Growth and Representation of Analytic and Harmonic Functions in the Unit Disc

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(Presented by M. M. Sheremeta)

Abstract. Let \( u(z) \) be harmonic in \( \{|z| < 1\} \), \( \alpha \geq 0 \), \( 0 < \gamma \leq 1 \). Let \( B(r,u) = \max \{ u(z) : |z| \leq r \} \), \( \omega(\delta, \psi) \) be the modulus of continuity of a function \( \psi \) defined on \([0, 2\pi]\). We prove that \( u(z) \) has the form

\[
u(re^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} P_\alpha(r, \varphi - t) \, d\psi(t),\]

where \( \psi \in BV[0, 2\pi] \) and \( \omega(\delta, \psi) = O(\delta^\gamma), \delta \downarrow 0 \), if and only if \( B(r,u) = O((1-r)^{\gamma-\alpha-1}), r \uparrow 1 \), and \( \sup_{0<r<1} \int_0^{2\pi} |u_\alpha(re^{i\varphi})| \, d\varphi < +\infty \). Here \( u_\alpha \) is the \( \alpha \)-fractional integral of \( u(re^{i\varphi}) \), \( P_\alpha(r,t) = \Gamma(1+\alpha) \Re(\frac{2}{(1-re^{it})^{\alpha+1}} - 1) \).

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1. Introduction and Main Results

1.1. Analytic Functions in the Unit Disc

Let \( D = \{ z \in \mathbb{C} : |z| < 1 \} \). Denote by \( A(D) \) the class of analytic functions in \( D \). For \( f \in A(D) \), let \( M(r,f) = \max \{|f(z)| : |z| = r\} \) be the maximum modulus, \( T(r,f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta, 0 < r < 1 \), the Nevanlinna characteristics, \( x^+ = \max\{x,0\} \).

Usually, the orders of the growth of analytic functions in \( D \) are defined by

\[
\rho_M[f] = \limsup_{r \uparrow 1} \frac{\log^+ \log^+ M(r,f)}{-\log(1-r)}, \quad \rho_T[f] = \limsup_{r \uparrow 1} \frac{\log^+ T(r,f)}{-\log(1-r)}.
\]

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It is well known that
\[ \rho_T[f] \leq \rho_M[f] \leq \rho_T[f] + 1, \quad (1.1) \]
and all cases are possible. This is in contrast to entire functions where the corresponding orders are equal. We recall a couple of results concerning (1.1).

Given \( \alpha > 1 \) and \( \rho \) satisfying \( \rho \leq \alpha \leq \rho + 1 \), C. N. Linden [6] constructed an analytic function in \( \overline{D} \setminus \{1\} \) of the form of the so-called Naftalevich–Tsuji product,
\[ g(z) = \prod_{n=1}^{\infty} E\left(\frac{1 - |a_n|^2}{1 - a_nz}, p\right), \quad \sum_n (1 - |a_n|)^{p+1} < \infty, \]
with the property that \( \rho_T[g] = \rho, \rho_M[g] = \alpha \). Here
\[ E(w, p) = (1 - w) \exp\{w + w^2/2 + \cdots + w^p/p\}, \quad p \in \mathbb{Z}_+, \]
is the Weierstrass primary factor, \( a_n \) are the zeros of \( g(z) \).

Another approach is used in the paper by M. Sheremeta [9] where, in particular, for given \( \alpha > 0 \), a class of analytic functions \( f \) represented by gap series (with Hadamard’s gaps) is extracted such that
\[
\int_0^1 (1 - r)^{1+\alpha} T(r, f) \, dr < +\infty \Leftrightarrow \int_0^1 (1 - r)^{1+\alpha} \log M(r, f) \, dr < +\infty.
\]

Prof. O. Skaskiv posed the following problem.

**Problem 1.1.** Given \( 0 \leq \rho \leq \alpha \leq \rho + 1 \), describe the class of analytic function in \( D \) such that \( \rho_T[f] = \rho, \rho_M[f] = \alpha \).

In order to solve Problem 1.1 one needs a parametric representation of functions that are analytic in \( D \) and have finite order of growth. Such a representation was obtained [1] in 1960th by M. M. Djrbashian using the Riemann–Liouville fractional integral.

