu} \) is the part of the restriction \( \mu \) to \( \{ |\zeta| = R_{k+1} \} \) such that \( \mu_k^{(3)}(\mathbb{C}) = 1 \). By construction, for all \( k \in \mathbb{N} \) we have the following:

1. \( \text{supp} \mu_k^{(1)} \subseteq Q_k, \mu_k^{(1)}(Q_k) \in 2\mathbb{Z}_+ \);
2. \( \Psi_1(R_k) \leq R_{k+1} \leq \Psi_2(R_k) \);
3. \( \text{supp} (\mu_k^{(2)} + \mu_k^{(3)}) \subseteq \{ \zeta : R_k < |\zeta| < R_{k+1} \} \);
4. \( 1 \leq (\mu_k^{(2)} + \mu_k^{(3)}) \{ \zeta : R_k < |\zeta| < R_{k+1} \} \leq 2 \).

Let \( \mu^{(1)} = \sum_{j=1}^{+\infty} \mu_j^{(1)} \), \( \mu^{(2)} = \sum_{j=1}^{+\infty} (\mu_j^{(2)} + \mu_j^{(3)}) \). From 3) and 4) it follows that \( \mu^{(2)}(\mathcal{D}_0(R)) = O(\log R), R \to +\infty \). Therefore, \( u_2(z) = \int_{\mathbb{C}} \log |1 - z/\zeta| \, d\mu^{(2)}(\zeta) \) is a subharmonic function in \( \mathbb{C} \). Let \( u_1(z) = u(z) - u_2(z) \). Then \( \mu_{u_1} = \mu^{(1)} \). We shall approximate \( u_1 \) and \( u_2 \) separately. It suffices to prove that

\[
I_n \overset{\text{def}}{=} \int_{2^n \in |z| \leq 2^{n+1}} |u(z) - \log |f(z)|| \, dm(z) = O(4^n \log \psi(2^n)), \quad n \to +\infty.
\]

Indeed, let \( R \in [2^n, 2^{n+1}] \). Then from (2.2) it follows that

\[
\frac{\int_{|z| < R} |u(z) - \log |f(z)|| \, dm(z)}{R^2} \leq \frac{\sum_{k=0}^{n} 4^k \log \psi(2^k)}{4^n} \leq 4C_1 \log \psi(2^n).
\]

2.2. Approximation of \( u_2(z) \). By construction, \( \mu^{(2)}(\mathbb{C}) = +\infty \). Put \( T_n = \sup \{ R > 0 : \mu^{(2)}(\mathcal{D}_0(R)) \leq 5n \} \), \( A_n = \{ \zeta : T_n < |\zeta| \leq T_{n+1} \} \). Let \( (A_n, \mu_n) \) be a partition of the measure \( \mu^{(2)} \) such that \( \mu^{(2)} = \sum_{k=1}^{+\infty} \mu_k \), \( \text{supp} \mu_n \subseteq A_n \), and \( \mu_n(A_n) = 5 \). We introduce \( r_n \) by the identities \( \log r_n = 5 \int_{A_n} \log |\zeta| \, d\mu_n(\zeta), n \in \mathbb{N}, \) and consider the formal product

\[
f_2(z) = \prod_{n=1}^{+\infty} \left( 1 - \frac{z}{r_n} \right)^5.
\]

Property 4) of the measures \( \mu_k^{(i)} \) implies that

\[
\Psi_1(T_n) \leq T_{n+1} \leq \Psi_0(T_n).
\]

Since \( r_{n+1}/r_{n-1} \geq T_{n+1}/T_n \geq \psi(T_n) \to +\infty, n \to +\infty \), the function \( f_2 \) is entire. Let

\[
d_k(\zeta) = \int_{A_k} \left( \log |1 - \frac{z}{\zeta}| - \log |1 - \frac{r_k}{\zeta}| \right) \, d\mu_k(\zeta)
\]

\[
= \int_{A_k} \left( \log |1 - \frac{z}{\zeta}| - \log |1 - \frac{r_k}{\zeta}| \right) \, d\mu_k(\zeta).
\]
Here we have used the choice of $r_k$. Fix $n \in \mathbb{N}$, and let $2^n \in [T_N, T_{N+1})$. Then for $k \geq N + 2$ and $\zeta \in A_k$ we have $|\zeta| \geq T_k \geq |z|T_k/T_{N+1} = 2^k/kT_{N+1}$. Consequently, $|\log|1 - \frac{z}{\zeta}|/|\zeta|, \log|1 - \frac{z}{r_k}|/|r_k| \leq 2|z|/r_k$. Therefore,

$$\int_{2^n \leq |z| \leq 2^{n+1}} \sum_{k=N+2}^{\infty} |d_k(z)| dm(z) \leq \int_{2^n \leq |z| \leq 2^{n+1}} \sum_{k=N+2}^{\infty} 20\frac{T_{N+1}}{T_k} dm(z),$$

(2.5)

$$\leq C_2 T_{N+1}^{N+1} = o(4^n), \quad n \to \infty.$$

Similarly, $|d_k(z)| \leq 20T_{k+1}2^{-n}$ for $k \leq N - 2$. Using (2.4), we obtain

$$\int_{2^n \leq |z| \leq 2^{n+1}} \sum_{k=1}^{N-2} |d_k(z)| dm(z) \leq o(4^n), \quad n \to \infty.$$

We estimate $\int_{2^n \leq |z| \leq 2^{n+1}} |d_k(z)| dm(z)$ for $k \in \{N - 1, N, N + 1\}$. By definition,

$$\int_{2^n \leq |z| \leq 2^{n+1}} |d_{N+1}(z)| dm(z) = \int_{2^n \leq |z| \leq 2^{n+1}} \int_{A_{N+1}} \left(\log|1 - \frac{z}{\zeta}| - \log\frac{z}{r_{N+1}}\right) d\mu_{N+1}(\zeta) dm(z).$$