For \( \alpha > 0 \), consider two subclasses of \( A(D) \),
\[ A_\alpha : \sup_{0 < r < 1} \int_0^{2\pi} \left( \int_0^{r} (r - t)^{\alpha-1} \log |f(te^{i\varphi})| \, dt \right) d\varphi < +\infty, \]
\[ A_\alpha^* : \sup_{0 < r < 1} \int_0^{2\pi} \left( \int_0^{r} (r - t)^{\alpha-1} \log^+ |f(te^{i\varphi})| \, dt \right) d\varphi < +\infty. \]
Obviously, \( A^*_\alpha \subset A_\alpha \). Note that \( f \in A^*_\alpha \) means that \( \int_0^1 T(t, f)(1 - t)^{\alpha - 1} dt < +\infty \), i.e., \( f \) belongs to the convergence class of order \( \alpha \).

Throughout this paper, \((1 - w)^\alpha, w \in D, \alpha \in \mathbb{R}\), means the branch of the power function such that \( (1 - w)^\alpha \big|_{w=0} = 1 \).

**Theorem A.** The class \( A_\alpha \) coincides with the class of functions represented in the form

\[
f(z) = C_\lambda z^\lambda B_\alpha(z) \exp \left\{ \frac{2\pi}{\int_0^\infty \frac{d\psi(\theta)}{(1 - e^{-i\theta}z)^{\alpha+1}}} \right\}
\]

\[\equiv C_\lambda z^\lambda B_\alpha(z) \exp \{g_\alpha(z)\}, \quad (1.2)\]

where \( \psi \in BV[0, 2\pi] \), \((z_k)\) is a sequence of zeros of \( f(z) \) such that \( \sum_k (1 - |z_k|)^{\alpha+1} < +\infty \); \( B_\alpha(z) = \prod_k (1 - \frac{z}{z_k}) \exp \{-W_\alpha(z, z_k)\} \) is a Djrbashian product,

\[
W_\alpha(z, \zeta) = \sum_k \frac{\Gamma(\alpha + k + 1)}{\Gamma(\alpha + 1)\Gamma(1 + k)} \times \left( \left( \frac{\bar{\zeta}z}{\zeta} \right)^k \int_0^{1 - x^{\alpha + 1}} dx \right) \left( \frac{\bar{\zeta}z}{\zeta} \right)^k \int_0^{1 - x^{\alpha + 1}} dx \right) .
\]

In this paper, we will restrict the considerations to the case where \( f(z) \) has no zeros and has a finite order of growth. Then \( f(z) = C_\alpha \exp \{g_\alpha(z)\} \) for some \( \alpha \geq 0 \).

Radial and non-tangential limits of \( g_\alpha(z) \) were investigated in many papers, e.g., D. J. Hallenbeck, T. H. MacGregor [2, 3], and M. M. Sheremeta [10], even for complex-valued functions \( \psi \) of bounded variation. It turns out that \( g_\alpha(z) \) admits the above estimates in terms of the modulus of continuity of \( \psi \). We cite a typical result [10].

Let \( S(\theta, \gamma) \) be the closed Stolz angle having the vertex at \( e^{i\theta} \) and the opening \( \gamma \), i.e., \( S(\theta, \gamma) = \{z \in D : |\arg(e^{i\theta} - z)| < \gamma/2\} \). A function \( g \) defined in \( D \) is said to have a nontangential limit at \( e^{i\theta} \) provided that \( \lim_{z \to e^{i\theta}, z \in S(\theta, \gamma)} g(z) \) exists for every \( \gamma \in (0, \pi) \).

**Theorem B.** Let \( \alpha > -1, \theta \in [0, 2\pi], \psi \in BV[0, 2\pi], \) and \( \omega \) be a nonnegative, increasing continuous, semi-additive function on \([0, +\infty)\), and \( \omega(0) = 0 \). If

\[
\int_0^1 t^{-\alpha - 2}\omega(t) \, dt = \infty, \quad |\psi(t) - \psi(\theta)| = o(\omega(|t - \theta|)), \quad t \to \theta,
\]
and \( g_\alpha \) is given by (1.2), then
\[
|g_\alpha(z)| = \frac{1}{|1-ze^{-i\theta}|} \int_0^1 t^{-\alpha-2}\omega(t) \, dt
\]
has a nontangential limit zero at \( e^{i\theta} \).

Since lower estimates for \( |g_\alpha(z)| \) are known only in particular cases (see [5], Theorem D and Remark 1.3 below), it is interesting to obtain results which give lower estimates for \( |g_\alpha(z)| \) in the general situation.

The main purpose of this paper is to describe the growth of \( |g_\alpha(z)| \) in terms of the modulus of continuity for \( \psi \) and find counterparts for harmonic functions in \( D \).

Problem 1.1 is not solved, but Theorem 3.3 and the corollary describe large classes of analytic functions \( f \) with the property \( \rho_T[f] = \rho, \rho_M[f] = \alpha, 0 \leq \rho \leq \alpha \leq \rho + 1 \). Theorem 3.2 yields asymptotic formulas for \( g_\alpha \) in Stolz angles when \( \psi \) is not continuous.

### 1.2. Representation and Growth of Harmonic Functions

We need to make some definitions. Let \( U_\theta(\delta) = \{ x \in [0, 2\pi] : |x - \theta| < \delta \}, \delta > 0 \). For \( \psi : [0, 2\pi] \to \mathbb{R} \), define the moduli of continuity by \( \omega(\delta, \theta; \psi) = \sup \{|\psi(x) - \psi(y)| : x, y \in U_\theta(\delta)\} \), \( \omega(\delta; \psi) = \sup_{\theta \in [0, 2\pi]} \omega(\delta, \theta; \psi) \).

Following [12] we say that \( \psi \in \Lambda_\gamma \) if \( \omega(\delta; f) = O(\delta^\gamma) (\delta \downarrow 0) \).

The fractional integral of order \( \alpha > 0 \) for \( h : (0, 1) \to \mathbb{R} \) is defined by the formulas [1]
\[
D^{-\alpha}h(r) = \frac{1}{\Gamma(\alpha)} \int_0^r (r-x)^{\alpha-1}h(x) \, dx,
\]
\[
D^0h(r) \equiv h(r), \quad D^\alpha h(r) = \frac{d^p}{dr^p} \{D^{-(p-\alpha)}h(r)\}, \quad \alpha \in (p-1; p], \ p \in \mathbb{N}.
\]

Let \( H(D) \) be the class of harmonic functions on \( D \). We put \( u_\alpha(re^{i\varphi}) = r^{-\alpha}D^{-\alpha}u(re^{i\varphi}) \), where the fractional integral is taken with respect to the variable \( r \). We define \( B(r, u) = \max\{u(z) : |z| \leq r\} \) for a subharmonic function \( u \) on \( D \).

Let
\[
S_\alpha(z) = \Gamma(1+\alpha)\left(\frac{2}{1-\alpha+1} - 1\right), \quad P_\alpha(r, t) = \Re S_\alpha(re^{it}).
\]
Remark 1.1. Note that $S_0(z)$ is the Schwartz kernel, $P_0(r,t)$ is the Poisson kernel; $P_\alpha(r,t) = D^\alpha(r^\alpha P_0(r,t))$.

Our starting point is the following two theorems.

**Theorem C (M. Djrbashian).** Let $u \in H(D)$, $\alpha > -1$. Then

$$u(re^{i\varphi}) = \int_0^{2\pi} P_\alpha(r, \varphi - \theta) d\psi(\theta),$$

(1.3)

where $\psi \in BV[0, 2\pi]$ if and only if

$$\sup_{0 < r < 1} \int_0^{2\pi} |u_\alpha(re^{i\varphi})| d\varphi < +\infty.$$  

**Remark 1.2.** Actually, for $\alpha = 0$, it is a classical result of Nevanlinna on representation of $\log |F(z)|$ if $F$ belongs to the Nevanlinna class $N$.

**Theorem D (Hardy–Littlewood).** Let $u \in H(D)$, $0 < \gamma \leq 1$. Then

$$u(re^{i\varphi}) = \int_0^{2\pi} P_0(r, \varphi - t)v(t) dt$$

(1.4)

for some function $v \in \Lambda_\gamma$ if and only if

$$B\left(r, \frac{\partial u}{\partial \varphi}\right) = O((1 - r)^{\gamma - 1}), \quad r \uparrow 1.$$  

**Remark 1.3.** Theorem D was originally proved in [4] for analytic functions (cf. Theorem 1.2).

Applying methods of proofs of Theorems B and C, we prove the following theorem which describes the growth of functions of form (1.3).