If $|\zeta| \geq 2|z|$ and $\zeta \in A_{N+1}$, then $\log|1 - \frac{z}{\zeta}| \leq \log 2$. Otherwise, we have $T_{N+1} \leq |\zeta| < 2|z| < 2^{n+2} \leq 4T_{N+1}$. Therefore, applying the Fubini theorem and changing the variables by the rule $T_{N+1} = \zeta, T_{N+1} = z$, we obtain

$$\int_{2^n \leq |z| \leq 2^{n+1}} \int_{A_{N+1}} \left|\log|1 - \frac{z}{\zeta}|\right| d\mu_{N+1}(\zeta) dm(z) \leq 2 \int_{2^n \leq |z| \leq 2^{n+1}} dm(z) + \int_{2^n \leq |z| \leq 2^{n+1}} \int_{T_{N+1} \in |\zeta| \leq 4T_{N+1}} \left|\log|1 - \frac{z}{\zeta}|\right| d\mu_{N+1}(\zeta) dm(z)$$

$$= O(4^n) + T_{N+1}^2 \int_{1 \leq |\eta| \leq 4} \int_{2^n/T_{N+1} \in |\xi| \leq 2^n/T_{N+1}} \left|\log|1 - \frac{\xi}{\eta}|\right| d\mu_{N+1}(T_{N+1}\eta) dm(\xi)$$

$$= O(4^n) + O(4^n) \int_{1 \leq |\eta| \leq 4} \int_{|\xi| \leq 2} \left|\log|1 - \frac{\xi}{\eta}|\right| d\mu_{N+1}(T_{N+1}\eta) dm(\xi).$$

(2.7)

However, elementary calculations show that

$$\int_{|\xi| \leq 2} \left|\log|1 - \frac{\xi}{\eta}|\right| dm(\xi) \leq C_3$$

for $1 \leq \eta \leq 4$. Recalling that $\mu_{N+1}(C) = 2$, from (2.7) we deduce the estimate

$$\int_{2^n \leq |z| \leq 2^{n+1}} \int_{A_{N+1}} \left|\log|1 - \frac{z}{\zeta}|\right| d\mu_{N+1}(\zeta) dm(z) = O(4^n), \quad n \to +\infty.$$
Similarly, if \( r_{N+1} \geq 2^{n+2} \geq 2|z| \), then \( |\log|1 - z/r_{N+1}|| \leq \log 2 \); otherwise, \( T_{N+1} \leq r_{N+1} < 2^{n+2} \), whence \( (T_{N+1} \xi = z) \)

\[
\int_{2^n \leq |z| \leq 2^{n+1}} \left| \log \frac{1 - z}{r_{N+1}} \right| \,dm(z)
\]

(2.9) \[= O(4^n) + O \left( 4^n \int_{1/2 \leq |\xi| \leq 1} \left| \log \frac{1 - \xi}{\eta} \right| \,dm(\xi) \right) = O(4^n), \quad n \to +\infty. \]

Now, we estimate the integral of \(|d_N(z)|\) over the annulus \( \{2^n \leq |z| \leq 2^{n+1}\} \). Putting \( \varphi(x) \equiv x/\psi(x) \), we have\[
\int_{A_N} \int_{2^n \leq |z| \leq 2^{n+1}} \left| \log \frac{z - \zeta}{\zeta} \right| \,dm(z) \,d\mu_N(\zeta)
\]

\[\leq \int_{2^n-1 \leq |\xi| \leq 2^{n+2}} \left( \int_{D_\zeta(\varphi(|\xi|))} \frac{\varphi(|\xi|)}{2} \log \psi(|\xi|) + \int_0^{|\xi|} s \,ds \right) \,d\mu_N(\zeta)
\]

\[+ \left( \int_{T_N \leq |\xi| \leq 2^{n+1}} \int_{2^n \leq |z| \leq 2^{n+1}} \left| \log \frac{z - \zeta}{\zeta} \right| \,dm(z) \,d\mu_N(\zeta) \right)
\]

\[\equiv I_{N,1} + I_{N,2} + I_{N,3} + I_{N,4}. \]

Integrating by parts and using the relation \( \varphi(x) = o(x), \ x \to +\infty \), we obtain \( I_{N,1} \)

\[= 2 \pi \int_{2^n-1 \leq |\xi| \leq 2^{n+2}} \left( \int_0^{\varphi(|\xi|)} \frac{|\xi|}{2} \log \psi(|\xi|) + \frac{\varphi(|\xi|)}{2} \right) \,d\mu_N(\zeta)
\]

\[= (\pi + o(1)) \int_{2^n-1 \leq |\xi| \leq 2^{n+2}} \varphi^2(|\xi|) \log \psi(|\xi|) \,d\mu_N(\zeta) = o(4^n), \quad n \to +\infty; \]

\[I_{N,2} \leq \int_{2^n-1 \leq |\xi| \leq 2^{n+2}} \int_{2^n \leq |z| \leq 2^{n+1}} \frac{|z - \zeta|}{\varphi(|\xi|)} \,dm(z) \,d\mu_N(\zeta)
\]

\[\leq \log \psi(2^{n+1}) 4^{n+1} \,d\mu_N(\{2^n \leq |\xi| \leq 2^{n+2}\}) = O(4^n \log \psi(2^n)), \quad n \to +\infty. \]

In the expressions for \( I_{N,3} \) and \( I_{N,4} \), the relationships between \( \zeta \) and \( z \) yield, respectively,

\[\left| \log \frac{z - \zeta}{\zeta} \right| \leq \log \frac{2^{n+1} + T_N}{T_N} \leq \log \left( 1 + 2 \frac{T_{N+1}}{T_N} \right) \]

\[\leq \log \psi(T_N) + O(1), \quad n \to +\infty. \]

\[\left| \log \frac{z - \zeta}{\zeta} \right| \leq 2 \frac{z}{\zeta} \leq 1. \]
Therefore, $I_{N,3} + I_{N,4} = O(4^n \log \psi(2^n))$, $n \to +\infty$. In a similar way we can obtain the estimate
\[
\int_{2^n \leq |z| \leq 2^{n+1}} \left| \log \left| 1 - \frac{z}{r_N} \right| \right| \, dm(z) = O(4^n \log \psi(2^n)), \quad n \to +\infty.
\]
Combining the estimates for $I_{N,j}$, $j \in \{1, \ldots, 4\}$, and the latter inequality, we obtain
\[
(2.10) \int_{2^n \leq |z| \leq 2^{n+1}} |d_N(z)| \, dm(z) = O(4^n \log \psi(2^n)), \quad n \to +\infty.
\]
From the definition (2.4) it follows that
\[
\int_{2^n \leq |z| \leq 2^{n+1}} |d_{N-1}(z)| \, dm(z)
\leq \int_{2^n \leq |z| \leq 2^{n+1}} \int_{A_{N-1}} \left| \log \left| 1 - \frac{\zeta}{z} \right| \right| \, d\mu_{N-1}(\zeta) \, dm(z).
\]
Much as in the treatment of the integral of $d_{N+1}(z)$, for $|\zeta| \leq |z|/2$, $\zeta \in A_{N-1}$ we observe that $|\log |1 - z/\zeta|| \leq \log 2$, whence
\[
\int_{2^n \leq |z| \leq 2^{n+1}} \int_{A_{N-1}} \left| \log \left| 1 - \frac{\zeta}{z} \right| \right| \, d\mu_{N-1}(\zeta) \, dm(z)
\leq \int_{2^n \leq |z| \leq 2^{n+1}} \int_{2^{n-1} \leq |\zeta| \leq T_N} \left| \log \left| 1 - \frac{\zeta}{z} \right| \right| \, d\mu_{N-1}(\zeta) \, dm(z) + O(4^n)
\leq \int_{T_N/2 \leq |\zeta| \leq T_N} \int_{2^n \leq |z| \leq 2^{n+1}} \left| \log \left| 1 - \frac{\zeta}{z} \right| \right| \, dm(z) \, d\mu_{N-1}(\zeta) + O(4^n)
\leq \frac{T_N^2}{4} \int_{2^n \leq |z| \leq 2^{n+1} / T_N} \left| \log \left| 1 - \frac{\zeta}{z} \right| \right| \, dm(T_N \zeta) \, d\mu_{N-1}(T_N \eta) + O(4^n)
\]
\[
= O(T_N^2) + O(4^n) = O(4^n), \quad n \to +\infty.
\]
Finally, from (2.8)–(2.10) and the latter relation we get
\[
(2.11) \int_{2^n \leq |z| \leq 2^{n+1}} \left| u_2(z) - \log |f_2(z)| \right| \, dm(z) = O(4^n \log \psi(2^n)), \quad n \to +\infty.
\]

2.3. Approximation of $u_1(z)$. We recall that $Q_k = \{ \zeta : R_k \leq |\zeta| \leq R_k \psi(R_k) \}$, $\text{supp} \, \mu_k^{(1)} \subset Q_k$, $\mu_k^{(1)}(Q_k) \in 2\mathbb{Z}_+$. Let
\[
P_k = \log Q_k = \{ s = \sigma + it : \log R_k \leq \sigma \leq \log R_k + \log \psi(R_k), 0 \leq t < 2\pi \},
\]
and let $l_k$ denote the ratio of the larger side of the rectangle $P_k$ to the smaller. For $k \geq k_0$ we have $l_k = \log \psi(R_k)/(2\pi) > 1$. Consider the measure $\mu_k^{(1)}$. If $\mu_k^{(1)}(\{p\}) \geq 2$ at some point $p$, from this measure we subtract the measure $\tilde{\mu}_k^{(1)}$ equal to $2|\mu_k^{(1)}(\{p\})|/2$ at every such $p$. The measure $\tilde{\mu}_k^{(1)} = \sum_k \tilde{\mu}_k^{(1)}$ is discrete, integer-valued, and finite on the compact sets in $C$. By the Weierstrass theorem, there exists an entire function $f_3$ with zeros of the corresponding multiplicity on the support of the measure $\tilde{\mu}_k^{(1)}$ (this support is an at most countable set of isolated points). We have $\mu_{\log_{|f_2|}} = \tilde{\mu}_k^{(1)}$. For every $k \in \mathbb{N}$, the measure $\mu^k = \mu_k^{(1)} - \mu_k^{(1)}|_{Q_k}$ satisfies the condition $\mu^k(\{p\}) < 2$ at every point $p \in Q_k$. By the
choice of the origin, on the rays emanating from it and on the circles centered at it there is at most one point \( p \) such that \( 0 < \mu^k(\{p\}) < 2 \), and at the same time the \( \mu^k \)-measure of the remaining part of either a ray or a circle is equal to zero. Under these conditions, the measures \( \nu^k \) defined by \( d\nu^k(s) = d\mu^k(e^s) \), \( s \in P_k \) (i.e., \( \nu^k(S) = \mu^k(\exp S) \) for every Borel set \( S \)), satisfy the conditions of Theorem 3 with \( \Pi = P_k \), \( k \in \mathbb{N} \). Applying that theorem, we obtain a system of rectangles \( P_k \) and of measures \( \nu_k \), \( 1 \leq m \leq N_k \). We have \( \nu_k(P_k) = 2 \), and every point \( s \) with \( |\text{Im} s| \leq 2\pi \) belongs to the interiors of at most four rectangles \( P_k \). We enumerate \( (P_k, \nu_k) \) by natural numbers, in an arbitrary way. As a result, we obtain \( (P_k, \nu_k) \) with \( \nu_k(P_k) = 2 \) and \( \text{supp} \nu_k \subset P_k \); then also the remaining properties listed in Theorem 3 are fulfilled. Let \( \omega_1^{(1)}, \omega_1^{(2)} \) be the solutions of the system of equations

\[
\begin{align*}
\omega_1^{(1)} + \omega_1^{(2)} &= \int_{P(1)} \omega \, d\nu(1)(\omega), \\
(\omega_1^{(1)})^2 + (\omega_1^{(2)})^2 &= \int_{P(1)} \omega^2 \, d\nu(1)(\omega),
\end{align*}
\]

(2.12)

and let

\[
\omega_1 = \frac{1}{2} \int_{P(1)} \omega \, d\nu(1)(\omega)
\]

(2.13)

be the center of mass of \( P(1) \), \( l \in \mathbb{N} \). From (2.12) we see that \( \omega_1^{(1)} + \omega_1^{(2)} = 2\omega_1 \),

\[
\int_{P(1)} \omega^2 \, d\nu(1)(\omega) = (\omega_1^{(1)})^2 + (2\omega_1 - \omega_1^{(1)})(2\omega_1 - \omega_1^{(1)})^2 = 2(\omega_1^{(1)} - \omega_1^2) + 2\omega_1^2.
\]

Therefore, by (2.13), for \( j \in \{1, 2\} \) we obtain

\[
|\omega_1^{(j)} - \omega_1| = \left| \frac{1}{2} \int_{P(1)} \omega^2 \, d\nu(1)(\omega) - \omega_1^2 \right|^\frac{1}{2}
\]

\[
= \left| \frac{1}{2} \int_{P(1)} (\omega - \omega_1)^2 \, d\nu(1)(\omega) \right|^\frac{1}{2} \leq \text{diam } P(1) \equiv d_1.
\]

Since \( \omega_1 \in P(1) \), we have

\[
\sup_{\omega \in P(1)} |\omega - \omega_1^{(j)}| \leq 2d_1, \quad j \in \{1, 2\}, \quad \sup_{\omega \in P(1)} |\omega - \omega_1| \leq d_1.
\]