**Theorem 1.1.** Let $u \in H(D)$, $\alpha \geq 0$, $0 < \gamma < 1$. Then $u(z)$ has form (1.3), where $\psi$ is a function of bounded variation on $[0, 2\pi]$ and $\psi \in \Lambda_\gamma$, if and only if

$$B(r, u) = O((1 - r)^{\gamma - \alpha - 1}), \quad r \uparrow 1,$$

and

$$\sup_{0 < r < 1} \int_0^{2\pi} |u_\alpha(re^{i\varphi})| d\varphi < +\infty.$$
Note that Theorem D corresponds to the case where $\psi$ is absolutely continuous, but Theorem 1.1 describes a general situation.

Similar to the way Theorem 1.1 was deduced from the proposition below, one can obtain the following generalization of Theorem D.

**Theorem 1.2.** Let $u \in H(D)$, $0 < \gamma < 1$, $\alpha \geq 0$. Then

$$u(re^{i\varphi}) = \int_0^{2\pi} P_\alpha(r, \varphi - t)v(t) dt$$

for some function $v \in \Lambda_\gamma$ if and only if

$$B(r, \frac{\partial u}{\partial \varphi}) = O((1 - r)^{\gamma - \alpha - 1}), \quad r \uparrow 1.$$

It is not difficult to prove a counterpart of the last theorem for analytic functions.

**Remark 1.4.** Similarly to [10], one can prove that if $u(z)$ is represented by (1.3), then

$$u(re^{i\vartheta}) = O\left(\int_{1-r}^{2\pi} \omega(\tau, \varphi; \psi) d\tau\right), \quad r \uparrow 1, \quad re^{i\vartheta} \in S(\varphi, \tau), \quad 0 \leq \tau < \pi.$$

**Problem 1.2.** To obtain necessary and sufficient conditions for local growth of $u \in H(D)$.

2. Proof of Theorem 1.1

2.1. The Case $\alpha = 0$.

First, let us prove an important particular case of Theorem 1.1, in spirit of Theorem D.

**Proposition 2.1.** Let $u \in H(D)$, $0 < \gamma \leq 1$. Then $u(z)$ has the form

$$u(re^{i\varphi}) = \int_0^{2\pi} P_0(r, \varphi - t) d\psi(t), \quad (2.1)$$

where $\psi$ has bounded variation on $[0, 2\pi]$ and $\psi \in \Lambda_\gamma$, if and only if

$$B(r, u) = O((1 - r)^{\gamma - 1}), \quad r \uparrow 1,$$

and

$$\sup_{0<r<1} \int_0^{2\pi} |u(re^{i\varphi})| d\varphi < +\infty.$$
In the sequel, the symbol $C$ with indices stands for some positive constants.

Proof of Proposition 2.1. First, we consider the case $\gamma = 1$. Note that the class $\Lambda_1$ consists of functions that are integrals of bounded functions. Thus it is sufficient to apply Theorem (6.3) [12, Ch. IV] which states that (1.4) holds if and only if $B(r, u)$ is bounded as $r \uparrow 1$.

Consider the case $\gamma \in (0, 1)$.

Necessity. The proof of necessity is standard (cf. [11, Ch. 8.2], [10]).

The following estimates of $P_0(r, t)$ are well known:

\[
|\frac{\partial}{\partial t} P_0(r, t)| \leq \frac{2}{(1-r)^2}, \quad |\frac{\partial}{\partial t} P_0(r, t)| \leq \frac{\pi^2}{t^2}, \quad r \geq \frac{1}{2}, \quad |t| \leq \pi.
\] (2.2)

We extend $\psi$ to $\mathbb{R}$ by the formula 
\[
\psi(t + 2\pi) - \psi(t) = \psi(2\pi) - \psi(0).
\]

Since $P_0(r, t)$ is a periodic and even function in $t$, we have

\[
u(re^{i\varphi}) = \int_{-\pi+\varphi}^{\pi+\varphi} P_0(r, \theta - \varphi) d(\psi(\theta) - \psi(\varphi))
\]

\[
= (\psi(\theta) - \psi(\varphi)) P_0(r, \theta - \varphi) |_{-\pi+\varphi}^{\pi+\varphi} - \int_{-\pi+\varphi}^{\pi+\varphi} \frac{\partial}{\partial \theta} (P_0(r, \theta - \varphi)) (\psi(\theta) - \psi(\varphi)) d\theta
\]

\[
= (\psi(2\pi) - \psi(0)) P_0(r, \pi) - \int_{-\pi}^{\pi} \frac{\partial}{\partial \tau} (P_0(r, \tau)) (\psi(\tau + \varphi) - \psi(\varphi)) d\tau.
\]