Put \( \zeta_1^{(j)} = \exp\{\omega_1^{(j)}\} \) and

\[
V(z) = \sum_{l} \int_{\zeta_1^{(1)}} \left( \log |1 - \frac{z}{\zeta_1^{(1)}}| - \frac{1}{2} \log |1 - \frac{z}{\zeta_1^{(2)}}| - \frac{1}{2} \log |1 - \frac{z}{\zeta_2^{(2)}}| \right) d\mu(1)(\zeta)
\]

\[
= \sum_{l} \int_{P(1)} \left( \log |1 - z e^{-\omega_1}| - \frac{1}{2} \log |1 - z e^{-\omega_1^{(1)}}| - \frac{1}{2} \log |1 - z e^{-\omega_1^{(2)}}| \right) d\nu(1)(\omega)
\]

\[
= \sum_{l} \Delta_1(z).
\]
Under the assumption that the series (2.15) converges absolutely, we need to prove that
\[ \int_{2^n \leq |z| \leq 2^{n+1}} |V(z)| \, dm(z) = O(4^n \log \psi(2^n)). \]

Let \( \mathcal{L}^+ \) be the set of \( l \)'s such that \( Q^{(l)} \subset D_0(2^{n-2}) \), let \( \mathcal{L}^- \) consist of all \( l \) with \( Q^{(l)} \subset \{ z : |z| > 2^{n+1} \} \), and let \( \mathcal{L}^0 = \mathbb{N} \setminus (\mathcal{L}^- \cup \mathcal{L}^+) \). We denote \( L(\omega) = \log(1 - ze^{-\omega}) \). For \( l \in \mathcal{L}^- \cup \mathcal{L}^+ \), \( \omega \in P^{(l)} \), and \( 2^n \leq |z| \leq 2^{n+1} \), the function \( L(\omega) \) is analytic. We shall use the following identities:

\[
L(\omega) - L(\omega^{(1)}) = \int_{\omega^{(1)}} L'(s) \, ds
\]

\[
= L'(\omega)(\omega - \omega^{(1)}) + \int_{\omega^{(1)}} L''(s)(\omega - s) \, ds
\]

\[
= L'(\omega)(\omega - \omega^{(1)}) + \frac{1}{2} L''(\omega)(\omega - \omega^{(1)})^2 + \frac{1}{2} \int_{\omega^{(1)}} L''(s)(\omega - s)^2 \, ds.
\]

(2.16)

Using the first identity in (2.16), we obtain

\[
|\Delta_l(z)| = |\text{Re} \int_{P^{(l)}} \left( L(\omega) - L(\omega^{(1)}) - \frac{1}{2}(L(\omega^{(2)}) - L(\omega^{(1)})) \right) \, d\nu^{(l)}(\omega)|
\]

\[
\leq \int_{P^{(l)}} \left| \int_{\omega^{(1)}} \frac{z}{e^s - z} \, ds \right| \, d\nu^{(l)}(\omega) + \frac{1}{2} \int_{P^{(l)}} \left| \int_{\omega^{(1)}} \frac{z}{e^s - z} \, ds \right| \, d\nu^{(l)}(\omega)
\]

\[
\leq \sup_{s \in P^{(l)}} \frac{1}{e^s - z} \left( \int_{P^{(l)}} (|\omega - \omega^{(1)}| + \frac{1}{2} |\omega^{(2)} - \omega^{(1)}|) \, d\nu^{(l)}(\omega) \right)
\]

\[
\leq \frac{3d_l |z|}{\inf_{s \in P^{(l)}} (e^s - z)}.
\]

Let \( l \in \mathcal{L}^- \). The rectangle \( P^{(l)} \) is of the form \( \{ s = \sigma + it : \sigma^- \leq \sigma \leq \sigma^+, 0 \leq t^- \leq t \leq t^+ \leq 2\pi \} \), i.e., the lengths of its sides are equal to \( \sigma^+ - \sigma^- \) and \( t^+ - t^- \). Let \( \lambda_l \) be the ratio of the maximal of these numbers to the minimal (\( \lambda_l \geq 1 \)). By condition 4) in Theorem 3, for \( \lambda_l > 3 \) we have \( t^+ - t^- = 2\pi, \sigma^+ - \sigma^- = 2\pi \lambda_l \). First, we estimate the quantities \( |\Delta_l(z)| \) with \( \lambda_l > 3 \). Then \( d_l = 2\pi \sqrt{1 + \lambda_l^2} \leq 2 \log \psi(e^{\sigma^+}) \), because \( \log R_k \leq \sigma^- < \sigma^+ \leq \log R_k + \log \psi(R_k) \) for some \( k \in \mathbb{N} \). From (2.17) it follows that

\[
\inf_{s \in P^{(l)}} |e^s - z| \geq \inf_{s \in P^{(l)}} \frac{e^s}{2} \geq \frac{e^{\sigma^-}}{2}, \quad l \in \mathcal{L}^-.
\]

But

\[
\int_{\sigma^-}^{\sigma^+} e^{-s} \, ds = e^{-\sigma^-} \left( 1 - e^{-\sigma^+ + \sigma^-} \right) \geq e^{-\sigma^-}/2.
\]
Therefore,

\begin{equation}
(2.18) \quad |\Delta_l(z)| \leq \frac{6d_l|z|}{e^{\sigma_l}} \leq 24|z| \int_{\sigma_l}^{\sigma_l^+} e^{-\sigma} \log \psi(e^\sigma) \, d\sigma.
\end{equation}

Since \( \psi(x) \) varies slowly, so is \( \log \psi(x) \); consequently,

\[ \log \psi(x^{2^{k+1}}) \leq (1 + \epsilon) \log \psi(x^{2^k}) \leq (1 + \epsilon)^{k+1} \log \psi(x), \quad \epsilon > 0, \quad x \geq x^* \cdot \]

Applying the above estimates and the fact that every point \( s \) belongs to the interiors of at most 4 of the rectangles \( P^{(l)} \), from (2.18) we deduce the inequality

\begin{equation}
(2.19) \quad \sum_{l \in \mathcal{L}^-, \lambda_l > 3} |\Delta_l(z)| \leq 24|z| \int_{\log 2^{n+1}}^{+\infty} e^{-\sigma} \log \psi(e^\sigma) \, d\sigma
\end{equation}

\[ \leq 24|z| \sum_{k=0}^{+\infty} \frac{\log 2^{n+2+k}}{2^{n+k+2}} \]

Using the first equation in (2.12), it is not difficult to obtain the following representation for \( \Delta_l \):

\begin{equation}
(2.20) \quad \Delta_l(z) = \int_{P^{(l)}} \left( \log \left| 1 - \frac{e^{\sigma_l^{(1)}}}{z} \right| - \frac{1}{2} \log \left| 1 - \frac{e^{\sigma_l^{(2)}}}{z} \right| - \frac{1}{2} \log \left| 1 - \frac{e^{\sigma_l^{(2)}}}{z} \right| \right) \, d\nu^{(l)}(\omega).
\end{equation}

We use (2.20) in order to estimate \( \Delta_l \) for \( l \in \mathcal{L}^+ \) with \( \lambda_l > 3 \). For the analytic function \( L_2(\omega) = \log(1 - \frac{\omega}{e^\lambda}) \) and \( z \in P^{(l)}, \quad l \in \mathcal{L}^+ \), identities (2.16) are true with \( L = L_2 \).

Therefore, as in (2.17), we obtain the estimate

\[ \sum_{l \in \mathcal{L}^+, \lambda_l > 3} |\Delta_l(z)| \leq \frac{3d_l}{\inf_{s \in P^{(l)}} \left| 1 - ze^{-s} \right|} \leq \frac{12e^{\sigma_l^+}d_l}{|z|}, \quad l \in \mathcal{L}^+.
\]

Since \( d_l \leq \log \psi(2^{n-1}) \), \( l \in \mathcal{L}^+ \), it follows that

\begin{equation}
(2.21) \quad \sum_{l \in \mathcal{L}^+, \lambda_l > 3} |\Delta_l(z)| \leq C_3 \frac{\log \psi(2^{n-1})}{|z|} \sum_{l \in \mathcal{L}^+, \lambda_l > 3} e^{\sigma_l^+} \int_{\sigma_l}^{\sigma_l^+} e^{-\sigma} \, d\sigma
\end{equation}

\[ \leq 4C_3 \frac{\log \psi(2^{n-1})}{|z|} \int_{\log 2^{n-1}}^{+\infty} e^{-\sigma} \, d\sigma \leq C_4 \log \psi(2^{n-1}).
\]

For \( l \in \mathcal{L}^+ \cup \mathcal{L}^- \), it remains to estimate the \( \Delta_l \) with \( 1 \leq \lambda_l \leq 3 \). Then \( d_l^2 \approx m(P^{(l)}) \), in particular, \( d_l^2 \leq 6\pi \). In this case, arguing as in [10] and using the second indentity in
(2.16), we get
\[
\Delta_l(z) = \text{Re} \int_{P(l)} \left( (\omega - \omega^{(1)}_l)L'((\omega^{(1)}_l)^2) - \frac{1}{2} L'((\omega^{(1)}_l)^2)(\omega^{(2)}_l - \omega^{(1)}_l) \right.
+ \int \frac{\omega^{(2)}_l}{(\omega^{(1)}_l)^2} \, ds - \frac{1}{2} \int \frac{\omega^{(2)}_l}{(\omega^{(1)}_l)^2} \, ds \right) d\nu^{(l)}(\omega)
\]
\[
= \text{Re} \left\{ -z \int_{P(l)} \frac{e^{s}(\omega - s)}{(e^{s} - z)^2} \, ds \, d\nu^{(l)}(\omega) + \frac{z}{2} \int_{P(l)} \frac{e^{s}(\omega^{(2)}_l - s)}{(e^{s} - z)^2} \, ds \, d\nu^{(l)}(\omega) \right\},
\]
whence
\[
(2.22) \quad |\Delta_l(z)| \leq 6d_l^2 |z| \sup_{s \in P(l)} \frac{|e^s|}{(e^s - z)^2}.
\]
If \( l \in \mathcal{L}^+ \), then \( |e^s - z| \geq |z|/2 \), and from (2.22) we deduce the inequality
\[
(2.23) \quad |\Delta_l(z)| \leq 24d_l^2 e^{\sigma^+_l} \leq \frac{C_4}{|z|} \int_{P(l)} e^s \, d\sigma \, dt.
\]
Applying (2.23), we obtain
\[
\sum_{l \in \mathcal{L}^+, \lambda_l \leq 3} |\Delta_l(z)| \leq \frac{C_4}{|z|} \sum_{l \in \mathcal{L}^+, \lambda_l \leq 3} \int_{P(l)} e^s \, d\sigma \, dt \leq 4C_4 \int_{0}^{2\pi} \frac{\log 2^{n-2}}{2} \int_{0}^{2\pi} e^s \, d\sigma \, dt
\]
\[
\quad \leq \frac{8\pi C_2 2^{n-2}}{|z|} = O(1).
\]
Now, let \( l \in \mathcal{L}^- \); then \( |e^s - z| \geq e^{\sigma^-}/2 \), and, therefore, (2.22) implies that
\[
|\Delta_l(z)| \leq 24d_l^2 |z|e^{-\sigma^-} \leq C_5 |z| \int_{P(l)} e^{-\sigma} \, d\sigma \, dt,
\]
\[
\sum_{l \in \mathcal{L}^-, \lambda_l \leq 3} |\Delta_l(z)| \leq 8\pi K_3 |z| \int_{\log 2^{n+1}}^{\infty} e^{-\sigma} \, d\sigma = 8\pi |z| 2^{-n-1}
\]
\[
\quad \leq C_6.
\]
Combining (2.19), (2.21), (2.24), and the latter estimate, we see that
\[
(2.25) \quad \sum_{l \in \mathcal{L}^+ \cup \mathcal{L}^-} |\Delta_l(z)| \leq C_7 \log \psi(2^{n+1}).
\]
Now, let \( l \in \mathcal{L}^0 \), i.e., \( Q^{(l)} \cap G_n \neq \emptyset \), where \( G_n = \{ z : 2^{n-2} \leq |z| \leq 2^{n+1} \} \). Among \( l \in \mathcal{L}^0 \), at most 8 are such that \( \lambda_l > 3 \). Indeed, for such \( l \) we have \( Q^{(l)} = \{ z : e^{\sigma^-} \leq |z| \leq e^{\sigma^+_l} \} \) and \( \sigma^+_l - \sigma^- \geq 6\pi \), so that \( e^{\sigma^+_l} - e^{\sigma^-} \geq 2^{n-1}(1 - e^{-6\pi}) \). Since the interiors of these \( Q^{(l)} \) form an at most 4-fold cover, for all \( l \) except at most 4 of them we have \( e^{\sigma^-} \geq 2^{n-1} \), and, except at most 8 of them, \( e^{\sigma^-} \geq e^{6\pi} 2^{n-1} > 2^{n+1} \), i.e., the intersection with \( G_n \) is empty.
We denote this exceptional set of indices \( l \) by \( \mathcal{L}_l^0 \). We must estimate \( \int_{G_n} |\Delta_l(z)| \, dm(z) \) for \( l \in \mathcal{L}_l^0 \). If \( |\zeta| \geq 3 \cdot 2^n \) (\( \zeta \in Q^{(l)} \), \( z \in G_n \)), then \( \log |1 - \frac{z}{\zeta}| \leq 2|z|/|\zeta| \leq 3 \). Otherwise, we have \( |\zeta| \leq 3 \cdot 2^n \), and