Hence, using (2.2), we obtain

\[
|u(re^{i\varphi})| \leq \frac{C_1(\psi)(1-r)}{1+r} + \left( \int_{|\tau| \leq 1-r} + \int_{1-r \leq |\tau| \leq \pi} \right) \left| \frac{\partial}{\partial \tau} P_0(r, \tau) \right| |\omega(|\tau|, \psi; \varphi)| d\tau
\]

\[
\leq o(1) + 2 \int_{|\tau| \leq 1-r} \frac{\omega(|\tau|, \psi; \varphi)}{(1-r)^2} d\tau + \int_{1-r \leq |\tau| \leq \pi} \frac{\pi^2}{\tau^2} |\omega(|\tau|, \psi; \varphi)| d\tau
\]

\[
\leq o(1) + 4 \frac{\omega(1-r, \psi; \varphi)}{1-r} + 2\pi \int_{1-r \leq \tau \leq \pi} \frac{\omega(\tau, \psi; \varphi)}{\tau^2} d\tau
\]

\[
\leq (2\pi^2 + 4) \int_{1-r \leq \tau \leq \pi} \frac{\omega(\tau, \psi; \varphi)}{\tau^2} d\tau + O(1), \quad r \uparrow 1.
\] (2.3)
Here, we used that the modulus of continuity is increasing. Since $\psi \in \Lambda_\gamma$, $\omega(\tau, \psi; \varphi) = O(\tau^\gamma)$ as $\tau \downarrow 0$. Thus, (2.3) yields

$$B(r, u) \leq C_2(\gamma)(1 - r)^{\gamma - 1}, \quad r \uparrow 1.$$ 

**Sufficiency.** Let $u(re^{i\varphi})$ be harmonic for $r < 1$, and $\int_0^{2\pi} |u(re^{i\varphi})| d\varphi \leq C_3$.

**Remark 2.1.** By Theorem C, we have (2.1), where $\psi \in BV[0, 2\pi]$, and one can take $\psi$ such that at any point $\theta$ of continuity of $\psi$, 

$$\psi(\theta) = \lim_{r_n \uparrow 1} \int_0^{\theta} u(r_n e^{i\varphi}) d\varphi \quad (2.4)$$

for some sequence $(r_n)$ ([1], [7, p. 57]).

Let $F(z)$ be an analytic function on $D_1$ such that $\Re F(z) = u(z)$. By the theorem of Zygmund [12, Th. (2.30), Ch. VII], $B(r, u) = O((1 - r)^{\gamma - 1})$ implies that $M(r, F) = O((1 - r)^{\gamma - 1})$ as $r \uparrow 1$.

Define the analytic function $\Phi(z) = \int_z^z F(\zeta) d\zeta$, $z \in D$. For any fixed $\varphi \in [0, 2\pi]$ and $0 < r' < r'' < 1$, we have

$$|\Phi(r'' e^{i\varphi}) - \Phi(r' e^{i\varphi})| = \left| \int_{r'}^{r''} F(\rho e^{i\varphi}) e^{i\varphi} d\rho \right|$$

$$\leq C_4 \int_{r'}^{r''} (1 - \rho)^{\gamma - 1} d\rho \leq \frac{C_4}{\gamma} (1 - r')^{1 - \gamma}.$$ 

Therefore, by Cauchy’s criterion, $\lim_{r \uparrow 1} \Phi(r e^{i\varphi}) \equiv \Phi(e^{i\varphi})$ exists uniformly in $\varphi$. Consequently, $\tilde{\Phi}(\varphi) \equiv \Phi(e^{i\varphi})$ is a continuous function on $[0, 2\pi]$.

Let us prove that $\tilde{\Phi} \in \Lambda_\gamma$. Let $h \in (0, 1)$, $z_0 = e^{i\varphi}$, $z_1 = (1 - h)e^{i\varphi}$, $z_2 = (1 - h)e^{i(\varphi + h)}$, $z_3 = e^{i(\varphi + h)}$.