\[
\int_{D_\zeta(2^n)} \| \log |1 - \frac{z}{\zeta}| \| \, dm(z) \\
\leq \int_{D_\zeta(2^n)} \| \log \frac{z - \zeta}{2^n} \| + \| \log \frac{\zeta}{2^n} \| \, dm(z) \\
(2.26) \\
\leq 2\pi \int_0^{2^n} \log \frac{2^n}{\tau} \, d\tau + \sup_{\zeta \in Q^{(l)}} \| \log |2^{-n}| \| \pi 2^n \\
\leq \pi^2 4^{n-1} + C_9 \log \psi(2^n) \cdot 4^n.
\]

Thus,

(2.27) \[
\int \sum_{l \in \mathcal{L}^*} \left( \int_{Q^{(l)}} + \int_{Q^{(l)} \cap \{|\zeta| \leq 3 \cdot 2^n\}} \right) \left( \int_{G_n} \| \log \frac{|z|}{|\zeta|} \| \, dm(z) \right) d\mu^{(l)}(\zeta) \\
\leq \sum_{l \in \mathcal{L}^*} \left( 3 \cdot 2m(G_n) \right) \\
+ \int_{Q^{(l)} \cap \{|\zeta| \leq 3 \cdot 2^n\}} \left( \int_{D_\zeta(2^n)} \| \log |1 - \frac{z}{\zeta}| \| dm(z) \right) \\
\leq K_9 4^n + K_9 \log \psi(2^n) \cdot 4^n \\
+ \int_{Q^{(l)} \cap \{|\zeta| \leq 3 \cdot 2^n\}} \sum_{G_n \setminus D_\zeta(2^n)} \log \frac{|z| + |\zeta|}{|\zeta|} \, dm(z) d\mu^{(l)}(\zeta) \\
\leq K_9 4^n \log \psi(2^n).
\]

For \( l \in \mathcal{L}_l^0 \setminus \mathcal{L}_l^0 \) we have \( \lambda_l \leq 3 \), i.e., all the corresponding rectangles \( P^{(l)} \) are “almost squares”; this allows us to apply the arguments used in [10, e.g.] Let \( D_l = \text{diam} Q^{(l)} \). Under the condition \( \text{dist}\{z, Q^{(l)}\} > 4D_l \), we can use the last identity in (2.16), and
argue as in (2.22) to obtain

\[
|\Delta_l(z)| = \left| \frac{1}{2} \text{Re} \int_{\Omega_l^{(1)}} z \int_{\omega_l^{(1)}} \frac{e^{s} z + e^{s}}{(e^{s} - z)^3} (\omega - s)^2 \, ds \, d\nu_l^{(1)}(\omega) \right|
\]

\[
+ \frac{1}{2} \text{Re} z \int_{\omega_l^{(1)}} e^{s} z + e^{s} (\omega_l^{(2)} - s)^2 \, ds \left| \frac{d_l^1 |z|^3}{|\zeta_l^{(1)} - z|^3} \leq D_l^3 \leq \frac{D_l^3}{|\zeta_l^{(1)} - z|^3}. \right.
\]

(2.28)

It follows that

\[
(2.29) \quad \int_{G_n} \sum_{l \in L^{0}, \lambda_l^{1} \leq 3} |\Delta_l(z)| \, dm(z) \leq \sum_{l \in L^{0}, \lambda_l^{1} \leq 3} \int_{G_n} |\Delta_l(z)| \, dm(z)
\]

\[
\leq \sum_{l \in L^{0}, \lambda_l^{1} \leq 3} \left( \int_{G_n \cap \{|z - \zeta_l^{(1)}| > 3D_l^{1}\}} |\Delta_l(z)| \, dm(z) + \int_{|z - \zeta_l^{(1)}| \leq 3D_l^{1}} \right) |\Delta_l(z)| \, dm(z)
\]

For the first sum in (2.29) we have

\[
(2.30) \quad \sum_{l \in L^{0}, \lambda_l^{1} \leq 3} D_l^3 \int_{G_n \cap \{|z - \zeta_l^{(1)}| > 3D_l^{1}\}} \frac{1}{|z - \zeta_l^{(1)}|^3} \, dm(z)
\]

\[
\leq \sum_{l \in L^{0}, \lambda_l^{1} \leq 3} D_l^3 2 \pi \int_{3D_l^{1}} \frac{1}{t^3} \, dt
\]

\[
\leq \sum_{l \in L^{0}, \lambda_l^{1} \leq 3} 2D_l^2 \leq \sum_{l \in L^{0}, \lambda_l^{1} \leq 3} m(Q_l^{(1)}).
\]

It remains to estimate \( \int_{|z - \zeta_l^{(1)}| \leq 3D_l^{1}} |\Delta_l(z)| \, dm(z) \). The definition of \( \Delta_l(z) \) and the relation \( \log |\zeta_l^{(1)}| + \log |\zeta_l^{(2)}| = \int_{Q_l^{(1)}} \log |\zeta| \, d\mu_l^{(1)}(\zeta) \) imply

\[
|\Delta_l(z)| = \int_{Q_l^{(1)}} \left( \log \left| \frac{\zeta - z}{3D_l} \right| - \frac{1}{2} \log \left| \frac{\zeta_l^{(1)} - z}{3D_l} \right| - \frac{1}{2} \log \left| \frac{\zeta_l^{(2)} - z}{3D_l} \right| \right) d\mu_l^{(1)}(\zeta).
\]
Thus,

\[
\int |\Delta_l(z)| \, dm(z) \\
\leq \int_{Q(l)} \left( \int_{|z - \zeta| \leq 3D_l} \left| \log \left( \frac{\zeta - z}{3D_l} \right) \right| + \frac{1}{2} \left| \log \left( \frac{\zeta^{(1)} - z}{3D_l} \right) \right| \\
+ \frac{1}{2} \left| \log \left( \frac{\zeta^{(2)} - z}{3D_l} \right) \right| \, dm(z) \right) \, d\mu^{(l)}(\zeta)
\]

\[= \int_{Q(l)} \left( \int_{|z - \zeta| \leq D_l} \left| \log \left( \frac{\zeta - z}{3D_l} \right) \right| \, dm(z) \right) \, d\mu^{(l)}(\zeta)
\]

\[+ \int_{|z - \zeta| < D_l} \log \frac{3D_l}{|\zeta^{(1)} - z|} \, dm(z)
\]

\[+ \left( \int_{|z - \zeta^{(1)}| \leq 3D_l} \left[ \int_{|z - \zeta^{(2)}| > 3D_l} \left| \log \left( \frac{\zeta^{(2)} - z}{3D_l} \right) \right| \, dm(z) \right] \right)
\]

\[\leq \int_{Q(l)} \left( \int_{0}^{D_l} \frac{3D_l}{\tau} \, d\tau + (3D_l)^2 \pi \log 3 \right) \, d\mu^{(l)}(\zeta)
\]

\[+ 2 \int_{0}^{D_l} \log \frac{3D_l}{\tau} \, d\tau + \pi (3D_l)^2 \log 3
\]

\[\leq 9(3\pi \log 3 + 1)D_l^2 \leq C_{11} m(Q(l)).
\]

The latter inequality and (2.29) yield

\[
\int_{G_n} \sum_{l \in \mathcal{L}^0} |\Delta_l(z)| \, dm(z) \leq \sum_{l \in \mathcal{L}^0 \setminus \mathcal{L}_n^0} K_{11} m(Q(l)) \leq K_{12} 2^n.
\]

Applying this inequality and (2.25) completes the proof of Theorem 1. Using Chebyshev's inequality, from (2.2) we can easily deduce Corollary 1.

§3. PROOF OF THEOREM 2'

Suppose that \( \sigma \) satisfies the conditions of the theorem. Without loss of generality, we may assume that \( \psi(r) = \exp \left\{ \int_0^r \frac{\psi(t)}{t} \, dt \right\} \) is unbounded. Obviously, \( \psi \in \Phi \). Let \( u_\psi \) be defined by formula (1.10) with \( \varphi = \psi \), \( \psi \in \Phi \). Suppose that there exists an entire function \( f \) and a constant \( \alpha \in [0, 1) \) satisfying

\[
\int_{|z| < R} \left| u_\psi(z) - \log |f(z)| - \alpha \log |z| \right| \, dm(z)
\]

\[< \varepsilon R^2 \log \psi(R), \quad R \geq R_\varepsilon,
\]

(3.1)
for arbitrary \( \varepsilon > 0 \). There is no loss of generality in assuming that \( f(0) \neq 0 \). By separating the term \( \frac{1}{2} \log |1 - z/r_1| \) from \( u_\psi \), we reduce case of \( \alpha \in [1/2, 1) \) to the case where \( \alpha \in [0, 1/2) \). Therefore, we consider the latter case in detail. We introduce the counting Nevanlinna characteristics of the Riesz masses \( u_\psi \) and \( f \):

\[
N(r, u_\psi) = \int_0^r \frac{n(t, u_\psi)}{t} \, dt,
\]

\[
N(r, f) = \int_0^r \frac{n(t, f)}{t} \, dt,
\]

where \( n(r, f) \) is the number of zeros of \( f \) in \( \overline{B}_0(r) \). By the Jensen formula (see [1, Chapter 3.9]), we have

\[
\frac{1}{2\pi} \int_0^{2\pi} (u_\psi(re^{i\theta}) - \log |f(re^{i\theta})| - \alpha \log r) \, d\theta = N(r, u_\psi) - N(r, f) - \alpha \log r - \log |f(0)|.
\]

If \( R \geq \tilde{R}_\varepsilon \) then, since the function \( \log \psi(R) \) varies slowly, we obtain

\[
\begin{align*}
&\int_R^{2R} |N(r, u_\psi) - N(r, f) - \alpha \log r - \log |f(0)|| r \, dr \\
&\leq \int_R^{2R} \frac{1}{2\pi} \int_0^{2\pi} \left| u_\psi(re^{i\theta}) - \log |f(re^{i\theta})| - \alpha \log r \right| d\theta \, dr \\
&\leq \frac{1}{2\pi} \int_{|z| \leq 2R} \left| u_\psi(z) - \log |f(z)| - \alpha \log |z| \right| \, dm(z) \\
&< \frac{\varepsilon}{2\pi} (2R)^2 \log \psi(2R) \\
&< \frac{2\varepsilon(1 + \varepsilon)}{\pi} R^2 \log \psi(R).
\end{align*}
\]

This implies that on \( [R, 2R] \) there exists \( r^* \) such that

\[
(3.2) \quad |N(r^*, u_\psi) - N(r^*, f) - \alpha \log r^*| \leq \frac{2\varepsilon(1 + \varepsilon)}{3} \log \psi(r^*) < \varepsilon \log \psi(r^*)
\]

for \( \varepsilon \in (0, 1/2) \). From (3.2) we shall derive the relation

\[
(3.3) \quad |n(r, u_\psi) - n(r, f) - \alpha| \leq \frac{1}{2}, \quad r \to +\infty.
\]

Suppose the contrary. If (3.3) fails, then there exists a sequence \( (r_k) \), \( r_k \to +\infty \) as \( k \to +\infty \), such that either i) \( n(r_k, u_\psi) - n(r_k, f) - \alpha < 1/2 \), or ii) \( n(r_k, f) - n(r_k, u_\psi) + \alpha > 1/2 \), \( k \to +\infty \). Consider case i). For arbitrary \( t \in [\tilde{r}_k, r_k] \), where \( \tilde{r}_k \psi(\tilde{r}_k) = r_k \), we have \( n(t, u_\psi) - n(\tau_k, u_\psi) \geq -1/2 \); therefore,

\[
n(t, u_\psi) - n(t, f) - \alpha \geq n(\tau_k, u_\psi) - n(\tau_k, f) - \alpha + n(t, u_\psi) - n(\tau_k, u_\psi) > 0.
\]

Since the values of \( n(t, u_\psi) - n(t, f) \) are integral multiples of \( 1/2 \), we have

\[
(3.4) \quad n(t, u_\psi) - n(t, f) - \alpha \geq 1/2 - \alpha > 0, \quad t \in [\tilde{r}_k, r_k].
\]
Choose \( t_k \in [\tau_k, 2\tau_k] \) and \( T_k \in [\tau_k/2, \tau_k] \) so that (3.2) be fulfilled for \( r^* \in \{t_k, T_k\} \). Then, using the definition of the function \( N(r, \cdot) \) and applying (3.2) and (3.4), for \( \epsilon \in (0, 1/4 - \alpha/2) \) we obtain

\[
\epsilon \log \psi(T_k) \quad > \quad |N(T_k, u_\psi) - N(T_k, f)| - \alpha \log T_k \\
\geq \int_{t_k}^{T_k} \frac{n(t, u_\psi) - n(t, f) - \alpha}{t} dt \quad > \quad |N(t_k, u_\psi) - N(t_k, f)| - \alpha \log t_k \\
> \quad \left( \frac{1}{2} - \alpha \right) \log \frac{T_k}{t_k} - \epsilon \log \psi(t_k) \\
= \quad \left( \frac{1}{2} - \alpha - \epsilon \right) \log \psi(t_k) > \epsilon \log \psi(T_k), \quad k \to +\infty.
\]