Then, by Cauchy’s theorem,

$$\Phi(z_3) - \Phi(z_0) = \int_{[0, z_3]} F(z) \, dz + \int_{[z_0, 0]} F(z) \, dz$$

$$= \left( \int_{[z_0, z_1]} + \int_{z_1}^{z_3} \right) F(z) \, dz.$$
For sufficiently small \( h > 0 \), we have
\[
\left| \int_{[z_0, z_1]} F(z) \, dz \right| \leq \int_{1-r}^{1} \frac{C_4}{(1-h)^{1-\gamma}} \, dr = \frac{C_4 h^{\gamma}}{\gamma}.
\]

Similarly, \( \left| \int_{[z_2, z_3]} F(z) \, dz \right| \leq \frac{C_4}{\gamma} h^{\gamma} \). It is obvious that \( \left| \int_{z_1}^{z_2} F(z) \, dz \right| \leq C_4 h^{\gamma} \).

Therefore, \( \left| \Phi(e^{i(\varphi+h)}) - \Phi(e^{i\varphi}) \right| \leq C_4 (\frac{2}{\gamma} + 1) h^{\gamma} \), so \( \tilde{\Phi} \in \Lambda_\gamma \).

2.2. The Case \( \alpha > 0 \).

Necessity. Let \( u \) have form (1.3), where \( \psi \in BV[0, 2\pi] \cap \Lambda_\gamma \). This implies
\[
u_{\alpha}(r e^{i\varphi}) = \int_{\varphi}^{\varphi + 2\pi} P_0(r, \varphi - \theta) \, d\psi(\theta).
\] (2.5)

By the proposition we have \( \sup_{r<1} \int_{0}^{2\pi} \left| u_{\alpha}(re^{i\varphi}) \right| \, d\varphi < +\infty \) and \( B(r, u_{\alpha}) = O((1-r)^{\gamma-1}) \) as \( r \uparrow 1 \).
We use the following formula [1, Chap. IX, (2.9)]:

\[
u(re^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} P_\alpha(r/\rho, \varphi - \theta) u_\alpha(\rho e^{i\theta}) d\theta, \quad 0 \leq r < \rho < 1.\]

Taking \( \rho = (1 + r)/2 \) and using the estimate \( B(r, u_\alpha) = O((1 - r)^{\gamma - 1}) \) \((r \uparrow 1)\) we obtain

\[
|u(re^{i\varphi})| = O \left( \int_0^{2\pi} \frac{(1 - \rho)^{\gamma - 1} d\theta}{|1 - \frac{r}{\rho} e^{i(\varphi - \theta)}|^{1+\alpha}} \right)
= O((1 - r)^{\gamma - 1}(\rho - r)^{-\alpha}) = O((1 - r)^{\gamma - 1 - \alpha}), \quad r \uparrow 1.
\]

The necessity is proved.

**Sufficiency.** Let \( \int_0^{2\pi} |u_\alpha(re^{i\varphi})| d\varphi < +\infty \) uniformly for all \( r \in (0, 1) \). Then, by Theorem C, we have (2.5), where \( \psi \in BV[0, 2\pi] \).

We need the following elementary lemma.

**Lemma 2.1.** Let, for all \( x \in [0, 1), f \in L[0, x], \) and \( 0 \leq \eta < \beta, |f(x)| = O((1 - x)^{-\beta}) \) as \( x \uparrow 1 \). Then \( |D^{-\eta}f(x)| = O((1 - x)^{\eta - \beta}) \) as \( x \uparrow 1 \).

**Proof.** Using the definition of the fractional integral and standard estimates we obtain

\[
|D^{-\eta}f(x)| = \left| \frac{1}{\Gamma(\eta)} \int_0^x f(t)(x-t)^{\eta-1} dt \right|
= O \left( \int_0^x \frac{(x-t)^{\eta-1}}{(1-t)^{\beta}} dt \right)
= O \left( \int_0^{x-2(1-x)} (1-t)^{\eta-1} dt + \int_{x-2(1-x)}^x \frac{dt}{(1-t)^{\beta}} \right)
= O((1 - x)^{\eta - \beta}), \quad x \uparrow 1.
\]

The lemma is proved. \( \square \)

Since \( B(r, u) = O((1 - r)^{\gamma - 1 - \alpha}) \) as \( r \uparrow 1 \), by the lemma we have that \( B(r, u_\alpha) = O((1 - r)^{\gamma - 1}) \) as \( r \uparrow 1 \). Therefore, by Proposition 2.1, \( \psi \in \Lambda_\gamma \).
3. Further Results on Analytic Functions

There is an analogue of Theorem C for analytic functions proved by M. Djrbashian [1]. The following theorem can be proved in the same way as the proposition and Theorem 1.1.