Thus, case i) is impossible. Similarly, in case ii) we have \( n(t, f) - n(t, u_\psi) + \alpha > 0 \) for \( t \in [\tau_k, \tau_k \psi(\tau_k)] \), whence \( n(t, f) - n(t, u_\psi) \geq \beta \), where \( \beta \) is a positive constant. Choosing \( t_k \in [\tau_k, 2\tau_k] \) and \( T_k \in [\tau_k \psi(\tau_k)/2, \tau_k \psi(\tau_k)] \) satisfying (3.2) instead of \( r^* \), and assuming that \( \epsilon \in (0, \beta/2) \), in the same way as before we obtain

\[
|N(T_k, f) - N(T_k, u_\psi) + \alpha \log T_k| \\
> \quad \int_{t_k}^{T_k} \frac{n(t, f) - n(t, u_\psi) + \alpha}{t} dt \quad > \quad |N(t_k, f) - N(t_k, u_\psi) + \alpha \log t_k| \\
> \quad (\beta - \epsilon) \log \psi(t_k).
\]

Therefore, case ii) is also impossible. Thus, relation (3.3) is proved.

Let \( \rho_k \) be the modulus of the \( k \)th zero of \( f \) (the zeros are enumerated in the order of nondecreasing of their moduli). Since the jumps of \( n(t, f) \) take natural values, and the jumps of \( n(t, u_\psi) \) take the value \( \frac{1}{2} \), relation (3.3) is possible only in the case when, starting with some \( k_0 \in \mathbb{N} \), between every two neighboring jump points \( \rho_k \leq \rho_{k+1} \) of the function \( n(t, f) \) there are points \( r_{m}, r_{m+1} \), in particular, \( \rho_k < \rho_{k+1} \). First, we consider the case where \( \alpha = 0 \). If \( \rho_k < r_{2k-1} \), then for \( r \in (\rho_k, r_{2k-1}) \) we have \( n(r, f) - n(r, u_\psi) \geq k - (2k - 2)/2 = 1 \). But if \( \rho_k > r_{2k} \), then \( n(r, u_\psi) - n(r, f) \geq 1 \) for \( r \in (\max\{r_{2k}, r_{k+1}\}, \rho_k) \). Therefore, \( r_{2k-1} \leq \rho_k \leq r_{2k} \) starting with some \( k \geq k_1 \). Thus,

\[
n(t, u_\psi) - n(t, f) = \begin{cases} \frac{1}{2} & \text{if } t \in [r_{2k-1}, \rho_k), \\ -\frac{1}{2} & \text{if } t \in [\rho_k, r_{2k}). \end{cases}, \quad k \geq k_1.
\]

If \( \rho_k \in [r_{2k-1}, \sqrt{r_{2k-1}r_{2k}}] \), then, choosing \( r^*_k \in [\rho_{k}, 2\rho_k] \) and \( t^*_k \in [r_{2k}/2, r_{2k}] \) such that (3.2) is satisfied with \( r^* \in \{r^*_k, t^*_k\} \), we obtain

\[
N(t^*_k, f) - N(t^*_k, u_\psi) \\
= \quad \int_{r^*_k}^{t^*_k} n(s, f) - n(s, u_\psi) ds + N(r^*_k, f) - N(r^*_k, u_\psi) \\
> \quad \frac{1}{2} \log \frac{r^*_k}{t^*_k} - \epsilon \log \psi(r^*_k) - \frac{1}{2} \log \frac{t^*_k}{\sqrt{r_{2k-1}r_{2k}}} - \epsilon \log \psi(r^*_k) \\
= \quad \left( \frac{1}{4} - \epsilon + o(1) \right) \log \psi(t^*_k), \quad k \to +\infty.
\]
which contradicts (3.2). If \( \rho_k \in [\sqrt{2^{r_k-1}}r_{2k}, r_{2k}] \), we choose \( r_k^* \in [r_{2k-1}, 2r_{2k-1}] \), \( t_k^* \in [\rho_k/2, \rho_k] \) satisfying (3.2). Since \( n(t, u_0) - n(t, f) = 1/2 \) for \( t \in (r_k^*, t_k^*) \), again we arrive at a contradiction with (3.2). Consequently, for \( \alpha = 0 \) the theorem is proved.

Now, let \( \alpha \in (0, 1/2) \). Relation (3.3) is possible only if the expression under the modulus sign takes the values \(-\alpha\) and \( \frac{1}{2} - \alpha \). Since the numbers \( r_k \) strictly increase, and the jump of \( n(r, u_\phi) \) equals \( \frac{1}{2} \) for \( r = r_k \) and the jump \( n(r, f) \) equals 1 for \( r = \rho_k \), we see that \( \rho_k = r_{2k} - k \geq k_2 \). Also, we have

\[
(3.5) \quad n(t, u_0) - n(t, f) = \begin{cases} 
0 & \text{if } t \in [r_{2k}, r_{2k+1}], \\
\frac{1}{2} & \text{if } t \in [r_{2k+1}, r_{2k+2}], 
\end{cases} \quad k \geq k_2.
\]

Choosing \( t_k^* \in [r_{2k-1}, 2r_{2k-1}] \) and \( r_k^* \in [r_{2k}/2, r_{2k}] \) that satisfy (3.2), using (3.5), and arguing as above, we arrive at a contradiction with (3.2). Therefore, for any \( \alpha \in [0, 1/2] \) there exists no entire function \( f \) with property (3.1); consequently, the same is true for \( \alpha \in (0, 1) \). Theorem 2’ is proved.

The proof of Theorem 2 is a word for word repetition of that of Theorem 2’ for \( \alpha = 0 \), with the difference that \( \psi \in \Phi \) is given by the condition of the theorem.

I would like to thank Professor O. Skaskiv, who read the paper and made valuable suggestions, as well as other participants of the Lviv seminar on the theory of analytic functions, for valuable comments, which contributed to the improvement of the initial version of the paper. Also, I am indebted to the Institute of Mathematics at the Jagellonian University for hospitality during my stay in April 2002 in Kraków, where a part of this paper was written.

References