**Theorem 3.1.** Let \( f(z) \) be an analytic function on \( D, \alpha \geq 0, 0 < \gamma < 1 \). Then \( f(z) \) has the form

\[
f(re^{i\varphi}) = \int_0^{2\pi} S_\alpha(r, \varphi - t) d\psi(t) + i\Im f(0),
\]

where \( \psi \) has bounded variation on \([0, 2\pi]\) and \( \psi \in \Lambda_\gamma \), if and only if

\[
B(r, |f|) = O((1-r)^{\gamma - \alpha - 1}), \quad r \uparrow 1,
\]

and

\[
\sup_{0 < r < 1} \int_0^{2\pi} |\Re f_\alpha(re^{i\varphi})| d\varphi < +\infty,
\]

where \( f_\alpha(re^{it}) = r^{-\alpha}D^{-\alpha}f(re^{it}) \).

Theorem 1.1 does not cover the case where \( \psi \in \Lambda_0 \), in particular, where \( \psi \) is not continuous. Here, following [10], we are able to prove a more precise result. It seems to be known, but I have not found it in the literature.

**Theorem 3.2.** Let \( f(z) \) have the form

\[
f(z) = \int_0^{2\pi} (1-ze^{-it})^{-\alpha} d\psi(t), \quad z \in D,
\]

where \( \alpha > 0, \psi \in BV[0, 2\pi] \). If \( \{t_k\} \) is the set of discontinuity points of \( \psi \) with jumps \( \{h_k\} \), then

\[
f(z) = \frac{h_k + o(1)}{(1-ze^{-it_k})^\alpha}, \quad z \to e^{it_k}, \quad z \in S(t_k, \tau), \quad \tau \in [0, \pi),
\]

and

\[
f(z) = \frac{o(1)}{(1-ze^{-it})^\alpha}, \quad z \to e^{it}, \quad z \in S(t, \tau), \quad t \notin \{t_k\}, \quad \tau \in [0, \pi).
\]
Proof. Since $\psi \in BV[0, 2\pi]$, the set $\{t_k\}$ is at most countable. Without loss of generality we can assume that $\psi$ is continuous from the right. Then $h_k = \psi(t_k) - \psi(t_k - 0)$. It is sufficient to prove (3.2) for $k = 1$.

We may assume that $t_1 \in (0, 2\pi)$. Let

$$H_1(t) = \begin{cases} 0, & 0 \leq t < t_1, \\ h_1, & t_1 \leq t \leq 2\pi. \end{cases}$$

We extend $\psi$ to $\mathbb{R}$ by the formula $\psi(t + 2\pi) = \psi(t)$, as well as $H_1$. The function $g(t) \overset{\text{def}}{=} \psi(t) - H_1(t)$ is continuous at the points $t = t_1 + 2\pi k$, $k \in \mathbb{Z}$, so $\omega(\delta, t_1, g) = o(1)$ as $\delta \downarrow 0$. We have

$$f(z) = \int_0^{2\pi} (1 - ze^{-it})^{-\alpha} \, dg(t) + \frac{h_1}{(1 - ze^{-it_1})^\alpha}. \quad (3.4)$$

Let $\omega_1(\delta) = \max\{\sqrt{\omega(\delta, t_1, g)}, \delta^{\alpha/2}\}$. It is easy to see that $\omega_1(\delta)$ satisfies the hypotheses of Theorem B on $\omega(\delta)$. Applying Theorem B to the integral in (3.4) we obtain (3.2) from (3.4).

Relationship (3.3) follows directly from Theorem B if we choose $\omega_2(\delta) = \max\{\sqrt{\omega(\delta, t, \psi)}, \delta^{\alpha/2}\}$. \qed

For $\psi \in BV[0, 2\pi]$, we define $\tau[\psi]$ to be $\sup \gamma$ satisfying $\psi \in \Lambda_\gamma$. In particular, $\omega(\delta, \psi) \in \Lambda_{\tau[\psi]-\varepsilon} \setminus \Lambda_{\tau[\psi]+\varepsilon}$.

**Theorem 3.3.** Let $F(z)$ be analytic in $D$, 

$$\log |F(re^{i\varphi})| = \int_0^{2\pi} P_\alpha(r, \varphi - t) \, d\psi(t),$$

where $\psi \in BV[0, 2\pi]$, $\tau[\psi] = \tau \in [0, 1)$. Then $\rho_M[F] = \alpha + 1 - \tau$, $\rho_T[F] \leq \alpha$. If, in addition, $\psi$ is not absolutely continuous, then $\rho_T[F] = \alpha$.

**Corollary 3.1.** Suppose that the conditions of Theorem 3.3 hold, and $\tau = 0$. Then $\rho_M[F] = \rho_T[F] + 1 = \alpha + 1$.

**Proof of Theorem 3.3.** First, let $\tau \in (0, 1)$. By Theorem 1.1,

$$\sup_{r < 1} \int_0^{2\pi} |u_\alpha(re^{i\varphi})| \, d\varphi < +\infty.$$ 

Since $\omega(\delta, \psi) \in \Lambda_{\tau-\varepsilon} \setminus \Lambda_{\tau+\varepsilon}$, $0 < \varepsilon \leq \min\{\tau, 1 - \tau\}$, by applying Theorem 1.1 again, we have

$$\log M(r, F) = B(r, \log |F|) = O((1 - r)^{\tau-\alpha-1-\varepsilon}),$$

where $B(r, \log |F|)$ is the integral of $|F|$ over $D$.
\[ \log M(r, F) \neq O((1 - r)^{1-\alpha-1+\varepsilon}), \quad r \uparrow 1, \]
i.e., \( \rho_M[F] = \alpha + 1 - \tau. \)

Further,

\[
T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{1}{2\pi} \int_0^{2\pi} P_\alpha(r, \varphi - t) \, d\psi(t) \right) \, d\varphi \\
\leq \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} P_\alpha^+(r, \varphi - t) \, d\psi(t) \, d\varphi \\
\leq \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{2}{|1 - re^{i\theta}|^{\alpha+1}} \, d\theta \\
= \begin{cases} 
O((1 - r)^{-\alpha}), & \alpha > 0, \\
O\left(\log \frac{1}{1-r}\right), & \alpha = 0, 
\end{cases} \quad r \uparrow 1,
\]
i.e., \( \rho_T[f] \leq \alpha. \)

In order to complete the proof of Theorem 3.3, we need the following result of F. A. Shamoian [8], which compares the classes \( A_\alpha \) and \( A^*_\alpha \).

**Theorem E ([8, Theorem 3]).** We have

\[ F(z) = \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} S_\alpha(ze^{-i\theta}) \, d\psi(\theta) \right\} \in A^*_\alpha \]

if and only if the following holds: 1) \( \psi \) is absolutely continuous;

2) \[ \int_0^{2\pi} \int_0^{2\pi} \frac{\left| \psi(\theta + t) - 2\psi(\theta) + \psi(\theta - t) \right|}{t^2} \, dt \, d\theta < +\infty. \]

As we noted above, \( F \in A^*_\alpha \) if and only if \( T(r, F) \) belongs to the convergence class of order \( \alpha \). Therefore, if \( \psi \) is not absolutely continuous, \( F \) has growth at least as the divergence class of order \( \alpha \), i.e., \( \rho_T[F] \geq \alpha. \)

If \( \tau = 0 \), then one can deduce in a similar way that \( \rho_T[f] \leq \alpha. \) Since \( \omega(\delta; \psi) \not\in \Lambda_\varepsilon, \varepsilon > 0, \)

\[ \log M(r, f) \neq O((1 - r)^{1-\alpha-1+\varepsilon}), \quad r \uparrow 1, \]
i.e., \( \rho_M[F] \geq \alpha + 1. \) Using the inequality \( \rho_M[F] \leq \rho_T[F] + 1 \), we obtain \( \rho_M[F] = \rho_T[F] + 1 = \alpha + 1, \) which is the statement of the corollary. \( \square \)
Remark 3.1. The condition $\tau < 1$ in Theorem 3.3 is essential. In fact, by the Cauchy theorem on residues,

$$\int_0^{2\pi} \frac{d\theta}{(1 - e^{-i\theta}z)^n} = 2\pi, \quad n \in \mathbb{N}, \ z \in D.$$ 

References

[8] F. A. Shamoian, Several remarks to parametric representation of Nevanlinna-Djrbashian’s classes // Mat. Zametki 52 (1992), N 1, 128–140.

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